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Operating on formal concept abstraction

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Abstract

The subject of this paper regards a procedure to obtain the abstract from of concepts, directly from their most natural form, thus these can be efficiently learned and the possibility of operating formally on them is reached. The achievement of said type of form results also useful to compute conceptual parameters symbolic and numerical in nature.
Key-words: Concept definitions; Fuzzy categories; Concept abstraction; Operations on concepts.

1 Introduction

One of the central problems of Artificial Intelligence as well of Cognitive Psychology is *formalising concepts* to get the possibility of operating on these. In the latter discipline, for better understanding what is intelligence; in the former, for developing intelligent algorithms. Concepts may derive either directly from structuring *entities* or from combining on other concepts, as is performed in the most effective way by human mind in which reality can be perceiving and efficiently managed. In knowledge-based technology entities are considered as *facts, objects*, or *situations* (Stepp, Michalski, 1986). Of fact "entity" corresponds to the Aristotle's "substance"; in the present context, it indicates nothing else than that which has reality and distinctness of being in facts or in thought. We use it to represent the world in distinct discrete units (Tomela, 1973). Concept learning has been frequently affirmed to be the principal basis for classification (Brachman, 1985; Chandrasekaran, Goel, 1988; Schmolze, Lipkis, 1983; Bobrow, Winograd, 1979) and rational reasoning (Dubois, 1991); the classes they give place to are *categories* and the entities these include constitute instances of the concepts themselves.

Many cognitivists stated the gradualness of efficiency in representing concepts (Roth and Mervis, 1983; Rosch, 1975; Vandierendock, 1991). The fuzzy nature of the concerning categories as well as the concepts themselves results so we evident. To learn concepts, consists of becoming acquantied with the entities that can instance these, i.e., that form the respective categories; in addition this also regards both acquiring concept symbolic and numerical parameters, such as *prototypes, typicality* of concept instances etc., and to derive concepts from other concepts, rationally. Thus to determine said parameter is urging. The formalism by which it would have been expected to fulfil this necessity was Fuzzy Set Theory (FST): a theory by which to represent experimental results regarding gradients of membership is direct. This theory, incepted by Zadeh in the sixties (Zadeh, 1965), has had a large success and a lot of applications, specially in the field of process control, as is reported in the current literature. Unfortunately, it has been ascertained that some incongruent structural foundations of FST, make this unsuitable to set up a concept algebra and, definitively, prevent a convenient use of such theory in Cognitive Sciences (see for example Osherson and Smith, 1981) and often in Artificial Intelligence, too.

In the work we are dealing with, we developed procedure to determine said parameters. This is carried out by putting into relation the *extensional definition* of a concepts (ED) with the *one intensional* (ID). The former being formed by the (possibly exhaustive) list of the entities that satisfy a considered concept, i.e., that can be tis instances. The latter consists instead of the disjunction of the (alternative) conjunctions of the necessary requirements to be ascertained in a given entity, so that this may be considered as an instance of the concept itself.

The mentioned forms of concept definitions as usual are not directly available: the intensional one can be derived from that extensional throughout an heavy algorithm and after the latter has been drawn from a concept natural form, which is a *rough definition* (RD): the form in which concepts more currently are expressed.

One procedure leading to ID from that ED has been studied by us. This has roots in the minimization algorithm of switching functions, introduced by McLuskey (McLuskey, 1965). We performed an appropriate adaptation of this, in formal terms (Arigoni, 1980 and 1982), after attemps to utilize it in analyzing semantical information (Arigoni & Balboni, 1974), by Zadeh and in formalizing concept (Zadeh, 1976).

Recently said algorithm has been more efficiently formalized into the framework of Abstraction Theory: *conceptual abstraction*, a function $\mathcal{F}$ permitting to draw ID through ED (Arigoni and Maniezzo, 1992) [1].

By this paper, it is shown the equivalence between function $\mathcal{F}$ and a simpler one, we develop herein; this permits abstracting ID directly from whichever RD, allowing so a more "economical" implementation of the basic procedure.

2 Fundamental definitions

Let us have an infinite universe X^∞ of entities x_i, where the latter result by the attributes denoted by binary interpretation of the values an infinite number of variables X_h take on them. Hence, to identify a specific subject σ^k through the entities by which these can be instanced, we consdier the entites to which a set of variable X_h that, singly denote either the possession or not of one same attribute (*definiens variables*) through their binary interpretation, Finally, we limit the atten-

[1] The mentioned theory has been incepted by Plaisted (1981); successively, this was enriched by others (Hobbs, 1985; Tenemberg, 1987; Saitta et all., 1991; Giunchiglia and Walsh, 1992; and others). Abstraction techniques are frequently used in knowledge-based systems to overcome problems of computational intractability.

tion exclusively to the values of additional l variables X_h $(h)1, \ldots, l)$ that characterize the defined subject significantly (*characterizing variables*). Thus the entities relative to each subject σ^k form a universe of discourse X^0 having cardinality 2^l and whose entities are l-tuples $x_\alpha = (x_{1\alpha}, \ldots, x_{h\alpha}, \ldots, x_{l\alpha})$. In these, for every h $x_{h\alpha}$ is either "0" or "1". Thus, whichever element of the power set $\mathcal{P}(X^0)$ may constitute the subset of entities –category– that defines extensionally one concept.

With regard to concept ID, the description of this is accomplished by evidencing the atributes that are irrelevant in the distinct entities; irrelevance evidenced by indicating such attributes by an asterisk, rather than by "0" and "1". This is illustrated in the following example.

Example 2.1 *We assume that in a universe X^∞ of objects, the subject "car" by means of an adequate number of definiens variables is identified. Then , through characterizing variables, specifically three of these, we compose one X^0. This includes so entities relative to eight different types of cars. By assuming that the considered variables are X_1 (price); X_2 (elegance); and X_3 (sturdiness), the obtained universe is:*

$$
\begin{aligned}
X^0 = \{x_1 &= 000\,(\text{inexpensive, not elegant, weak}),\\
x_2 &= 001\,(\text{inexpensive, not elegant, sturdy}),\\
x_3 &= 010\,(\text{inexpensive, elegant, weak}),\\
x_4 &= 011\,(\text{inexpensive, elegant, sturdy}),\\
x_5 &= 100\,(\text{expensive, not elegant, weak}),\\
x_6 &= 101\,(\text{expensive, not elegant, sturdy}),\\
x_7 &= 110\,(\text{expensive, elegant, weak}),\\
x_8 &= 111\,(\text{expensive, elegant, sturdy}),\}
\end{aligned}
$$

On X^0 we define extensionally the concept "I: interesting car". This takes place by following a specific criterion. The obtained ED can also called concrete definition of I, because of the concreteness of its elements –entities. The so formed category is assumed as including the wole X^0, x_5 excluded. This means that, according to I, it is sufficient that a car is conform to one of the alternative conjunctions of necessary requirements, to be considered as interesting; thus the interesting *cars form the category:*

$$
\begin{aligned}
I^c = \{x_1 &= 000\,(\text{inexpensive, not elegant, weak}),\\
x_2 &= 001\,(\text{inexpensive, not elegant, sturdy}),\\
x_3 &= 010\,(\text{inexpensive, elegant, weak}),\\
x_4 &= 011\,(\text{inexpensive, elegant, sturdy}),\\
x_6 &= 101\,(\text{expensive, not elegant, sturdy}),\\
x_7 &= 110\,(\text{expensive, elegant, weak}),\\
x_8 &= 111\,(\text{expensive, elegant, sturdy}),\}
\end{aligned}
$$

By applying $\mathcal{F}$ on concept I, or better, on the category I^c that I itself underlies, we abstract the concept ID, $I^{\mathcal{A}}$, this, which can also called abstract definition, and because of the abstractness of its elements, is:

$$
\begin{aligned}
I^{\mathcal{A}} = \mathcal{F}(I^c) \;=\; &\{0 * *\,(\text{inexpensive}, *, *),\\
&* * 1\,(*, *, \text{sturdy}),\\
&*1 *\,(*, \text{elegant}, *),\}
\end{aligned}
$$

The latter means that in order a car is considered as interesting, it is sufficient the fulfillment by this, of one of the listed disjoint conjunctions forming $I^{\mathcal{A}}$, which is simpler than verifying the more numerous and redundant conjunctions appearing in $I^{\mathcal{C}}$.

It is remarked that because of the different relevance that the attributes can have in the distinct entities of $I^{\mathcal{C}}$, as it can be revealed from $I^{\mathcal{A}}$, $I^{\mathcal{C}}$ itself is fuzzy.

The concision of the form of $I^{\mathcal{A}}$ is well evident; however, the complexity of the procedure to achieve it may be high, specially if: (a) the number l of considered variables X_h is greater than the one considered in the example here reported. When it is so, in fact, due to the exponential dependence on l by the cardinality of X^0, this may reach high levels and likewise the extension of the concepts ED performed on it, may result cumbersome; (b) the definition of concepts into consideration is not its ED, as above it has been assumed the eventuality, but it is an RD. In this case, which is quite common, before applying $\mathcal{F}$, the *available RD has to be transformed backwar into the one ED, which makes the task even more laborious.*

To exemplify the hypothesized case, we report about one of the numerous forms of RD that may be the one available for the concept I in issue. This is:

$$
\begin{aligned}
RD(I) = I^{\mathcal{A}} \quad = \{ \quad & 00*\,(\text{inexpensive, not elegant}, *), \\
& **1\,(*, *, \text{sturdy}), \\
& 011\,(\text{inexpensive, elegant, sturdy}), \\
& 01*\,(\text{inexpensive, elegant}, *), \} \\
& 11*\,(\text{expensive, elegant}, *), \}
\end{aligned}
$$

It is remarked that concept RD is that which can be given by an inquired human agent. This in that, the latter currently is not capable of keeping rigorous account of possible irrelevancies on the information it gives.

The direct abstraction of $I^{\mathcal{A}}$ from $I^{\mathcal{R}}$ of our example does not present particular difficulty to be carried out intuitively; nevertheless this would be impracticable where X^0 was greater and the defined concept possibly more extended.

3 The conceptual universe

In the last section, by the symbol "$*$" for irrelevance, it has been introduced an attribute notation additional to those "0" and "1". Thus, also a new notation for the entities forming the possible concept definitions and for the relative universe is needed. By this all the abstract entities we already used for ID and RD in Example 2.1, are added to those of X^0 earlier considered. Said new entities form a set we indicate by X^*; then by the union of this to X^0 is finally obtained the set of all the entities, either abstract or concrete, by which to define concepts in all their possible forms: ED, ID and RD. These may so be formed with entities of $X = X^0 \cup X^*$, and each of these appears in the elements of the power set $\mathcal{P}(X)$.

