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Interactive Decision-Making in Multiobjective Fuzzy Programming

Jose M. Cadenas[a], Fernando Jiménez[b]
Dpto. de Informática y Sistemas
Facultad de Informática. Universidad de Murcia
30071-Espinardo, Murcia
e-mail: [a]*jcadenas@dif.um.es*, [b]*fernan@dif.um.es*

Abstract

We present an interactive decision support system which aids in solving a general multiobjective fuzzy problem, that is, a multiobjective programming problem with fuzzy goals subject to a fuzzy constraint set.

The interactive decision support system is proposed. After eliciting the fuzzy goals of the decision maker for each objective function and the fuzzy elements for each constraint, the satisfactory solutions for the decision maker were derived by interactively updating the reference membership values set by the decision maker. We discuss here multiobjective fuzzy linear programming and interactive multiobjective fuzzy linear programming.

Keywords: Multiobjective Programming, Fuzzy Sets, Interactive decission support system.

1 Introduction

Models considering multiple criteria are of increasing importance in decision support [4]. The decision maker is no longer forced to restrict his considerations to one main aspect but can take into account different points of view. In multiple objective programming models the criteria are described by objective functions and possible alternatives are implicity determined by constraints.

When the fuzzy goals and fuzzy contraints of the decision maker for the linear programming problem are elicited by linear membership functions, the problem can be solved as an ordinary linear programming problem [13]. We can obtain a fuzzy solution when the parametric method is utilized. This solution is a subset of the efficient solutions to a problem. The solution with the highest degree of membership to the fuzzy decision set is a maximizing decision using the min-operator (in the terminology of Bellman-Zadeh). That is, a fuzzy decision D is the fuzzy set of all those elements which belong to the fuzzy sets Z_l, $l = 1, \ldots, k$, describing goals and to the fuzzy sets R_i, $i = 1, \ldots, m$, describing constraints:

$$D = \bigcap_{l=1}^{k} Z_l \cap \bigcap_{i=1}^{m} R_i$$

Choosing an optimal element, then, means selecting the element x^0 which has the highest degree of membership to the fuzzy decision set:

$$\mu_D(x^0) = \max_{x \in X} \mu_D(x) = \max_{x \in X} \min \left\{ \min_{l=1,\ldots,k} \mu_{Z_l}(x), \min_{i=1,\ldots,m} \mu_{R_i}(x) \right\}$$

We consider linear models:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) = (c_1 x, c_2 x, \ldots, c_k x) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij} x_j \underset{\sim}{<} b_i, \; i = 1, \ldots, m \\ & x_j \geq 0, \; j = 1, \ldots, n \end{array}$$

If the goals and constraints are described by membership functions $\mu_{z_l}(x)$ and $\mu_{R_i}(x)$ respectively, the optimal decision is x^0 with

$$\mu_D(x^0) = \max_{x \in X} \min \{\mu_{z_1}(x), \ldots, \mu_{z_k}(x), \mu_{R_1}(x), \ldots, \mu_{R_m}(x)\}$$

An equivalent formulation is given by the following crisp mathematical programming model

$$\begin{array}{llll} Max & \alpha & & \\ s.t.: & & & \\ & z_l(x) \geq \overline{z}_l - (1-\alpha)(\overline{z}_l - \underline{z}_l) & & , \; l = 1, \ldots, k \\ & \sum_{j=1}^{n} a_{ij} x_j \leq \; b_i + d_i(1-\alpha) & & , \; i = 1, \ldots, m \\ & x_j \geq 0, \quad \alpha \in [0,1] & & , \; j = 1, \ldots, n \end{array}$$

when the $\mu_{z_l}(\cdot)$, $\mu_{R_i}(\cdot)$, $l = 1, \ldots, k$, $i = 1, \ldots, m$ are linear membership functions.

2 A General Multiobjective Fuzzy Linear Programming Model

In this section we propose a general multiobjective Fuzzy Linear Programming (FLP) model in which all the elements describing the model are imprecise, that is, a problem

$$
\begin{array}{ll}
\underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\
s.t.: & \\
 & \sum_{j=1}^{n} \widetilde{a}_{ij} x_j \underset{\sim}{<} \widetilde{b}_i, \; i \in M = \{1, \ldots, m\} \\
 & x_j \geq 0, \; j \in N = \{1, \ldots, n\}
\end{array} \tag{1}
$$

where the following fuzzy elements are considered:

i) For each objective function, $\exists \mu_{z_l}^0 \in F(R)$ such that $\mu_{z_l}^0 : R \to [0,1]$, $l = 1, \ldots, k$, defining the fuzzy goals,

