

Information systems in categories of valued relations

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Abstract

The paper presents a categorical version of the notion of information system due to D. Scott. The notion of information system is determined in the framework of ordered categories with involution and division and the category of information systems is constructed. The essential role in all definitions and constructions play correlations between inclusion relations and entailment relations.

0 Introduction

The axiomatic definition of information system introduced by D. Scott in [S82] is closely related with deep results of the theory of approximation lattices developed by the same author. An important idea due to Scott is to study discrete objects using continuous ones. This idea is closely related with the fundamental methodology conception of fuzzy

*This paper has been written while the author was an invited professor at the Centre de Recerca Matemàtica, Institut d'Estudis Catalans, Universitat Autònoma de Barcelona.

mathematics. Realization of Scott's theory in the fuzzy case leads to an approach to the notion of fuzzy computability.

In this paper the first step of such a realization is proposed. We introduce the notions of E-structure (entailment structure) and construct the category of E-structures. Roughly speaking an E-structure is a "data object" endowed with inclusion relation (or approximation relation) i , and entailment relation s satisfying "elementless" analogues of Scott's axioms. To define morphisms of E-structure we use specific techniques of so-called correspondence categories. Formula

$$s[i, i] = [s, s]$$

(first applied in [G88]) is a keypoint of our constructions. This formula allows to translate the definition of morphism of information systems into the language of correspondence categories and so to avoid the use of elements.

The further expansion of Scott's theory of computability to the fuzzy case is related with investigation of categories of E-structures in ordered categories with involution and division endowed with some additional operations, for example with a functor "directed family of finite subsets" or something like that.

All constructions are developed in the framework of abstract ordered categories with involution. However the essential role is played by categories of valued relations.

The paper is organized as follows.

In section 1 basic notions, constructions and results concerning ordered categories with involution and categories of L -fuzzy relations are presented.

In section 2 the main example (that of an information system in the category of L -fuzzy relations) is analyzed.

In section 3 we construct the category of information systems over a general correspondence category with division.

1 OI-categories

In terminology, basic notions and results we follow [CGR84], [GT84]. Well composed and well compiled exposition can be also found in

[FS90] (ch. 2, 2.1-2.3).

By an ordered category with involution (or shortly OI-category) C we mean a category endowed with an involution $^\circ$ and order relations \subset on sets of morphisms $C(X, Y)$ where X, Y are objects from C such that the following conditions hold:

- 1) involution is a contravariant functor such that $X^\circ = X$, $f^{\circ\circ} = f$, $(fg)^\circ = g^\circ f^\circ$ for all objects X and morphisms f, g whenever fg is defined;
- 2) composition and involution are monotone, i.e. if $f \subset g$ then $f^\circ \subset g^\circ$ and $hf \subset hg$ and $fh \subset gh$ whenever compositions are defined.

A morphism $f : X \rightarrow Y$ in an OI-category is called:

- functional if $f^\circ f \subset 1$ and $ff^\circ \supset 1$;
- injection if f is functional and $ff^\circ \subset 1$;
- projection if f is functional and $f^\circ f \supset 1$.

Functional morphisms of OI-category C form a category which will be denoted by $Fun(C)$.

A morphism $e : X \rightarrow X$ is called:

- reflexive if $e \supset 1$;
- symmetric if $e^\circ = e$;
- transitive if $ee \subset e$;
- coreflexive if $e \subset 1$;
- cotransitive if $ee \supset e$;
- preordering if e is reflexive and transitive;
- ordering if 1 is the greatest morphism h such that $e \supset h$ and $e^\circ \supset h$;

- equivalence if e is reflexive, symmetric and transitive;
- coequivalence if e is coreflexive, symmetric and cotransitive.

Equivalence e on X is a congruence if e can be presented in the form $e = ff^\circ$ where $f : X \rightarrow Y$ is a functional morphism. If f is a projection then Y is a factorobject of X with respect to e . In this case e is called realizable.

Dually, coequivalence e is a cocongruence if e is this presentable in the form $e = f^\circ f$ where f is functional. If e is of the form $f^\circ f$ for an injection f then e is realizable.