From the previous section, it is clear that every concept $\mathcal{S}$ can be identified, at different levels of abstraction, by one same number of elements –subsets forming a category– $S^{\mathcal{C}}$, $S^1, \ldots, S^{\mathcal{A}}$ of definitions (semantically) equivalent one to any other, as it occurs for $I^{\mathcal{C}}, \ldots, I^{\mathcal{A}}$ of Example 2.1. The equivalence relation that links such possible forms of definition, indicated by "$<>$", introducess on $\mathcal{P}(X)$ a partition as it results from the quotient set $\mathcal{P}(X)/<>$: for every possible concept, one part of $\mathcal{P}(X)$ includes all the elements that define this; the other, those which do not do it. The set of elements of $\mathcal{P}(X)$ forming one same part of this, is exactly the formalization of what we call *concept*. Since $\mathcal{P}(X)$ comes down from the definiens variables relative to one specific subject σ^k, each definable concept will regard such subject. We define the set of such concepts as *k-conceptual universe* and indicate it by $\Xi^k = \{\mathcal{R}, \mathcal{S}, \mathcal{T}, \ldots\}$. It is noted that once the subject in issue is fixed, its index k can be dropped so to simplify the notations.

The elements –concepts– forming one same conceptual universe Ξ, can be partially ordered by one relation. This is indicated by "$\prec$" and depends on the meaning of the entities of X that are included in the categories regarding the different concepts of Ξ and that subsume one another: possibility by a concept of leading another into a more extended frame (Wos, 1984), and having a more extended ED (Arigoni and Rossi, 1994).

4 Used symbols and structure of the paper

In the coming sections the treatment of the subject proceeds in a strictly formal way; to possibly facilitate its fluency, in this brief section we outline the structure of the paper and precise the used symbols.

$X^\infty = \{x_i : i = 1, 2, \ldots\}$: infinite universe of discourse, where the x_i constitute the elements –entitites.

X_h: definiens and characterizing variables; these take values on the x_i. The former serve to individuate the x_i relative to one same subject σ^k, by taking on the entities of this invariably one same value; through the latter, that are a finite number l, the distinct x_i relative to one same subject are distinguished one from another by the different values the X_h themselves assume.

x_{hi}: attributes denoted by the values of the X_h, in the infinite entities x_i; thus, for every i, $x_i = (x_{1i}, x_{2i}, \ldots)$.

$x_{h\alpha}$: attributes denoted by characterizing variables ($h = 1, \ldots, l$).

$x_\alpha = (x_{1\alpha}, \ldots, x_{h\alpha}, \ldots, x_{l\alpha})$: finite entities.

$X^0 = \{x_\alpha : \alpha = 1, \ldots, n = 2^l : \forall h, x_{h\alpha} \in \{0, 1\}\}$.

$X^* = \{x_\alpha : \exists x_{h\alpha} = *\}$.

$X = X^0 \cup X^* = \{x_\alpha : \alpha = 1, \ldots, n = 3^l : x_{h\alpha} \in \{0, 1, *\}\}$.

R, S, ...: elements of the power set $\mathcal{P}(X)$, or else, categories that define concepts; in particular: $I^{\mathcal{C}}$ is the concrete definition and $I^{\mathcal{A}}$ is the abstract one.

$<>$: concrete equivalence relation among the elements of $\mathcal{P}(X)$.

$\Xi^k = \mathcal{P}(X)/ <>$: quotient-set each class of which contains the possible definitions of one same concept, i.e., set of all different possible concepts that can be defined on $\mathcal{P}(X)$, about one subject σ^k.

$\mathcal{R}, \mathcal{S}, \ldots$: concepts that can be defined on the subject σ^k.

By Section 1 it has been underlined that the aimed purpose of operating formally on concepts is specified by abstracting these. Sections 2 and 3 have puntualized the different concrete/abstract levels at which concepts can be defined and the universe these form. Particularly, we committed to an example to precise: (a) how a (finite) conceptual universe is derived from an infinite universe of discourse; (b) how to define a concept, that is, how to specifies all the elements to identify this, concretely, abstractly, or at intermediate levels, also eventually roughly.

The development of the paper goes on with developing the outlined basic elements. In Section 5 there are refined elements previously introduced. In Section 6 is considered an algorithm by which leading to the intensional, or abstract, definition of a given concept, on the basis of the extensional, or concrete, definition of the concept itself. Differently, Section 7 regards how to achieve the same concept abstract definition; herein this is accomplished in a more economic way, that is, by deriving such an abstraction from whichever of the various possible natural forms of concept definition: the ones rough. The same algorithm is deeply analyzed in Section 8. Finally, the procedure considered in Section 9 permits obtaining the abstract definition of the negation of a concept, directly from the same type of definition of such concept.

5 Additional definitions

In the present section we formalize further the previously introduced elements furthermore.

Definition 5.1 *Given two entities $x_\alpha, x_\beta \in X$, we say that x_α* equals *x_β iff $\forall h = 1, \ldots, l$ $x_{h\alpha} = x_{h\beta}$. We indicate this by $x_\alpha = x_\beta$ and, differently, $x_\alpha \neq x_\beta$.*

Definition 5.2 *Given two entities $x_\alpha, x_\beta \in X$, we say that x_α and x_β are* isosignificant *and indicate this by $x_\alpha \tilde{=} x_\beta$ iff $x_\alpha = x_\beta$, or $\exists! k : x_{k\alpha} \neq x_{k\beta}$ and $\forall h \neq k$ $x_{h\alpha} = x_{h\beta}$, given $x_{k\alpha}, x_{k\beta} \in \{0, 1\}$.*

Definition 5.3 *Given two entities $x_\alpha, x_\beta \in X$, we say that x_α and x_β are* strictly isosignificant *and indicate this by $x_\alpha \sim x_\beta$ iff $x_\alpha \tilde{=} x_\beta$ and $x_\alpha \neq x_\beta$.*

Example 5.1

0010 ∼ 0011; 1101 ∼ 1001; 1010 ∼ 1011; 1∗01 ∼ 1∗11;
∗001 ∼ ∗000; ∗11∗ ∼ ∗01∗; 10∗∗ ∼ 00∗∗.

Definition 5.4 *Given two entities x_α and x_β such that $\forall h$ $x_{h\alpha} = x_{h\beta}$ or $x_{h\beta} = *$, we say that x_β* generalizes *x_α, or conversely that x_α* concretizes *x_β. This relation is indicatd by $x_\alpha \geq x_\beta$; in addition, if $x_\alpha \geq x_\beta$ and $x_\alpha \neq x_\beta$ then we shall write $x_\alpha > x_\beta$.*

Example 5.2

0001>∗001; 1010>101∗; 1110>11∗0; 101∗>10∗∗;
∗110>∗11∗; 0101>01∗∗; 1∗00>∗∗0∗; ∗001>∗∗∗1.

Proposition 5.1 *$(X, \geq)$ is a partially oredered set; we denote its minimum element by $x_{\#}$ and the maximum by x_*. The former represent the null string (Arigoni and Rossi, 1994), the latter an l-tuple all elements of which are "*".*

The proof is very simple since is based merely on definition of $\geq$.

Definition 5.5 *Given an entity $x_\beta \in X$, we call* concretized *x_β the set:*

$$\{x_\beta\}^C = \{x_\alpha \in X^0 : x_\alpha \geq x_\beta\}.$$

Example 5.3

$$\begin{array}{ll} x_\beta = 001*, & \{x_\beta\}^C = \{0010, 0011\}; \\ x_\beta = 1*0*, & \{x_\beta\}^C = \{1000, 1001, 1100, 1101\}; \\ x_\beta = 1***, & \{x_\beta\}^C = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}; \end{array}$$

The sense of Definition 5.5 can be extended to subsets of entities as in the following

Definition 5.6 *We call* concretized *an element C of $\mathcal{P}(X)$ and indicate it by C^C, the set of the concretized entities x_β of C.*

Formally: $C^C = \bigcup_{x_\beta \in C} \{x_\beta\}^C$.

Definition 5.7 *$\sigma : \mathcal{P}(X) \to \mathcal{P}(X \times X)$ is a function defined as follows:*

$$\begin{array}{rcl} \sigma(S) & = & S^+ \cup S^-; \\ S^+ & = & \{(x_\alpha, x_\beta) \in S \times S : x_\alpha \sim x_\beta\}; \\ S^- & = & \{(x_\alpha, x_\alpha) \in S \times S : \nexists x_\beta \in S : x_\alpha \sim x_\beta\}; \end{array}$$

Definition 5.8 *The function $\oplus : \sigma(X) \to X$, defined as follows:*

$\oplus(x_\alpha, x_\beta = x_\delta$ *such that* $\forall h \neq k\; x_{h\delta} = x_{h\alpha} = x_{h\beta}$ *and* $x_{k\delta} = *$

$\oplus(x_\alpha, x_\alpha = x_\alpha$

is called conflation operator [2]

[2]The notion of conflation is due to Tenenberg (Tenenberg, 1987) and can be accomplished on indistinguishable entities (Hobbs, 1985). In this paper it corresponds formally to a particular type of synthesis performed in function of the meaning of such entities (Arigoni, 1980).

In the following we shall write $x_\alpha \oplus x_\beta$, instead of $\oplus(x_\alpha, x_\beta)$.
Conflation can be schematized as follows:

$$\begin{array}{lll} x_\alpha = (0,1,0,\ldots,1,\ldots,0) & & \\ & \text{conflation} & x_\delta = (0,1,0,\ldots,*,\ldots,0) \\ x_\beta = (0,1,0,\ldots,0,\ldots,0) & & \end{array}$$

Proposition 5.2 *For every $x_\alpha, x_\beta \in X$ such that $x_\alpha \stackrel{\sim}{=} x_\beta$*

1) $x_\alpha \oplus x_\beta = x_\beta \oplus x_\alpha$

2) $x_\alpha \geq x_\alpha \oplus x_\beta$ *and* $x_\beta \geq x_\alpha \oplus x_\beta$.

that is, the conflation operator is commutative and the result of every conflation generalizes both the conflated entities. The proof derives directly from definition.

It exists a particular relation linking any pair C, D of elements of $\mathcal{P}(X)$ which have the same concrete definition, as specified below.

Definition 5.9 *Let be given two subsets C and D such that $C^C = D^C$, this relation is defined as* concrete equivalence *and is indicated by $C <> D$.*

Definition 5.10 $\Xi = \mathcal{P}(X)/ <>$ *is the set of all concepts that can be defined on X about one subject identified by the definiens variables.*

In the following the elements of Ξ, i.e., each class that originates from the application of relation $<>$ on $\mathcal{P}(X)$, shall be indicated by $\mathcal{C}, \mathcal{D}, \mathcal{R}, \mathcal{S}, \ldots$. As indication of one particular representing we can put $\mathcal{C} = [C^C]$, $\mathcal{D} = [D^C]$, $\mathcal{R} = [R^C]$, and so on.