$$
\mu_{z_l}^0(z_l(x)) = \begin{cases} 0 & \text{si } z_l(x) \leq \underline{z}_l \\ \frac{z_l(x) - \underline{z}_l}{\overline{z}_l - \underline{z}_l} & \text{si } \underline{z}_l \leq z_l(x) \leq \overline{z}_l \\ 1 & \text{si } z_l(x) \geq \overline{z}_l \end{cases}
$$

where $\underline{z}_l$, $\overline{z}_l$ are the worst value and the best value of the l-th objective function respectively;

ii) for each row, constraint $\exists \mu_i \in F(R)$ such that $\mu_i : R \to [0,1]$, $i \in M$, which defines the fuzzy number on the right hand side;

iii) for each $i \in M$ and $j \in N$, $\exists \mu_{ij} \in F(R)$ such that $\mu_{ij} : R \to [0,1]$, defining the fuzzy numbers in the technological matrix;

iv) for each row, $\exists \mu^i \in F(F(R))$ such that $\mu^i : F(R) \to [0,1]$, giving, $\forall x \in R^n$, the accomplishment degree of the fuzzy number $\widetilde{a}_{i1}x_1 + \widetilde{a}_{i2}x_2 + .. + \widetilde{a}_{in}x_n$, $i \in M$ with respect to the i-th constraint, that is, the adequacy between this fuzzy number and the corresponding one $\widetilde{b}_i$ with respect to the i-th constraint.

Evidently, this problem generalizes the problems proposed in the specialized literature, where the decision maker is allowed an uncertain knowledge about some elements of the problem.

3 A method for solving a General Multiobjective FLP Model

We propose a method for solving the general model which consists in reemplacing the constraint set by a fuzzy convex set.

Let $A, B \in F(R)$. A simple method for ranking fuzzy number consists in defining a ranking function that maps each fuzzy number into the real line, $g : F(R) \to R$. This function $g(\cdot)$ is known, then

$$g(A) < g(B) \Rightarrow A \text{ is less that } B$$
$$g(A) > g(B) \Rightarrow A \text{ is great that } B$$
$$g(A) = g(B) \Rightarrow A \text{ is equal that } B$$

Usually, g is called the linear ranking function if

a) $g(A+B) = g(A) + g(B),\ \forall A, B \in F(R)$

b) $g(rA) = rg(A),\ r \in R,\ r > 0,\ A \in F(R)$

Using the results introduced in [2], we obtain the following auxiliary problem to solve (1):

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} \widetilde{a}_{ij} x_j \leq_g \widetilde{b}_i + \widetilde{d}_i(1-\alpha) \\ & x_j \geq 0,\ \alpha \in [0,1],\ i \in M,\ j \in N \\ & g \text{ linear ranking function} \end{array} \tag{2}$$

To solve this problem, we can utilize several comparison relations of fuzzy numbers. This allows us to obtain an efficient solution to the problem (1).

3.1 Solving the General Model

In this section, different approaches to studying a solution for the general multi-objective FLP problem will presented. These different approaches will use linear ranking functions for the substitution of the set of constraints in the same ways as we have studied before. If the fuzzy numbers are generated by Homogeneous Linear Fuzzy Function, then using a Linear Ranking Function an auxiliary multiobjective LP problem is easily obtained. In general, using some Linear Ranking Function, an auxiliary multiobjective LP problem is always obtained.

The first approaches assume that the fuzzy numbers of the problem have been generated by linear homogeneous functions, [8], and using the linear ranking functions the fuzzy solution of the problem can be obtained. The following approaches use the general fuzzy numbers (not generated by homogeneous linear fuzzy functions).

3.1.1 Homogeneous Linear Fuzzy Functions

We consider the multiobjective FLP problem (1) and the auxiliary problem (2).

In the following it will be shown that (1) (the auxliary problem (2)) can be transformed into a special equivalent multiobjective FLP problem if the fuzzy numbers are assumed to be defined as Homogeneous Linear Fuzzy Functions.

As it is clear, (2) may be written as

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} f(a_{ij})x_j \leq_g f(b_i) + f(d_i)(1-\alpha) \\ & x_j \geq 0, \ \alpha \in [0,1], \ i \in M, \ j \in N \\ & g \text{ linear ranking function} \end{array} \quad (3)$$

where $f : R \longrightarrow F(R)$ is a homogeneous linear fuzzy function with $f(t) = t\tau$, $t \in R$ and $\tau \in F(R)$, [8].