Let $f : X \rightarrow Z, g : Y \rightarrow Z$ be morphisms in an OI-category C . By f/g is denoted the greatest morphism (if exists) $h : X \rightarrow Y$ such that $hg \subset f$. Dually, if $f : Z \rightarrow X, g : Z \rightarrow Y$ we put $g \setminus f = (f^\circ/g^\circ)^\circ$. (Note that morphisms of the form f/g and $g \setminus f$ are widely used in the theory of fuzzy relational equations).

OI-category C is a category with division (shortly OID-category) if for every pair of cofinal morphisms f, g in C there exists f/g .

Obviously $l = f/f$ is preordering and $lf = f$. Further

$$h \setminus (f/g) = (h \setminus f)/g$$

for all morphisms f, g, h whenever all quotient exist.

OI-category C is called modular (allegory in [FS90]) if it satisfies the following axioms.

11. All sets $C(X, Y)$ are lower semilattices.
12. If $f : X \rightarrow Y, s : X \rightarrow Z, t : Y \rightarrow Z$ are morphisms in C then the following inequality holds

$$(r \cap st^\circ)t \supset rt \cap s.$$

A modular category is called correspondence category if it satisfies the following axiom.

10. For every morphism $r : X \rightarrow Y$ in C there exists a functional factorization, i.e., there exist functional morphisms f, g such that $r = f^\circ g$.

A reason to call OI-categories satisfying I0-I2 correspondence categories provides the theorem which states that every small correspondence category can be embedded in the category of sets and relations with the natural composition, involution and ordering.

Note that hom-sets in modular OID-categories are Heyting algebras.

In order to have “enough” subobjects and factorobjects any OI-category can be completed by symmetric idempotents. In early 70th this construction was introduced in a serie of papers by the author and M. Tsalenko dealing with homological constructions and prevarietes in non-abelian categories. Later it was applied by the same authors in the general theory of fuzzy systmes [GT84].

Let C be an OI-category. We shall describe the completion of C by symmetric idempotents which will be denoted by $SI(C)$ (by a symmetric idempotent on X we mean a morphism $a : X \rightarrow X$ such that $a = aa = a^\circ$).

The objects of $SI(C)$ are pairs (X, a) where X is an object from C and a is a symmetric idempotent on X . If $(X, a), (Y, b)$ are objects in $SI(C)$ then morphisms from (X, a) to (Y, b) in $SI(C)$ are triples $(f; a, b)$ such that $f : X \rightarrow Y$ is a morphism in C and $af = f = fb$. The structure of an OI-category in $SI(C)$ is defined by

$$\begin{aligned} (f; a, b)(g; b, c) &= (fg; a, c); \\ (f; a, b)^\circ &= (f^\circ; b, a); \\ (f; a, b) \subset (g; a, b) &\leftrightarrow f \subset g \end{aligned}$$

where $(f; a, b), (g; b, c)$ are morphisms in $SI(C)$. Morphisms of the form $(a; a, a)$ are units in $SI(C)$. In $SI(C)$ every equivalence is a realizable congruence, and, dually, every coequivalence is a realizable cocongruence. Any functional morphism in $SIC(C)$ can be presented as the composition of a projection followed by an injection.

Two full subcategories of $SI(C)$ are of special interest. The category of equivalence $Eq(C)$ and the category of coequivalence $Coeq(C)$. Objects of $Eq(C)$ (resp. of $Coeq(C)$) are equivalences (resp. coequivalences) from C . In $Eq(C)$ every equivalence is realizable and $Eq(C)$ is the minimal completion of C satisfying this property. The dual is true for $Coeq(C)$.

If C is divisible then so is $SI(C)$: given $(f; a, c), (g; b, c)$ in $SI(C)$ then

$$(f; a, b)/(g; b, c) = (a(f/g)b; a, b).$$

If C is a modular OI-category then it can be checked that $SI(C)$ is also modular.

If C satisfies the following axiom

I0'. Every morphism r in C is representable in the form $r = f \circ c g$ where f, g are functional and c is a coequivalence,

then $SI(C)$ satisfies I0.

So if C is a modular category satisfying I0' then $SI(C)$ is a correspondence category. Note that in this case $SI(C)$ is equivalent to $Eq(Coeq(C))$.