6 The abstraction algorithm

Herein we consider the procedure yielding the abstract –intensional– definition of a concept, once the one concrete –extensional– is given.

Definition 6.1 *The function $a : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as follows:*

$$a(S) = \{x_\delta : x_\delta = x_\alpha \oplus x_\beta \text{ and } (x_\alpha, x_\beta) \in \sigma(S)\}$$

is called partial abstraction function.

Note that $a(S) = \oplus(\sigma(S))$, i.e., the image of $\sigma(S)$ in the function $\oplus$.

Moreover, given the commuativity of $\oplus$ and the symmetry of $\sim$, only one of the two pairs (x_α, x_β), (x_β, x_α) belonging to $\sigma(S)$, can be considered.

To complete the definition we have to assume that $a(\emptyset) = \# = \{x_\#\}$, where $x_\#$ stands as the null string.

Definition 6.2 *Let $S^C \subseteq X^0$ be given, we denote*

$$\begin{aligned} a^0(S^C) &= S^C = S^0 \\ a^1(S^C) &= a(S^C) = S^1 \\ a^2(S^C) &= a(a(S^C)) = S^2 \\ &\vdots \\ a^j(S^C) &= S^j \end{aligned}$$

Definition 6.3 *Given an entity $x_\alpha \in X$, we denote the number of "$*$" in x_α by μ_α. Moreover we put $\mu_S = \max_{x_a \in S}\{\mu_\alpha\}$.*

Lemma 6.1 $\forall x_\alpha \in S^i\ \exists x_\gamma \in S^{i+1}$ *such that* $x_\alpha \geq x_\gamma$.

Proof. Let us assume that $x_\alpha \in S^i$, then:

1) if $\nexists x_{\beta \in S^i}$ such that $x_\alpha \sim x_\beta$ then $x_\alpha \in S^{i+1}$ and $x_\alpha \geq x_\alpha$

2) if $\exists x_{\beta \in S^i}$ such that $x_\alpha \sim x_\beta$ then $\exists x_\gamma \in S^{i+1}$ such that $x_\gamma = x_\alpha \oplus x_\beta$ i.e. $x_\alpha \geq x_\gamma$.

Corollary 6.1.1. *$\forall i \in N$ the following holds:*

$$\text{either} \quad \mu_{S^{i+1}} = \mu_{S^i} \quad \text{or} \quad \mu_{S^{i+1}} = \mu_{S^i} + 1$$

Proof. Let $x_\alpha \in S^i$ and $\mu_\alpha = \mu_{S^i}$, from Lemma 6.1 it follows that $\exists x_\gamma \in S^{i+1}$ such that $x_\alpha \geq x_\gamma$, therefore either $x_\alpha \in S^{i+1}$ or $x_\gamma \in S^{i+1}$ and $x_\alpha > x_\gamma$. In the first case it follows that $\mu_{S^i+1} = \mu_{S^i}$; in the second, $\mu_{S^i+1} = \mu_{S^i} + 1$.

Corollary 6.1.2. $\forall i \ \mu_{S^i} \leq i$.

Proof. This is given by induction on i. The thesis is obvious for $i = 0$, since $\mu_{S^0} = 0$ considering that $S^0 = S^C \subseteq X^0$. Let us suppose, as an inductive hypothesis, that $\mu_{S^{i-1}} \leq i - 1$; by Corollary 6.1.1, $\mu_{S^i} \leq \mu_{S^{i-1}} + 1$, then $\mu_{S^i} \leq \mu_{S^{i-1}} + 1 \leq (i-1) + 1 = i$.

Corollary 6.1.3. *If $i < j$ then $\mu_{S^i} \leq \mu_{S^j}$.*

Lemma 6.2 *If $x_\alpha, x_\beta \in S^i$ and $x_\alpha \sim x_\beta$ then $\mu_\alpha = \mu_\beta = i$.*

Proof. From the Corrollary 6.1.2 it follows that $\mu_\alpha = \mu_\beta \leq i$ ($\mu_\alpha = \mu_\beta$ is implicit if $x_\alpha \sim x_\beta$); so we have to prove the absurdity of $\mu_\alpha = \mu_\beta < i$.

Let us suppose that $\mu_\alpha = \mu_\beta = j < i$, then $x_\alpha, x_\beta \in S^j$, and since $x_\alpha \sim x_\beta$, $x_\alpha \notin S^{j+1}$ and $x_\beta \notin S^{j+1}$. It follows that such x_α and x_β do not belong to any one of the successive abstractions and this is absurd because of the hypothesis according to which $x_\alpha, x_\beta \in S^i$ and $i > j$.

Theorem 6.3. *Given $S^C \subseteq X^0$ let $a^j(S^c) = S^j$ and $\mu_{S^j} = m$, if $m < j$, then $\forall i \in N$ $a^{m+i}(S^c) = a^m(S^c) = a^j(S^c)$ holds.*

Proof. Let us suppose that $a^m(S^c)^+ \neq \emptyset$, then $\exists x_\alpha, x_\beta \in a^m(S^c)$ such that $x_\alpha \sim x_\beta$ and by Lemma 6.2, $\mu_\alpha = \mu_\beta = m$. It follows that $x_\gamma = x_\alpha \oplus x_\beta \in a^{m+1}(S^c)$ and $\mu_\gamma = m + 1$, therefore $\mu_{S^{m+1}} \geq (m+1)$.

By Corollary 6.1.2 $\mu_{S^{m+1}} \leq (m+1)$, so that $\mu_{S^{m+1}} = (m+1)$; but this is absurd in that from $m < j$ it follows that $(m+1) \leq j$ and due to Corollary 6.1.3 it must be $\mu_{S^{m+1}} \leq \mu_{S^i}$; definitively, it results $(m+1) \leq m$ which is absurd.

This shows that $a^m(S^c)^+ = \emptyset$, consequently

$$a^{m+1}(S^c) = a^m(S^c)$$

By induction it can be shown that $a^{m+i}(S^c) = a^m(S^c)$; in fact, assuming $a^{m+i-1}(S^c) = a^m(S^c)$:

$$a^{m+i}(S^c) = a(a^{m+i-1}(S^c)) = a(a^m(S^c)) = a^{m+1}(S^c) = a^m(S^c)$$

Moreover if the equality holds $\forall i \in N$, it holds for $i = j - m$, then

$$a^m(S^c) = a^{m+j-m}(S^c) = a^j(S^c).$$

We can now give the following

Definition 6.4 $\mathcal{A}(S) = a^m(S^i)$ *is called* global abstraction function. *Note that* $m(\leq l)$ *is the smaller index such that* $a^m(S^c) = a^{m+1}(S^c)$.

Theorem 6.3 assures the existence of one such index m; moreover $m \leq l$, since in each entity there are at the most l asterisks.

Definition 6.5 $\forall S \in \mathcal{P}(X)$, *if it exists* $i \leq l$ *such that* $S = S^i = a^i(S^c)$, *then* S *is the i-th* normal *definition of the concept, otherwise* S *is one possible rough definition of the concept.*

In particular $\mathcal{A}(S)$ *is indicated also by* $S^{\mathcal{A}}$.

7 The incremental algorithm

The procedure considered in this section still regards the achievement of the intensional definition of a concept; this, however, is derived from any given concept rough definition and does not require to go throughout the extensional definition of the concept itself.

Definition 7.1 $\forall x_\alpha, x_\beta \in X$

$$\delta_h(x_\alpha, x_\beta) = \begin{cases} 1, & \textit{if } (x_{h\alpha}, x_{h\beta}) = (0,1) \textit{ or } (1,0) \\ 0, & \textit{otherwise} \end{cases}$$

$\Delta(x_\alpha, x_\beta) = \sum_{h=1}^{l} \delta_h(x_\alpha, x_\beta)$ *is the distance between the two entities.*

Definition 7.2 *The function* $\sigma_\mu : \mathcal{P}(X) \to \mathcal{P}(X \times X)$ *is defined as follows:*

$$\sigma_\mu(S) = \{(x_\alpha, x_\beta) \in S \times S : \Delta(x_\alpha, x_\beta) = 1\}$$

Example 7.1 *Given* $S = \{x_1 = 000{*}{*}{*}, x_2 = 101{*}01, x_3 = 100{*}00, x_4 = 000000, x_5 = {*}{*}0100\}$

Δ	x_1	x_2	x_3	x_4	x_5
x_1	*0*	*2*	*1*	*0*	*0*
x_2	*2*	*0*	*2*	*3*	*2*
x_3	*1*	*2*	*0*	*1*	*0*
x_4	*0*	*3*	*1*	*0*	*1*
x_5	*0*	*2*	*0*	*1*	*0*

Since $\Delta((000{*}{*}{*}, 100{*}00) = 1$, $\Delta(10{*}11, 1101{*}) = 1$, $\Delta(11000, {*}1{*}10) = 1$, $\sigma_\mu(S) = \{(x_1, x_3), (x_3, x_4), (x_4, x_5)\}$.

Definition 7.3 *The function* $+ : \sigma_\mu(X) \to X$ *is defined as follows:*

$$+(x_\alpha, x_\beta) = x_\gamma$$

such that:

$$x_{k\gamma} = * \text{ and } \forall h \neq k \; x_{k\gamma} = \begin{cases} x_{h\alpha}, & \text{if } x_{h\alpha} \neq * \\ x_{h\beta}, & \text{otherwise} \end{cases}$$

in the hypothesis that $\delta_k(x_\alpha, x_\beta) = 1$ *and* $\forall h \neq k \;\; \delta_h(x_\alpha, x_\beta) = 0$.

As an example let us consider:

$$\begin{array}{lcl} x_\alpha = 000{*}{*}{*} & & \\ & \backslash & \\ & & +(x_\alpha, x_\beta) = {*}0000{*} \\ & / & \\ x_\beta = 10000{*} & & \end{array}$$

Definition 7.4 *For every subset* S *of* X*, the set defined as follows*

$$r(S) = S \backslash \{x_\alpha \in S : \exists x_\beta \in S \text{ and } x_\alpha > x_\beta\}$$

is called reduction *of* S*. Note that* $r(\emptyset) = \emptyset$.

Lemma 7.1 *If* $\forall x_\alpha \in S\ \exists x_\beta \in S \cup T$ *such that* $x_\alpha > x_\beta$*,t hen* $\forall x_\alpha \in S$ $\exists x_\epsilon \in T \backslash S$ *such that* $x_\alpha > x_\epsilon$.