Theorem 1 *Let f be a homogeneous linear fuzzy function and let g be a linear ranking function. Then the solution of (2) is either*

i) the solution of the following multiobjective FLP problem if $g(\tau) \geq 0$,

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \leq b_i + d_i(1-\alpha) \\ & x_j \geq 0, \ \alpha \in [0,1], \ i \in M, \ j \in N \end{array} \quad (4.1)$$

ii) the solution of the following multiobjective FLP problem if $g(\tau) \leq 0$,

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \geq b_i + d_i(1-\alpha) \\ & x_j \geq 0, \ \alpha \in [0,1], \ i \in M, \ j \in N \end{array} \quad (4.2)$$

Proof.

If the linear ranking function, g, is applied to the constraint, one has

$$\begin{aligned} \sum_{j=1}^{n} f(a_{ij})x_j \quad &\leq_g \quad f(b_i) + f(d_i)(1-\alpha) \Leftrightarrow \\ &\Leftrightarrow \quad g(\sum_{j=1}^{n} f(a_{ij})x_j) \leq g(f(b_i) + f(d_i)(1-\alpha)) \Leftrightarrow \end{aligned}$$

$$\Leftrightarrow \quad g(\tau)\sum_{j=1}^{n} a_{ij}x_j \leq g(\tau)(b_i + d_i(1-\alpha))$$

which is true $\forall x_j \geq 0$, $j \in N$, $i \in M$.

Therefore it follows that the nonfuzzy constraint set is one of the following sets:

$C_1 = \{x \in R^n /\ x \geq 0,\ \sum_{j=1}^{n} a_{ij}x_j \leq b_i + d_i(1-\alpha),\ i \in M,\ \alpha \in [0,1]\}$, si $g(\tau) \geq 0$.

$C_2 = \{x \in R^n /\ x \geq 0,\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i + d_i(1-\alpha),\ i \in M,\ \alpha \in [0,1]\}$, si $g(\tau) \leq 0$.

This proves our assertion. ∎

This approach generalize the fuzzy goal multiobjective problems with/without fuzzy constraints.

- If $f(t) = t\tau$ with $\tau = \widetilde{1}$ ($\widetilde{1} = (1,1,1,1)$), we obtain the problem:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2, (x) \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \underset{\sim}{<} b_i \\ & x_j \geq 0, i \in M, j \in N \end{array}$$

- If $f(t) = t\tau$ with $\tau = \widetilde{1}$, $\alpha \in [0,1]$ ($\widetilde{1} = (1,1,1,1)$), we obtain the problem:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ & x_j \geq 0,\ i \in M,\ j \in N \end{array}$$

3.1.2 Linear Ranking Functions

We consider the multiobjective FLP problem (1), but the fuzzy coefficients have not been obtained from Homogeneous Linear Fuzzy Functions. Let g be the Linear Ranking function that is used. Then to solve (1), the following associated multiobjective LP problem can be considered.

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} g(\widetilde{a}_{ij})x_j \leq g(\widetilde{b}_i) + \widetilde{d}_i(1-\alpha)) \\ & x_j \geq 0,\ \alpha \in [0,1],\ i \in M,\ j \in N \end{array}$$

from which the following equivalent model (where g is linear) can be written:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} g(\widetilde{a}_{ij})x_j \leq g(\widetilde{b}_i) + g(\widetilde{d}_i)(1-\alpha) \\ & x_j \geq 0,\ \alpha \in [0,1],\ i \in M,\ j \in N \end{array} \tag{5}$$

Evidently, according to the characteristic of the several fuzzy number linear ranking functions, [14], different models of Multiobjective LP problems for solving the former problem are obtained.

This approach generalizes the fuzzy goal multiobjective problems with/without fuzzy constraints.

- If $g(\widetilde{u}) = u$, we obtain the problem:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \underset{\sim}{<} b_i \\ & x_j \geq 0,\ i \in M,\ j \in N \end{array}$$

- If $g(\widetilde{u}) = u,\ \alpha \in [0,1]$, we obtain the problem:

$$\begin{array}{ll} \underset{\sim}{Max} & (z_1(x), z_2(x), \ldots, z_k(x)) \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ & x_j \geq 0,\ i \in M,\ j \in N \end{array}$$

4 Interactive Decision Making in Multiobjective Fuzzy Programming

Many models and methods dealing with these problems can be found in the specialized literature. But although FLP is one of the most studied in the fuzzy context, contrary to what has happened with the theorical developments, software implementations of the achieved results are not so frequent [10, 7, 6, 9, 2]. From this point of view, the previous discussion justifies the necessity of developing further work and making an effort in this area.