Let $(L, *, \leq)$ be a commutative completely lattice-ordered integral monoid with zero (see [B73]). That is (L, \leq) is a complete lattice with the top 1 and the bottom 0, and $(L, *)$ is a commutative monoid, 1 is a unit and 0 is a zero for $*$, and $*$ is monotone with respect to \leq . If in addition $*$ is completely distributive with respect to sups, i.e.

$$a * \sup(A) = \sup(a * A)$$

for all a in L , $A \subset L$, then according to [HS91] we shall say that L is a ruler (for measuring fuzziness).

If L is a ruler then L is residuated with implication

$$a \rightarrow b = \sup\{x | x * a \leq b\}.$$

Let X, Y be sets. Functions $r : X \times Y \rightarrow L$ are called L -fuzzy relations (from X to Y , $r : X \rightarrow Y$). The involution of r is defined by $r^\circ(y, x) = r(x, y)$. If $r : X \rightarrow Y$, $s : Y \rightarrow Z$ are L -fuzzy relations then the composition rs is defined by

$$rs(x, z) = \sup\{r(x, y) * s(y, z) | y \in Y\}.$$

The composition of L -fuzzy relations is associative iff L is a ruler. In what follows we assume that L is a ruler.

Taking sets to be objects and L -fuzzy relations to be morphisms we obtain an OI-category which will be denoted by $L\text{-Fur}$ or simply by \mathbf{Fur} (L will be omitted if it makes no confusion). Category \mathbf{Fur} is divisible: if $r : X \rightarrow Z$, $s : Y \rightarrow Z$ are fuzzy relations then r/s is defined by

$$(r/s)(x, y) = \inf\{s(y, z) \rightarrow r(x, z) \mid z \in Z\}.$$

Categories $\text{Fun}(SI(L\text{-Fur}))$, $\text{Fun}(Eq(L\text{-Fur}))$, $\text{Fun}(Coeq(L\text{-Fur}))$ and its subcategories form various categories of L -fuzzy sets, L -fuzzy sets with fuzzy equalities and so on. Not going into details we refer to the exhaustive analysis in [HS91].

2 Information systems in categories of L -fuzzy relations

Throughout this section we assume that L is a Heyting algebra.

Let $p : D \rightarrow A$ be an L -fuzzy relation treated as a “property-object” relation. Let Fin be the set of functions from D to L with finite supports. For u in Fin , a in A we put

$$m(u, a) = u(a).$$

So m is the membership relation. The inclusion relation on Fin is defined by

$$i(u, v) = \inf\{v(d) \rightarrow u(d) \mid d \in D\}.$$

We define the incidence relation q from Fin to A by

$$q(u, a) = \inf\{u(d) \rightarrow p(d, a) \mid d \in D\}.$$

The family of consistent subsets in Fin is defined by the membership function

$$Con(u) = \sup\{q(u, a) \mid a \in A\}.$$

Further, define the entailment relation from Fin to D by

$$t(u, d) = \inf\{q(u, a) \rightarrow p(d, a) \mid a \in A\},$$

and the relation from Fin to Fin generated by t (which also will be called entailment relation) by

$$s(u, v) = \inf\{v(d) \rightarrow t(u, d) \mid d \in D\}.$$

If u, v are in Fin we put $w = \max\{u, v\}$ for w defined by

$$w(d) = \max\{u(d), v(d)\}.$$

It can be checked that

$$s(u, v) = \inf\{q(u, a) \rightarrow q(v, a) \mid a \in A\}.$$

Now we shall enumerate some properties of the above mentioned relations:

- (is1) s is a preordering;
- (is2) $i \subset s$;
- (is3) $Con(u) \wedge s(u, v) \wedge Con(v) \leq Con(\max\{u, v\})$;
- (is4) $s(u, v) \wedge s(u, v') \leq s(u, \max\{v, v'\})$;
- (is5) $i(u', u) \wedge s(u, v) \wedge i(v, v') \leq s(u', v')$

for all u, u', v, v' in Fin .

Using this properties we arrive immediately to the fuzzy version of the notion of information system.

Let D be a set and Con be a fuzzy family of fuzzy subsets of D with finite supports. Let i be an inclusion relation. The structure of information system on Con is defined by an entailment relation i.e. by relation s satisfying (is1)-(is5).

In terms of the relational algebra conditions (is3)-(is5) can be replaced by:

$$\begin{aligned} s'^{\circ} s' &\subset s'^{\circ} i'; \\ s' &\subset (s' \cap i'^{\circ}) i', \end{aligned}$$

where i' and s' are restrictions of r and s to Con (i.e. $i'(u, v) = Con(u) \wedge i(u, v) \wedge Con(v)$ and the same for s').