Proof. Let us suppose that $\exists x_\epsilon \in T \backslash S\ x_\alpha > x_\epsilon$, then, from the put hypothesis, $\forall x_\alpha \in S\ \exists x_\beta \in S \backslash T$ such that $x_\alpha > x_\beta$, and this is absurd; in fact, given $x_\alpha, x_\beta \in S$ and $x_\alpha > x_\beta$, it must exists an $x_\delta \in S$ such that $x_\alpha > x_\beta > x_\delta$ and so on, so that $x_* \in S$; since an element strictly more general than x does not exist, we obtain the absurd.

Theorem 7.2. *Given the elements* S, T *of* $\mathcal{P}(X)$*; if* $\forall x_\alpha \in S\ \exists x_\beta \in S \cup T$ *such that* $x_\alpha > x_\beta$ *then* $r(S \cup T) = r(T)$.

Proof. $r(S \cup T) \subseteq r(T)$:
if $x_\delta \in r(S \cup T)$ then $x_\delta \in S \cup T$ and $\not\exists x_\beta \in S \cup T$ such that $x_\delta > x_\beta$; it follows that $x_\delta \in T$ and, in addition, $\not\exists x_\beta \in T$ such that $x_\delta > x_\beta$, so that $x_\delta \in r(T)$.
$r(T) \subseteq r(S \cup T)$:
if $x_\delta \in r(T)$ then $x_\delta \in T$ and $\not\exists x_\beta \in T$ such that $x_\delta > x_\beta$; we have to show that $\not\exists x_\beta \in S$ such that $x_\delta > x_\beta$; this follos from lemma 7.1, which states that, in the theorem hypotesis, $\forall x_\alpha \in S\ \exists x_\epsilon \in T \backslash S$ $x_\alpha > x_\epsilon$, i.e. $\exists x_\epsilon \in T$ such that $x_\delta > x_\epsilon$, but this is absurd since $x_\delta \in r(T)$.

Corollary 7.2.1. $\forall S, T \in \mathcal{P}(X)$, $r(r(S) \cup T) = r(S \cup T)$.

Proof. Since $r(S) = S \backslash \{x_\alpha \in S : \exists x_\beta \in S$ and $x_\alpha > x_\beta\}$, it follows that:

$$S = r(S) \cup \{x_\alpha \in S : \exists x_\beta \in S \text{ and } x_\alpha > x_\beta\};$$

therefore $r(S \cup T) = r(r(S) \cup \{x_\alpha \in S : \exists x_\beta \in S$ and $x_\alpha > x_\beta\} \cup T)$.

Since $\forall x_\alpha \in \{x_\alpha \in S : \exists x_\beta \in S$ and $x_\alpha > x_\beta\}$ it exists $x_\beta \in S$ such that $x_\alpha > x_\beta$, then $\forall x_\alpha \in \{x_\alpha \in S : exists x_\beta \in S$ and $x_\alpha > x_\beta\}$ $exists x_\beta \in S \cup T$ such that $x_\alpha > x_\beta$.

Since $S \cup T = r(S) \cup T \cup \{x_\alpha \in S : \exists x_\beta \in S$ and $x_\alpha > x_\beta\})$, by Theorem 7.2:

$$r(S \cup T) = r(r(S) \cup T) \cup \{x_\alpha \in S : \exists x_\beta \in S \text{ and } x_\alpha > x_\beta\}) = r(r(s) \cup T).$$

Definition 7.5 *The function* $m : \mathcal{P}(x) \rightarrow \mathcal{P}(X)$ *defined as follows*

$$m(S) = r(S \cup S\Delta_1)$$

where

$$S\Delta_1 = \{x_\alpha + x_\beta : x_\alpha, x_\beta \in S \textit{ and } \Delta(x_\alpha, x_\beta) = 1\} = +(\sigma_\mu(S))$$

is called partial $\mathcal{M}$-abstraction function*.*

To complete the definition we put $m(\emptyset) = \# = \{x_\#\}$*.*

Example 7.2

$$\begin{aligned}
S &= \{1110**, 1111*0, *11011, *11010\} \\
\sigma_\mu(S) &= \{(1110**, 1111*0), (1111*0, *11010), (*11011, *11010)\} \\
S\Delta_1 &= \{111**0, 111*10, *1101*,\} \\
S \cup S\Delta_1 &= \{(1110**, 1111*0, *11011, *11010, 111**0, 111*10, \\
&\quad *1101*,\} \\
r(S \cup S\Delta_1) &= \{(1110**, 111**0, *1101*\} = m(S)
\end{aligned}$$

Theorem 7.3. $m(S) = S$ *iff* $\exists i > 1$ $m^i(S) = S$*.*

Proof.

$\Rightarrow$ If $m(S) = S$ then $m^2(S) = m(m(S)) = m(S) = S$ and by induction on i, we have $\forall i \in N$ $m^i(S) = S$.

$\Leftarrow$ $m(S) = r(S \cup S\Delta_1)$

$m^2(S) = r(m(S) \cup m(S)\Delta_1) = r(r(S \cup S\Delta_1) \cup m(S)\Delta_1) = r(S \cup S\Delta_1 \cup m(S)\Delta_1)$ in the same way (by the use of Corollary 7.2.1) we get

$$\begin{aligned}
m^i(S) &= r(S \cup S\Delta_1 \cup m(S)\Delta_1 \cup m^2(S)\Delta_1 \cup \ldots \cup m^{i-1}(S)\Delta_1) \\
m^{i+1}(S) &= r(S \cup S\Delta_1 \cup m(S)\Delta_1 \cup m^2(S)\Delta_1 \cup \ldots \cup m^{i-1}(S)\Delta_1 \\
&\cup \quad m^i(S)\Delta_1)
\end{aligned}$$

In the hypothesis that $\exists i$: $m^i(S) = S$ it follows that $m^i(S)\Delta_1 = S\Delta_1$ (Definition 7.5); therefore the last term in $m^{i+1}(S)$ equals the second and it can be canceled. It follows that

$$S = m^i(S) = m^{i+1}(S)$$

and then

$$S = m^{i+1}(S) = m(m^i(S)) = m(S).$$

Theorem 7.4. *(convergence)* $\forall S \in \mathcal{P}(X)\ \exists h \in N$ *such that* $\forall i > 0$ $m^{h+i}(S) = m^h(S)$.

Proof. Since $\forall i$, $m^i(S)$ is an element of $\mathcal{P}(X)$, which is finite, there must exist two indexes h, k such that

$$m^h(S) = m^k(S) \text{ with for example } 0 < h < k$$

Let $r = k - h$, then $k = h + r$ and

$$m^r(m^h(S)) = m^k(S) = m^h(S)$$

We are now in the hypothesis of Theorem 7.3: it suffices to put $m^h(S) = T$ and to observe that T is reduced, being this an $\mathcal{M}$-abstraction.

It follows that $m(T) = T$, i.e., $m^{h+1}(S) = m^h(S)$, and the theorem is proved for $i = 1$.

By supposing that

$$m^{h+(i-1)}(S) = m^h(S)$$

then:

$$m^{h+i}(S) = m(m^{h+(i-1)}(S)) = m(m^h(S)) = m^{h+1}(S) = m^h(S)$$

The theorem is so proved by induction.

Definition 7.6 *The set* $\mathcal{M}(S) = m^h(r(S))$, *where* h *is the smaller index such that* $m^h(r(S)) = m^{h+1}(r(S))$, *is defined as* global $\mathcal{M}$-abstraction *of* S.

We put $\mathcal{M}(\emptyset) = \#$.

The existence of the above index is assured by the Theorem 7.4.

Definition 7.7 $\rho(X) = \{S \in \mathcal{P}(X) : S = r(S)\}$ *is the set of all reduced sets.*

Definition 7.8 $\forall S, T \in \rho(X)$ *we define the following relation:*

$$S >> T \Leftrightarrow \forall x_\alpha \in S\ \exists x_\beta \in T \text{ such that } x_\alpha \geq x_\beta.$$

Proposition 7.5. *The introduced relation is a partial ordering on $\rho(X)$.*

Proof. (reflexivity) $S >> S\ \forall S \in \rho(X)$: obvious in that $x_\alpha \geq x_\alpha$ $\forall x_\alpha \in S$.
(antisymmetry) Let suppose that $\forall x_\alpha \in S\ \exists x_\beta \in T$ such that $x_\alpha \geq x_\beta$ and $\forall x_\gamma \in T\ \exists x_\delta \in S$ such that $x_\gamma \geq x_\delta$. If there exist $x_\alpha \in S$ and $x_\beta \in T$ such that $x_\alpha \in x_\beta$, then $\exists x_\delta \in S$ such that $x_\beta \geq x_\delta$ and, by the transitivity of $\geq$, $x_\alpha > x_\delta$ with $x_\alpha, x_\beta \in S$; which is absurd in that $S = r(S)$.

It follows that $\forall x_\alpha \in S\ \exists x_\beta \in T$ such that $x_\alpha \geq x_\beta$ but not $x_\alpha > x_\beta$; therefore it must be $x_\alpha = x_\beta$.

This solve that $S \subseteq T$, and in the same way it can be shown that $T \subseteq S$; in conclusion, $S = T$.
(transitivity) This follows directly from the transitivity of $\geq$.

Proposition 7.6. $r(S^{\mathcal{A}}) = S^{\mathcal{A}}$ *and* $r(S)^C = S^C$.

The proof immediately follows by definitions.

Proposition 7.7. *Given two entities x_α and x_β:*

$$x_\alpha \geq x_\beta \text{ iff } \{x_\alpha\}^C \subseteq \{x_\beta\}^C.$$

Proof. The assertion is demostrated in two parts:

i) As $\{x_\alpha\}^C = \{x_\delta \in X^0 : x_\delta \geq x_\alpha\}$ and $\{x_\beta\}^C = \{x_\delta \in X^0 : x_\delta \geq x_\beta\}$, due to the transitivity of $\geq$ it follows that $\forall x_\delta : x_\delta \geq x_\alpha$, if $x_\alpha \geq x_\beta$ then $x_\delta \geq x_\beta$, i.e.: for all $x_\delta \in \{x_\alpha\}^C$, $x_\delta \in \{x_\beta\}^C$ or else $\{x_\alpha\}^C \subseteq \{x_\beta\}^C$.

ii) In the hypothesis that $\{x_\alpha\}^C \subseteq \{x_\beta\}^C$ it follows that $\forall x_\delta \in \{x_\alpha\}^C$

(a) $x_\delta \geq x_\alpha$, i.e., for all h, either $x_{h\delta} = x_{h\alpha}$ or $x_{h\alpha} = *$

and

(b) $x_\delta \geq x_\beta$, i.e., for all h, either $x_{h\delta} = x_{h\beta}$ or $x_{h\beta} = *$.