We proposed an interactive processes for solving Multiobjective FLP problems. The system, through consecutive questions to the decision maker, shows him the different solutions which can be obtained. It interacts with the decision maker, who selects several methods or modifies the elements of the problem, and evaluates their solutions until he is satisfied. The decision maker will select the one he is interested in according to his preferences (criteria).

When the fuzzy numbers and fuzzy constraints are defined and a ranking function is chosen, we calculate both the minimum and maximun individual values for each objective function on the constraint set, [12].

In other words, using the information in the following table of the extreme solutions

	c_1	c_2	c_3	...	c_k
x^{01}	c_1x^{01}	c_2x^{01}	c_3x^{01}	...	c_kx^{01}
x^{02}	c_1x^{02}	c_2x^{02}	c_3x^{02}	...	c_kx^{02}
...	...	...	...	...	...
x^{0k}	c_1x^{0k}	c_2x^{0k}	c_3x^{0k}	...	c_kx^{0k}
x^{11}	c_1x^{11}	c_2x^{11}	c_3x^{11}	...	c_kx^{11}
x^{12}	c_1x^{12}	c_2x^{12}	c_3x^{12}	...	c_kx^{12}
...	...	...	...	...	...
x^{1k}	c_1x^{1k}	c_2x^{1k}	c_3x^{1k}	...	c_kx^{1k}

the system suggests membership functions for objective functions ($x^{01}, \ldots, x^{0k}$, extreme solutions for each objective function with $\alpha = 0$, $x^{11}, \ldots, x^{1k}$, extreme solutions for each objective function with $\alpha = 1$). The individual optimal ones in the diagonal of the upper half. The pessimistic values can occur either in the upper or in the lower half.

With $\bar{c}_l = \sum_j c_{lj} x_j^0$ and $\underline{c}_l = \min \left\{ \min\limits_{l'} \sum_j c_{l'j} x_j^0 \, , \, \min\limits_{l'} \sum_j c_{l'j} x_j^1 \right\}$, linear membership functions are determined which contain information on the structure of the model, [12].

$$\mu^0_{z_l}\left(\sum_j c_{lj}x_j\right) = \begin{cases} 0 & , \sum_j c_{lj}x_j < \underline{c}_l \\ \frac{\sum_j c_{lj}x_j - \underline{c}_l}{\overline{c}_l - \underline{c}_l} & , \underline{c}_l \leq \sum_j c_{lj}x_j \leq \overline{c}_l \\ 1 & , \overline{c}_l \leq \sum_j c_{lj}x_j \end{cases}$$

The decision maker takes account of his degree of satisfaction within the range of the maximum and minimun values of each objective function, and defines the membership function $\mu^0_{z_l}(\sum_j c_{lj}x_j)$. The defining domain of the membership function ranges from the values of objective function $\underline{c}_l^{user}$ for 0 degree of satisfaction to the values of objective function $\overline{c}_l^{user}$ for 1 degree of satisfaction (between $\underline{c}_l$ and $\overline{c}_l$).

In the interactive decision, if the decision maker is not satisfied with the achievement level of the objective function of the solution, he can change the elements of problem step by step until a satisfactory solution is obtained.

An interactive algorithm for finding a satisfactory solution can be constructed as follows. (see fig. 1)

Interactive algorithm for Fuzzy Multiobjective Linear Programming

1. Model Formulation.
2. Decide the membership function for each fuzzy number of the constraints and for each fuzzy constraint (if the fuzzy numbers are assumed to be defined as a homogeneous linear fuzzy function, decide the function f).
3. Choose a linear ranking function.
4. Find the individual minimum and maximum values for each objective function in the given constraint domain.
5. Decide the membership function for each objective function.
6. Solve the multiobjective fuzzy problem (4) (or (5)).
7. If satisfied with the current solution, stop. Otherwise, change the parameters describing the problem. Go to step 2.

The interactive steps (2,3,5,7) permit the decision maker to test several elements until a solution is satisfactory.

5 Conclusions

We have proposed a solution method for a general Multiobjective Fuzzy Linear Programming problem according to Bellman and Zadeh's maximizing decision. An

Figure 1: The Interactive System's Flowchart

interactive method is introduced which aids in solving the multiobjective fuzzy programming problem, transformed into an ordinary linear programming problem.

The interactive is used to choose the appropiate function to transform the fuzzy constraint set into an ordinary contraint set and to define the membership functions for each fuzzy element.

The satisfactory solutions for the decision maker were derived by interactively updating the elements of the problem.

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