Let (Con_1, s_1) , (Con_2, s_2) be information systems. A morphism from (Con_1, s_1) to (Con_2, s_2) is a fuzzy relation f from Con_1 to Con_2 (or from Fin_1 to Fin_2) such that the following conditions hold:

$$(mis1) \quad Con_1(u) \wedge Con_2(v) \wedge Con_2(v') \wedge f(u, v) \wedge f(u, v') \leq \\ Con_2(\max\{v, v'\}) \wedge f(u, \max\{v, v'\});$$

$$(mis2) \quad s_1(u', u) \wedge f(u, v) \wedge s_1(v, v') \leq (u', v').$$

The composition of fuzzy relations provides the composition of morphisms of information systems.

Now let $f_i : X \rightarrow Y_i$, $i = 1, 2$ be fuzzy relations. We define fuzzy relation $[f_1, f_2]$ from X to $Y_1 \times Y_2$ by

$$f(x, y_1, y_2) = f(x, y_1) \wedge f(x, y_2).$$

In the definition of an information system condition $s' \subset (s' \cap i'^o)i'$, is equivalent to the following equality

$$s[i, i] = [s, s].$$

Fuzzy relation f defines a morphism of information systems (Con_1, s_1) and (Con_2, s_2) iff the following equalities hold:

$$\begin{aligned} s'_1 f &= f = f s'_2; \\ f[i_2, i_2] &= [f, f]. \end{aligned}$$

3 Information systems in correspondence categories

Let C be a correspondence category with division in which all equivalences are realizable (for example, let $C = SI(L-\mathbf{Fur})$ where L is a Heyting algebra).

By an E -structure we mean a triple (C, i, s) where C is an object and i, s are morphisms in C satisfying the following conditions:

- (is0') i is an ordering;
- (is1') s is a preordering;
- (is2') $i \subset s$;
- (is3') $s^\circ s \subset i^\circ i$;
- (is4') $s \subset (s \cap i^\circ)i$.

In the general categorical framework an E -structure can be generated like in the previous section.

Let $p : D \rightarrow A$, $m : F \rightarrow D$ be morphisms in C . Put

$$q = m^\circ \setminus p$$

Further, let $k : C \rightarrow F$ be an injection such that

$$k^\circ k = 1 \cap qq^\circ$$

(k is a “domain” of q). Let i be an “inclusion” generated on C by m , i.e.

$$i = k(m/m)k^\circ,$$

and let s be an “entailment” generated by q ; i.e.

$$s = k(q/q)^\circ k^\circ.$$

We also can define t by

$$t = k(p/q)k^\circ$$

and show that

$$s = t/km.$$

Conditions (is0')-(is2') hold without any additional assumptions. To make (is3') and (is4') be true we have to impose F to be “similar to the power-set”. Nevertheless it can be seen that all components of an E -structure can be reconstructed in OID-categories starting with the basic relation.

Assume that for any pair of objects Y_1, Y_2 there exists in $C(Y_1, Y_2)$ the greatest morphism. This morphism can be presented in the form

$u_1^o u_2$ where $u_1 : U \rightarrow Y_1$, $u_2 : U \rightarrow Y_2$ are functional morphisms such that $u_1 u_1^o \cap u_2 u_2^o = 1$ (note that in this case U is a direct product of Y_1 and Y_2 in $Fun(C)$).

For morphisms $f_i : X \rightarrow Y_i$; $i = 1, 2$, put

$$[f_1, f_2] = f_1 u_1^o \cap f_2 u_2^o.$$

It can be shown that $[f_1, f_2]$ is the greatest morphism such that $f u_i \subset f_i$, $i = 1, 2$. Further, condition (is4') in the definition of information system is equivalent to

$$s[i, i] = [s, s].$$

Given E -structures (C_1, i_1, s_1) , (C_2, i_2, s_2) , we say that $f : C_1 \rightarrow C_2$ is a morphism of E -structures if the following conditions hold:

$$(mis1') s_1 f = f s_2;$$

$$(mis2') f[i_2, i_2] = [f, f].$$

The composition of morphisms of E -structures is induced by the composition in C . Using the specific techniques of correspondence categories it can be shown that this definition is correct.

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