Relatively to attribute $x_{h\alpha}$ and to its homologous $x_{h\beta}$ the four following cases are possible:

1) $x_{h\alpha} \in \{0,1\}$ and $x_{h\beta} \in \{0,1\}$;

2) $x_{h\alpha} \in \{0,1\}$ and $x_{h\beta} = *$;

3) $x_{h\alpha} = *$ and $x_{h\beta} = *$;

4) $x_{h\alpha} = *$ and $x_{h\beta} \in \{0,1\}$.

Since for every h and every $x_\delta \in \{x_\alpha\}^C \subseteq \{x_\beta\}^C x_{h\delta} \in \{0,1\}$, incase 1), from a) and b) it follows that $x_{h\delta} = x_{h\alpha} = x_{h\beta}$. The fourth case is absurd, in fact, in the hypothesis that $x_{h\beta} = 1(0)$, no conflation is feasible on $\{x_\beta\}^C$ with respect to the h-variable. Thus, being $\{x_\alpha\}^C$ a subset of $\{x_\beta\}^C$, conflations are impossible also on $\{x_\alpha\}^C$. It follows that $x_{h\alpha} = 1(0)$.

Cases 1) and 3) tell us that $x_{h\alpha} = x_{h\beta}$ whereas in case 2) it results $x_{h\beta} = *$. In conclusion, for all h, either $x_{h\alpha} = x_{h\beta}$ or $x_{h\beta} = *$ thatis $x_\alpha \geq x_\beta$.

Corollary 7.7.1. *If for all $x_\delta \in \{x_\alpha\}^C x_\delta \geq x_\beta$ then $x_\alpha \geq x_\beta$.*

Proof. If $x_\delta \in \{x_\alpha\}^C$ and $x_\delta \geq x_\beta$ then $x_\delta \in \{x_\beta\}^C$ thus $\{x_\alpha\}^C \subseteq \{x_\beta\}^C$ and $x_\alpha \geq x_\beta$.

Theorem 7.8. *$S^C \subseteq T^C \Leftrightarrow S^{\mathcal{A}} >> T^{\mathcal{A}}$.*

Proof. To prove the theorem, we show that:

i) if $\forall x_\alpha \in S^{\mathcal{A}} \exists x_\beta \in T^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$ then $S^C \subseteq T^C$;

ii) if $S^C \subseteq T^C$ then $\forall x_\alpha \in S^{\mathcal{A}} \exists x_\beta \in T^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$.

i) From Lemma 7.7 and for the hypothesis put in the theorem it follows that for every $x_\alpha \in S^{\mathcal{A}}$ it exists at least one $x_\beta \in T^{\mathcal{A}}$ such that $\{x_\alpha\}^C \subseteq \{x_\beta\}^C$, so that $\{x_\alpha\}^C \subseteq \{x_\beta\}^C \subseteq T^C$. Therefore $S^C = \cup_\alpha \{x_\alpha\}^C$ results by elements all belonging to T^C, and then $S^C \subseteq T^C$.

ii) Let $x_\alpha \in S^{\mathcal{A}}$ be given.

By Definition 5.6 $\{x_\alpha\}^C \subseteq S^C$ and from the put hypothesis ($S^C \subseteq T^C$) we have $\{x_\alpha\}^C \subseteq T^C$. By considering the abstractionprocess on

T^C, since every conflation feasible on $\{x_\alpha\}^C$ remain still feasible on T^C, it exists an entity x_β in $T^{\mathcal{A}}$ which generalizes at least each entity of $\{x_\alpha\}^C$. Thus, by Corollary 7.7.1 it exists $x_\beta \in T^{\mathcal{A}}$ that generalizes $x_\alpha \in S^{\mathcal{A}}$.

Corollary 7.8.1. *If $S^{\mathcal{A}} \subseteq T^{\mathcal{A}}$, then $S^C \subseteq T^C$.*

Proof. Since for every x_α of X, $x_\alpha \geq x_\alpha$, if $S^{\mathcal{A}} \subseteq T^{\mathcal{A}}$ then $\forall x_\alpha \in S^{\mathcal{A}}$, $\exists x_\beta \in T^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$ (such an x_β equals x_α); then from the theorem we have $S^C \subseteq T^C$.

Theorem 7.9. $\mathcal{A}(\{x_\alpha\}^{\mathcal{C}}) = \{x_\alpha\}$.

The proof is given by induction on μ_α and considering Theorem 7.8.

Theorem 7.10. *$\forall T \in [S^C]$, i.e. for all T such that $T^C = S^C$, if $T = r(T)$, then $T >> S^{\mathcal{A}}$.*

Proof. $\forall x_\alpha \in T$ we have that $\{x_\alpha\}^C \subseteq T^C = S^C$ and by Theorem 7.8 $\forall x_\alpha \in \mathcal{A}(\{x_\alpha\}^C)$ $\exists x_\beta \in T^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$; by Theorem 7.9 $\mathcal{A}(\{x_\alpha\}^C) = \{x_\alpha\}$, therefore $\forall x_\alpha \in T$ $\exists x_\beta \in T^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$. Since $T^C = S^C$, $\mathcal{A}(T^C) = \mathcal{A}(S^C)$, that is $S^{\mathcal{A}} = T^{\mathcal{A}}$.

The theorem characterize $S^{\mathcal{A}}$ with respect to the others elements belonging to the same equivalence class: it results the minimum in the relation $>>$.

Note that if T is not reduced, we still can state that $\forall x_\alpha \in T$ $\exists x_\beta \in S^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$, since the theorem is true for the reduction of T.

Theorem 7.11. *If $\exists x_\beta \in X \backslash S^{\mathcal{A}}$ and $\exists x_\alpha \in S^{\mathcal{A}}$ such that $x_\alpha > x_\beta$, then $\exists x_\delta \in \{x_\beta\}^C$ such that $x_\delta \notin S^C$.*

Proof. Let us suppose that $\forall x_\delta \in \{x_\beta\}^C$, $x_\delta \in S^C$ then $\{x_\beta\}^C \subseteq S^C$ and by Theorems 7.8, and 7.9 $\exists x_\gamma \in S^{\mathcal{A}}$ such that $x_\beta \geq x_\gamma$; then $x_\alpha > x_\beta \geq x_\gamma$, i.e. $x_\alpha > x_\gamma$, with $x_\alpha, x_\gamma \in S^{\mathcal{A}}$, which is absurd because of $S^{\mathcal{A}} = r(S^{\mathcal{A}})$ (Prop. 7.6).

Lemma 7.12. *$\forall x_\beta \in S$ $\exists x_\gamma \in m(S)$ such that $x_\beta \geq x_\gamma$.*

Proof. By considering that $m(S) = r(S \cup S\Delta_1)$, two cases are possible:

1) if $x_\beta \in m(S)$ then $x_\beta \geq x_\beta$.

2) if $x_\beta \in S \backslash m(S)$, this means that x_β was deleted, which is possible only when $\exists x_\gamma \in m(S)$ such that $x_\beta > x_\gamma$.

Corollary 7.12.1. $\forall x_\beta \in S \ \exists x_\gamma \in \mathcal{M}(S)$ *such that* $x_\beta \geq x_\gamma$.

Corollary 7.12.2. *If* $x_\alpha \in S^C$ *then* $\exists x_\gamma \in \mathcal{M}(S)$ *such that* $x_\alpha \geq x_\gamma$.

Proof. If $x_\alpha \in S^C$ then $\exists x_\beta \in S$ such that $x_\alpha \geq x_\beta$ and by Lemma 7.12 $\exists x_\gamma \in \mathcal{M}(S)$ such that $x_\beta \geq x_\gamma$, thus $x_\alpha \geq x_\gamma$.

Lemma 7.13. *If* $x_\alpha, x_\beta \in X$ *are such that* $\Delta(x_\alpha, x_\beta) = 1$, *then*

$$\{x_\alpha + x_\beta)^C \subseteq \{x_\alpha\}^C \cup \{x_\beta\}^C$$

Proof. Since $\Delta(x_\alpha, x_\beta) = 1$ we can suppose that $\exists! k$ such that $x_{k\alpha} = 0$ and $x_{k\beta} = 1$. Let us consider the attributes of $x_\delta = x_\alpha + x_\beta$ as it follows from the definition of $+$:

1) $x_{h\delta} = 0$ iff $x_{h\alpha} = 0$ and $x_{h\beta} = *$ or $x_{h\alpha} = x_{h\beta} = 0$ or $x_{h\beta} = 0$ and $x_{h\alpha} = *$;

2) $x_{h\delta} = 1$ iff $x_{h\beta} = 1$ and $x_{h\alpha} = *$ or $x_{h\alpha} = x_{h\beta} = 1$ or $x_{h\alpha} = 1$ and $x_{h\beta} = *$;

3) $x_{h\delta} = *$ iff $x_{h\alpha} = x_{h\beta} = *$ or $h = k$.

Let $x_\epsilon \in \{x_\delta\}^C$; by noting that $\forall h \ x_{h\epsilon} \in \{0, 1\}$ and $\forall h$ such that $x_{h\delta} \in \{0, 1\}$ we get $x_{h\epsilon} = x_{h\delta}$, so that we can show:

a) if $x_{k\epsilon} = 0$ then $x_\epsilon \geq x_\alpha$ i.e. $x_\epsilon \in \{x_\alpha\}^C$;

b) if $x_{k\epsilon} = 1$ then $x_\epsilon \geq x_\beta$ i.e. $x_\epsilon \in \{x_\beta\}^C$;

In the first case $x_{k\epsilon} = x_{k\alpha} = 0$ and for all $h \neq k$, if $x_{h\epsilon} = x_{h\delta} = 0$ then by 1) $x_{h\alpha} = 0$ or $x_{h\alpha} = *$; if $x_{h\epsilon} = x_{h\delta} = 1$ then by 2) $x_{h\alpha} = 1$ or $x_{h\alpha} = *$. In conclusion, $x_{h\epsilon} = x_{h\alpha}$ or $x_{h\alpha} = *$ i.e. $x_\epsilon \geq x_\alpha$.

In the second case $x_{k\epsilon} = x_{k\beta} = 1$ and for all $h \neq k$, if $x_{h\epsilon} = 0$ then by 1) $x_{h\beta} = 0$ or $x_{h\beta} = *$; if $x_{h\epsilon} = x_{h\delta} = 1$ then by 2) $x_{h\beta} = 1$ or $x_{h\beta} = *$. In conclusion, $x_{h\epsilon} = x_{h\beta}$ or $x_{h\beta} = *$ i.e. $x_\epsilon \geq x_\beta$.

Lemma 7.14. $\forall S \in \mathcal{P}(X)\ (\mathcal{M}(S))^C = S^C$.

Proof. We can show that $\forall S (m(S))^C = S^C$ holds; from $m(S) = r(S \cup S\Delta_1)$, it follows that $(m(S))^C = (S \cup S\Delta_1)^C$, since $\forall S$, $r(S)^C = S^C$ (Proposition 7.6).

Thus:

$$(S \cup S\Delta_1)^C = S^C \cup S\Delta_{1^C} = \cup_{x_\alpha \in S}\{x_\alpha\}^C \cup \cup_{x_\gamma \in S\Delta_1}\{x_\gamma\}^C = S^C$$

since $\forall x_\gamma \in S\Delta_1\ \exists x_\alpha, x_\beta \in S$ such that $x_\alpha + x_\beta = x_\gamma$ and by Lemma 7.13 $\{x_\gamma\}^C \subseteq \{x_\alpha\}^C \cup \{x_\beta\}^C$.

Since $\mathcal{M}(S) = m^h(S)$, it follows that $(\mathcal{M}(S))^C = S^C$.

Note that S is considered reduced since $(r(S))^C = S^C$.

From Theorem 7.14 it follows that $\mathcal{M}(S) \in [S^C]$, thus Theorem 7.10 states that:

$$\mathcal{M}(S) >> S^{\mathcal{A}}$$

that is: $\forall x_\alpha \in \mathcal{M}(S)\ \exists x_\beta \in S^{\mathcal{A}}$ such that $x_\alpha \geq x_\beta$.

8 $\mathcal{M}(S) - \mathcal{A}(S)$ equivalence

We start this section by a characterization of $\mathcal{M}(S)$:

$\forall x_\alpha, x_\beta \in \mathcal{M}(S)$ the following three cases are possible:

1) $\Delta(x_\alpha, x_\beta) = 0$;

2) $\Delta(x_\alpha, x_\beta) = 1$ and $\exists x_\gamma \in \mathcal{M}(S)$ such that $x_\alpha + x_\beta \geq x_\gamma$;

3) $\Delta(x_\alpha, x_\beta) > 1$.

This follows directly from the Definition 7.6 of $\mathcal{M}(S)$, by considering that in all others situations we can not have $m^h(S) = m^{h+1}(S)$.

Theorem 8.1 *Let $x_\beta \in X$. If $\{x_\beta\}^C \subseteq S^C$ then $\exists x_\gamma \in \mathcal{M}(S)$ such that $x_\beta \geq x_\gamma$.*

Proof. By induction on μ_β, i.e. on the number of "*" in x_β.

If $\mu_\beta = 1$ then $\{x_\beta\}^C$ contains two elements x_{β_1} and x_{β_2} such that $x_{\beta_1} \oplus x_{\beta_2} = x_\beta$. Since $\{x_\beta\}^C \subseteq S * C$ and for every element of S^C one exists in S more general, we have three possible cases:

1) $x_\beta \in S$;

2) $x_{\beta_1}, x_{\beta_2} \in S$;

3) $\exists x_{\beta_3}, x_{\beta_4} \in S$ such that $x_{\beta_1} \geq x_{\beta_3}$ and $x_{\beta_2} \geq x_{\beta_4}$.

In the first case the theorem follows directly from Lemma 7.12. In the second, either $x_\beta = x_{\beta_1} \oplus x_{\beta_2} \in m(S)$ or in $m(S)$ it exists an element which is more general than x_β. Again by Lemma 7.12 the theorem is proved.

In the third case:

let $x_{1\beta_1} = 0$, $x_{1\beta_2} = 1$, $x_{1\beta} = *$ and $\forall h = 2, 3, \dots, l$ $x_{h\beta_1} = x_{h\beta_2} = x_{h\beta}$.

Since $x_{\beta_1} \geq x_{\beta_3}$ and $x_{\beta_2} \geq x_{\beta_4}$ it follows that $\forall h = 2, 3, \dots, l$

$$x_{h\beta_3} = x_{h\beta_1} \text{ or } x_{h\beta_3} = *$$

and

$$x_{h\beta_4} = x_{h\beta_2} \text{ or } x_{h\beta_4} = *$$

If either $x_{1\beta_3} = *$ or $x_{1\beta_4} = *$, then:

$x_\beta \geq x_{\beta_3}$ or $x_\beta \geq x_{\beta_4}$; thus from Lemma 7.12 the assertion is proved.

If $x_{1\beta_3} \neq *$ and $x_{1\beta_4} \neq *$ then $x_{1\beta_3} = x_{1\beta_1} = 0$ and $x_{1\beta_4} = x_{1\beta_2} = 1$, and since $\forall h = 2, 3, \dots, l$ $x_{h\beta_3} = x_{h\beta}$ or $x_{h\beta_3} = *$ and $x_{h\beta_4} = x_{h\beta}$ or $x_{h\beta_4} = *$ it follows that $\Delta(x_{\beta_3}, x_{\beta_4}) = 1$ and $x_\beta \geq x_{\beta_3} + x_{\beta_4}$.

Thus $x_{\beta_3} + x_{\beta_4} \in m(S)$ or in $m(S)$ it exists an element more general than $x_\beta - 3 + x_{\beta_4}$. Since $x_{\beta_3} + x_{\beta_4}$ is more general than x_β, from Lemma 7.12 it follows that the theorem is proved for $\mu_\beta = 1$.

Let us suppose, now, the theorem true for $\mu_\beta = n - 1$.

Let x_{α_1} and $x_{\alpha 2}$ two elements of X such that $x_{\alpha 1} \oplus x_{\alpha 2} = x_\beta$. It directly follows that $\mu_{\alpha_1} = \mu_{\alpha_2} = n - 1$ and, for the inductive hypothesis, there exist x_{γ_1} and $x_{\gamma_2} \in \mathcal{M}(S)$ such that $x_{\alpha 1} \geq x_{\gamma_1}$ and $x_{\alpha 2} \geq x_{\gamma_2}$.

Let $x_{1\alpha_1} = 0$, $x_{1\alpha_2} = 1$, $x_{1\beta} = *$ and $\forall h = 2, 3, \dots, l$, $x_{h\alpha_1} = x_{h\alpha_2} = x_{h\beta}$.

Since $x_{\alpha_1} \geq x_{\gamma_1}$ and $x_{\alpha 2} \geq x_{\gamma_2}$ it follows that $\forall h = 2, 3, \dots, l$

$$x_{h\gamma_1} = x_{h\alpha_1} \text{ or } x_{h\gamma_1} = *$$

and

$$x_{h\gamma_2} = x_{h\alpha_2} \text{ or } x_{h\gamma_2} = *$$

If $x_{1\gamma_1} = *$ or $x_{1\gamma_2} = *$, then from the above we get either $x_\beta \geq x_{\gamma_1}$ or $x_\beta \geq x_{\gamma_2}$ and the theorem is proved.

If $x_{1\gamma_1} \neq *$ and $x_{1\gamma_2} \neq *$ then $x_{1\gamma_1} = x_{1\alpha_1} = 0$ and $x_{1\gamma_2} = x_{1\alpha_2} = 1$, and since $\forall h = 2, 3, \ldots, l$ $x_{h\gamma_1} = x_{h\beta}$ or $x_{h\gamma_1} = *$ and $x_{h\gamma_2} = x_{h\beta}$ or $x_{h\gamma_2} = *$ we have $\Delta(x_{\gamma_1}, x_{\gamma_2}) = 1$. Thus due to the properties of $\mathcal{M}(S)$, it follows that $\exists x_\gamma \in \mathcal{M}(S)$ such that $(x_{\gamma_1} + x_{\gamma_2}) \geq x_\gamma$; since $x_{1\gamma} = *$ and $\forall h = 2, 3, \ldots, l$ $x_{h\gamma} = x_{h\beta}$ or $x_{h\gamma} = *$, it follows that $x_\beta \geq x_\gamma$, i.e. the theorem is proved.

Due to this important property of $\mathcal{M}$-abstraction we are now ready to show the main result:

Theorem 8.2 *$\forall S \in \mathcal{P}(X)$ we have $\mathcal{M}(S) = \mathcal{A}(S)$.*

Proof. Theorem 7.14 states that $\mathcal{M}(S)^C = S^C$, that is, by Theorem 7.10:

$$\mathcal{M}(S) >> \mathcal{A}(S)$$

We show that $\mathcal{A}(S) >> \mathcal{M}(S)$, so that $\mathcal{M}(S) = \mathcal{A}(S)$ follows due to the antisymmetry of $>>$.

Let $x_\alpha \in \mathcal{A}(S)$ then $\{x_\alpha\}^C \subseteq S^C$ and, by Theorem 8.1, $\exists x_\beta \in \mathcal{M}(S)$ such that $x_\alpha \geq x_\beta$. It follows that $\forall x_\alpha \in \mathcal{A}(S)$, $\exists x_\beta \in \mathcal{M}(S)$ such that $x_\alpha \geq x_\beta$, i.e.:

$$\mathcal{A}(S) >> \mathcal{M}(S).$$

Corollary 8.1 *If $S^C = T^C$, then $\mathcal{M}(S) = \mathcal{M}(T)$.*

Proof. If $S^C = T^C$, then $\mathcal{A}(S) = \mathcal{A}(T)$, so that $\mathcal{M}(S) = \mathcal{M}(T)$.

Corollary 8.2 *$\mathcal{M}(S) = \mathcal{M}(S^C)$.*

The proof follows directly from the above corollary, by considering that $(S^C)^C = S^C$.

Corollary 8.3 *If $S = S_1 \cup S_2$, then $\mathcal{M}(S) = \mathcal{M}(\mathcal{M}(S_1) \cup S_2)$.*

Proof. $S^C = (S_1 \cup S_2)^C = S_1^C \cup S_2^C$ and $\mathcal{M}(S_1)^C = S_1^C$; by Corollary 8.2.2 we get: $\mathcal{M}(S) = \mathcal{M}(S^C) = \mathcal{M}(\mathcal{M}(S_1)^C \cup S_2^C) = \mathcal{M}(\mathcal{M}(S_1) \cup S_2)$.

Because of the last corollary, the algorith of $\mathcal{M}$-abstraction can be named *incremental algorithm*: When the abstraction of aconcept is given, we can add to this a set and directly obtain the new abstraction. With respect to $\mathcal{A}$-algorithm, $\mathcal{M}$-algorithm can operate directly on concept rough definitions, rather than on the nrmal ones.

Example 8.1 *Given* $S = \{1110**, 1111*0, *11011, *11010\}$

S^C	$a^1(S^C)$	$a^2(S^C)$	$\mathcal{A}(S)$
111000	11100*	1110**	1110**
111001	1110*0	1110**	111**0
111010	111*00	11100*	*1101*
111011	1110*1	11100*	
111100	11101*	*1101*	
111110	111*10		
011011	*11011		
011010	1111*0		
	01101*		
S	$m(S)$		
1110**	1110**		
1111*0	111**0		
*11011	*1101*		
*11010			

Note that $m(S)$ *equals* $\mathcal{M}(S)$ *since* $S\Delta_1 = \{111**0, 111*10, *1101*\}$.

9 The decremental algorithm

In this last section we make up an algorithm through which for any concept defined abstractly, it can be obtained the abstract definition of the negation of the concept itself. This constitutes an useful complement to the operations introduced in the previous sections.

Definition 9.1 $\forall x_\alpha, x_\beta \in X$:

$$x_\alpha \vee x_\beta = \begin{cases} x_\gamma, & \text{if } \Delta(x_\alpha, x_\beta) = 0 \\ x_\#, & \text{if } \Delta(x_\alpha, x_\beta) > 0 \end{cases} \quad \text{where } x_{h\gamma} = \begin{cases} x_{h\alpha}, & \text{if } x_{h\alpha} \neq * \\ x_{h\beta}, & \text{otherwise.} \end{cases}$$

$x_\alpha \vee x_\beta$ is the more general entity that concretizes both x_α and x_β.

Proposition 9.1 $\forall x_\alpha, x_\beta \in X$ $\{x_\alpha \vee x_\beta\}^C = \{x_\alpha\}^C \cap \{x_\beta\}^C$.

Proof. Let us consider the two cases a) $\Delta(x_\alpha, x_\beta) = 0$ and b) $\Delta(x_\alpha, x_\beta) > 0$.

a) $\Delta(x_\alpha, x_\beta) = 0$; $\forall h$ three cases are possible:

1) $x_{h\alpha} = x_{h\beta} \in \{0, 1, *\}$

2) $x_{h\alpha} = *$ and $x_{h\beta} \in \{0, 1\}$

3) $x_{h\alpha} \in \{0, 1\}$ and $x_{h\beta} = *$

$x_\delta = x_\alpha \vee x_\beta$ concretize both x_α and x_β by definition, i.e., $x_\delta \geq x_\alpha$ and $x_\delta \geq x_\beta$.

Thus, $\{x_\delta\}^C \subseteq \{x_\alpha\}^C$ and $x_\delta\}^C \subseteq \{x_\beta\}^C$, that is: $\{x_\delta\}^C \subseteq \{x_\alpha\}^C \cap \{x_\beta\}^C$.

Let $x_\epsilon \in \{x_\alpha\}^C \cap \{x_\beta\}^C$, i.e., $x_\epsilon \geq x_\alpha$ and $x_\epsilon \geq x_\beta$, we show that $\forall x_\epsilon, x_\epsilon \geq x_\delta$; if $x_\epsilon \in \{x_\alpha\}^C \cap \{x_\beta\}^C$, then $\forall h$ such that $x_{h\alpha} \in \{0, 1\}$ $x_{h\epsilon} = x_{h\alpha}$, and for these indexes we have also that $x_{h\delta} = x_{h\alpha}$, i.e.: $x_{h\epsilon} = x_{h\delta}$.

If $x_{h\alpha} = *$ and we are in case 3) $x_{h\delta} = *$ holds, while in case 2) $x_{h\delta} = x_{h\beta} \in \{0, 1\}$, and also $\forall h$ such that $x_{h\beta} \in \{0, 1\}$ $x_{h\epsilon} = x_{h\beta}$ in that $x_\epsilon \in \{x_\beta\}^C$.

In conclusion, either $x_{h\epsilon} = x_{h\delta}$ or $x_{h\delta} = *$, that is, $x_\epsilon \geq x_\delta$ with $x_\epsilon \in S^C$; this shows that $x_\epsilon \in \{x_\delta\}^C$ i.e. $\{x_\alpha\}^C \cap \{x_\beta\}^C \subseteq \{x_\delta\}^C$.

b) $\Delta(x_\alpha, x_\beta) > 0$, that is, $\exists k$ such that

1) $x_{k\alpha} = 0$ and $x_{k\beta} = 1$ or

2) $x_{k\alpha} = 1$ and $x_{k\beta} = 0$

In the first case: $\forall x_\epsilon \in \{x_\alpha\}^C$ $x_{k\epsilon} = 0$ and $\forall x_\gamma \in \{x_\beta\}^C x_{k\gamma} = 1$ so that $\{x_\alpha\}^C \cap \{x_\beta\} = \emptyset$.

In the second: $\forall x_\epsilon \in \{x_\alpha\}^C$ $x_{k\epsilon} = 1$ and $\forall x_\gamma \in \{x_\beta\}^C x_{k\gamma} = 0$ so that $\{x_\alpha\}^C \cap \{x_\beta\}^C = \emptyset$. $\{x_\#^C = \emptyset$ by definition, thus $\{x_\alpha\}^C \cap \{x_\beta\}^C = \{x_\delta\}^C = \emptyset$.

Definitively, the proposition states that $\forall x_\alpha, x_\beta, x_\alpha \vee x_\beta$ is an element that concretizes both x_α and x_β and between these elements such $x_\alpha \vee x_\beta$ is the more general.

Definition 9.2 $\forall x_\alpha, x_\beta \in X$:

$$\overline{x}_{h\alpha} = \begin{cases} 0 & \textit{if } x_{h\alpha} = 1 \\ 1 & \textit{if } x_{h\alpha} = 0 \\ & \textit{if } x_{h\alpha} = * \end{cases}$$

$x_{\alpha k\beta} = x_\gamma$ *where* $x_{k\gamma} = x_{k\beta}$ *and* $\forall h \neq k$ $x_{h\gamma} = x_{h\alpha}$;
$x_{\alpha k\beta} = x_\delta$ *where* $x_{k\delta} = \overline{x}_{k\beta}$ *and* $\forall h \neq k$ $x_{h\delta} = x_{h\alpha}$.

Proposition 9.2 $\forall x_\alpha, x_\beta \in X$ *such that* $x_{k\beta} \neq *$:

1) $\{x_{\alpha k\beta}\}^C \cap \{x_{\alpha k\overline{\beta}}\}^C = \emptyset$;

2) if $\{x_{k\alpha} = *$ *then* $\{x_{\alpha k\beta}\}^C \cup \{x_{\alpha k\overline{\beta}}\}^C = \{x_\alpha\}^C$.

Proof. Let us denote $x_{\alpha k\beta}$ by x_δ and $x_{\alpha k\overline{\beta}}$ by x_ϵ.

1) $x_{k\beta} \neq *$ thus $x_{k\beta} \in \{0,1\}$; if we suppose that $x_{k\beta} = 0$, then $x_{k\delta} = 0$ and $x_{k\epsilon} = 1$; it follows that $\{x_{\alpha k\beta}\}^C \cap \{x_{\alpha k\overline{\beta}}\}^C = \emptyset$. The same happens if $x_{k\beta} = 1$.

2) in this case $\forall h \neq k$ $x_{h\gamma} = x_{h\epsilon} = x_{h\alpha}$, and $x_{k\alpha} = *$; thus $x_\alpha < x_{\alpha k\beta}$ and $x_\alpha < x_{\alpha k\overline{\beta}}$; it follows that $\{x_{\alpha k\beta}\}^C \cup \{x_{\alpha k\overline{\beta}}\}^C \subseteq \{x_\alpha\}^C$. The inverse inclusion follows directly by considering that for every $x_\xi \in \{x_{\alpha k\beta}\}^C$ and if $x_{k\xi} = 1$, then $x_\xi \in \{x_{\alpha k\overline{\beta}}\}^C$ (if $x_{k\beta} = 0$)

Corollary 9.1 $\{x_{\alpha k\overline{\beta}}\}^C = \{x_\alpha\}^C \backslash \{x_{\alpha k\beta}\}^C$.

Lemma 9.1 $\forall x_\alpha, x_\beta \in X$ *such that* $x_\beta \geq x_\alpha$, $\bigcap_{k: x_{k\alpha} \neq x_{k\beta}} \{x_{\alpha k\beta}\}^C = \{x_\beta\}^C$.

Proof. Let be d =card $\{k : x_{k\alpha} \neq x_{k\beta}\}$; note that if $x_{k\alpha} \neq x_{k\beta}$, in the hypothesis that $x_\beta \geq x_\alpha$, then $x_{h\alpha} = *$ and $x_{k\beta} \in \{0,1\}$.

We proceed by induction on d.

If $d)1$ i.e. $\exists! k$ such that $x_{k\alpha} \neq x_{k\beta}$, when we replace $x_{k\alpha}$ by $x_{k\beta}$, we obtain x_β, in that $\forall h \neq k$ $x_{h\alpha} = x_{h\beta}$.

It follows that

$$\{x_{\alpha k\beta}\}^C = \{x_\beta\}^C.$$

Let us suppose the theorem is true for $d = n - 1$.
By ordering set $\{k : x_{k\alpha} \neq x_{k\beta}\}$ we can put

$$\bigcap_{k=1}^{n} \{x_{\alpha k\beta}\}^C = \bigcap_{k=1}^{n-1} \{x_{\alpha k\beta}\}^C \bigcap \{x_{\alpha n\beta}\}^C$$

from which, by inductive hypothsesis:

$$\bigcap_{k=1}^{n} \{x_{\alpha k\beta}\}^C = \{x_\beta\}^C \cap x_{\alpha n\beta}\}^C.$$

Since by the hypothesis that $x_\beta geq x_\alpha$ it follows directly that $x_\beta \geq x_{\alpha n\beta}$, that is

$$\{x_\beta\}^C \subseteq \{x_{\alpha n\beta}\}^C,$$

$\{x_\beta\}^C\{x_{\alpha n\beta}\}^C = \{x_\beta\}^C$ i.e. $\bigcap_{k=1}^{n}\{x_{\alpha k\beta}\}^C = \{x_\beta\}^C$.

Definition 9.3 *$\forall x_\alpha, x_\beta \in X$ such that $x_\beta \geq x_\alpha$ we define*

$$\{x_\alpha \backslash x_\beta\} = \bigcup_{k: x_{k\alpha} \neq x_{k\beta}} \{x_{\alpha k\overline{\beta}}\}).$$

Lemma 9.2 *$\forall x_\alpha, x_\beta \in X$ such that $x_\beta \geq x_\alpha$ $\{x_\alpha \backslash x_\beta\}^C = \{x_\alpha\}^C \backslash \{x_\beta\}^C$.*

Proof.

$$\begin{aligned}\{x_\alpha \backslash x_\beta\}^C &= \left(\bigcup_{k: x_{k\alpha} \neq x_{k\beta}} \{x_{\alpha k\overline{\beta}}\} \right)^C = \bigcup_{k: x_{k\alpha} \neq x_{k\beta}} \{x_{\alpha k\overline{\beta}}\}^C = \\ &= \bigcup_{k: x_{k\alpha} \neq x_{k\beta}} \left(\{x_\alpha\}^C \backslash \{x_{\alpha k\beta}\}^C \right) = \{x_\alpha\}^C \backslash \bigcup_{k: x_{k\alpha} \neq x_{k\beta}} \{x_{\alpha k\beta}\}^C = \\ &= \{x_\alpha\}^C \backslash \{x_\beta\}^C.\end{aligned}$$

Definition 9.4 *$\forall A, B \in \mathcal{P}(X)$ $A = \bigcup_{h=1}^{r}\{a_h\}$ and $B = \bigcup_{k=1}^{p}\{b_k\}$ we define:*

$$A \vee B = r \left(\bigcup_{\substack{h=1,\ldots,r \\ k=1,\ldots,p}} \{a_h \vee b_k\} \right)$$

Theorem 9.1 *$\forall A, B \in \mathcal{P}(X)$ $A = \bigcup_{h=1}^{r}\{a_h\}$ and $B = \bigcup_{k=1}^{p}\{b_k\}$:*

$$(A \vee B)^C = A^C \cap B^C$$

Proof.

$$\begin{aligned}(A \vee B)^C &= \left(r\left(\bigcup_{\substack{h=1,\dots,r\\k=1,\dots,p}}\{a_h \vee b_k\}\right)\right)^C = \left(\bigcup_{\substack{h=1,\dots,r\\k=1,\dots,p}}\{a_h \vee b_k\}\right)^C = \\ &= \bigcup_{\substack{h=1,\dots,r\\k=1,\dots,p}}\{a_h \vee b_k\}^C = \bigcup_{\substack{h=1,\dots,r\\k=1,\dots,p}}\{a_h\}^C \cap \{b_k\}^C = \\ &= \left(\bigcup_{h=1}^{r}\{a_h\}\right)^C \cap \left(\bigcup_{k=1}^{p}\{b_k\}\right)^C = A^C \cap B^C.\end{aligned}$$

Definition 9.5 $\forall S \in \mathcal{P}(X)$ $S = \bigcup_{i=1}^{n}\{x_i\}$

$$\mathcal{D}(S) = \mathcal{M}\left(\bigvee_{i=1}^{n}\{x_*\backslash x_i\}\right)$$

Theorem 9.2 *$\forall S \in \mathcal{P}(X)$ $S = \bigcup_{i=1}^{n}\{x_i\}$, $\mathcal{D}(S)^C = X^0\backslash S^C = \overline{S^C}$.*

Proof. Let $S_i = \{x_*\backslash x_i\}$:

by Theorem 7.14 $\mathcal{D}(S)^C = \left(\bigvee_{i=1}^{n} S_i\right)^C$ and by Theorem 9.5 $\left(\bigvee_{i=1}^{n} S_i\right)^C = \bigcup_{i=1}^{n} S_{i^C}$;

by Lemma 9.4 $S_{i^C} = X^0\backslash\{x_i\}^C$ so that, by the De Morgan's laws:

$$\mathcal{D}(S)^C = \bigcap_{i=1}^{n} X^0\backslash\{x_i\}^C = X^0\backslash\bigcup_{i=1}^{n}\{x_i\}^C = \overline{S^C}.$$

Theorem 9.3 *$\forall S \in \mathcal{P}(X)$ $\mathcal{D}(S) = \mathcal{A}(\overline{S^C})$.*

Proof. By Theorem 8.2 $\mathcal{M}(\mathcal{D}(S)) = \mathcal{A}(\mathcal{D}(S))$, but $\mathcal{M}(\mathcal{D}(S)) = \mathcal{D}(S)$, being $\mathcal{D}(S)$ and $\mathcal{M}$-abstraction; thus

$$\mathcal{D}(S) = \mathcal{A}(\mathcal{D}(S)) = \mathcal{A}(\mathcal{D}(S)^C) = \mathcal{A}(\overline{S^C}).$$

The $\mathcal{D}$-operator can constitute a basis to make up an algorithm yielding the abstraction of the negation of one concept, starting from one rough definition of this. By considering the way in which is such an algorithm operates, this can be called *decremental algorithm*.

Corollary 9.2 $\forall T \in [S^C]$ *it is easy to prove that:*

1) $\mathcal{D}(T) = \mathcal{D}(S)$

2) $\mathcal{D}(S^C) = \mathcal{D}(S)$

By the algorithms we introduced (see Section 7 and 9) and in consideration of the results these yield, given $S, T \in \mathcal{P}(X)$, the following equalities are true:

1) $\mathcal{A}(\overline{S^C}) = \mathcal{D}(S)$;

2) $\mathcal{A}(S^C \cup T^C) = \mathcal{A}(S \cup T) = \mathcal{M}(S \cup T)$ since $(S \cup T)^C = S^C \cup T^C$;

3) $\mathcal{A}(S^C \cap T^C) = \mathcal{A}(\overline{\overline{S^C} \cup \overline{T^C}}) = \mathcal{D}(\overline{S^C} \cup \overline{T^C}) = \mathcal{D}((\mathcal{A}(\overline{S^C}) \cup \mathcal{A}(\overline{T^C}))^C) = \mathcal{D}(\mathcal{A}(\overline{S^C}) \cup \mathcal{A}(\overline{T^C})) = \mathcal{D}(\mathcal{D}(S) \cup \mathcal{D}(T))$;

4) $\mathcal{A}(S^C \cap \overline{T^C}) = \mathcal{A}(\overline{\overline{S^C} \cup T^C}) = \mathcal{D}(\overline{S^C} \cup T^C) = \mathcal{D}(\mathcal{D}(S) \cup T)$.

These show that by means of the mentioned algorithms, it is possible to obtain directly the abstraction of any subset resulting from operations on elements of $\mathcal{P}(X)$. We can put, for example,

1) $\sim S = \mathcal{A}(\overline{S^C}) = \mathcal{D}(S)$;

2) $S \sqcup T = \mathcal{A}(S^C \cup T^C) = \mathcal{M}(S \cup T)$;

3) $S \sqcap T = \mathcal{A}(S^C \cap T^C) = \mathcal{D}(\mathcal{D}(S) \cup \mathcal{D}(T))$.

10 Conclusions

One of the relevant results obtained by the paper was determining the structural characteristics of the conceptual universe Ξ. These derive from the relation existing among its elements –concepts. Such characteristics are complementary to the reticular structure of Ξ, earlier evidenced. This derives from the linking that "$\prec$" operates on the concepts forming Ξ because of the possible relations among concepts, subsuming one another: possibility by a concept of leading another into a more extended frame, i.e., having a more extended extensional definition.

By considering the introduced algorithms and the results these yield, we have achieved the possibility that given any two elements of $\mathcal{P}(X)$, that is, two concepts, we can acquire knowledge of their negation, union and possibly intersection. The outcoms still are concepts and are given directly in abstract form. Thus, it is put the basis for structuring elements that formalize concepts, as an algebra. The abstract form of any considered concept is also useful to compute conceptual parameters as well symbolic as numerical, thorough its comparison with the one concrete.

References

Arigoni, A.O. and Balboni, E. (1974): Definizione ricorsiva di significato soggettivo, Procced. XXIV *Nat. Con. of Philosofy (Logics Sect.)*, L'Aquila, Italy.

Arigoni A, O. (1980): Mathematical development arising from 'semantical implication' and the evaluation of membership characteristic functions, *Fuzzy Sets and Systems*, **4**, 167-183.

Arigoni A, O. (1982): Transformational-generative grammar for describing formal properties, *Fuzzy Sets and Systems*, **8**, 311-322.

Arigoni, A.O. and Maniezzo, V. (1992): Conceptual abstraction: a context-depending simplification of concept representations, *Working Notes of the Approximation and Abstraction of Computational Theories '92"*. San Jose, Calif.

Arigoni, A.O. Furlanello, C. and Maniezzo, V. (1992): "Incremental learning of roughy represented concepts", 4th Int. Conf. on Information Processing and management of Uncertinty in Knowledge-Based Systems, Palma de Mallorca, Spain, (Proc. Springer-Verlag).

Arigoni, A.O. (1993): "Typicality of concept instances: a semiotic way for its evaluation", in: E.P. Klement and W. Slany Ed.

Fuzzy Logic in Artificial Intelligence, Lecture Notes on Artificial Intelligence (625) Springer-Verlag, Vien.

Bobrow, D.G., Winograd, T.: KRL: another prospective, *Cognitive Science*, 3, 29-42, 1979.

Brachmann, R.J.: 'I lied about the trees' or, defaults and definitions in knowledge representation, *The AI magazine*, pag. 90-93, 1985.

Chandrasekaran, B., Goel, A.: From numbers to symbols to knowledge structures: artificial intelligence perspectives on the classification task, *IEEE trnasactions on systems, man and cybernetics*, vol. 18, n. 3, 1988.

Dubois, D. (1991): Semantic et Cognition: catégories, prototypes, typicalité, *Editions du CNRS*, Paris.

Giunchiglia, F. and Walsh, T. (1992): A theory of Abstraction, *Artificial Intelligence*, **57**, 323-389.

Hobbs, J.R. (1985): Granularity, in Proceed. of 96h IJCAI Conference, 432-435, *Int. Joint. Conf. on Artificial Intelligence.*

McLuskey, E.J. (1965): *Introduction to the Theory of Switching Circuits*, New York, McGraw-Hill.

Plaistel, D.A. (1981): Theorem proving with abstraction, *Artificial Intelligence*, **16**, 47-108.

Rosch, E. (1975): Cognitive representation of semantic categories, *J. Exp. Pshychology: general*, **104**.

Roth, E.M. and Mervis, C.B. (1983): Fuzzy set theory and class inclusion relations in semantic categories, *J. of Verbal Learning and Verbal Behaviour*, **22**, 509-525.

Saitta, L., Roverso, D. and Giordana, A. (1991): Abstracting background knowledge for concept learning, *Proc. of Machine learning. EWSL-91,* 1-13, Springer Verlag.

Schmolze, J.G., Lipkis, T.A.: Classification in the KL-ONE knowledge representation system, *IJCAI-83*, 1983, pp. 330-332.

Stepp, R.E. and Michalski, R.S. (1986): Conceptual clustering of structured objects: a goal-oriented approach, *Artificial Intelligence*, **28**, 43-69.

Tabossi, P. (1985): La rappresentazione mentale dei concetti, *Giornale Italiano di Psicologia*, **1**, 45-70.

Tenenberg, J.D. (1987): Preserving Consistency Across Abstraction Mappings, *IJCAI-10.*

Tomela, R. (1973): Theoretical Concepts, Springer-Verlag, Wien, New York.

Vandierendonck, A. (1991): Are category membership decision based on concept gradness?, *Eur. Jour. of Cogni. Psych.* **3**, 343-362.

Wos, L. (1984): *Automated reasoning - Introduction and Application*, Prentice-Hall Inc., New Jersey.

Zadeh, L.A. (1976): A fuzzy approach to the definition of complex or imprecise concepts, *Int. J. Man-Machine Studies*,**8**, 249-291.