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Sugeno's negations and t-norms

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Abstract

A functional characterization of Sugeno's negations is presented and as a consequence, we study a family of non strict Archimedean t-norms whose (vertical-horizontal) sections are straight lines.

Keywords: non strict Archimedean t-norm, strong negation, Sugeno's negation, functional equation.

1 Introduction

Since K. Menger (1942), B. Schweizer and A. Sklar (1961) introduced t-norms in the setting of probabilistic metric spaces ([4], [5]), many authors have studied this class of binary operations on [0,1] (see [2], [6]). Apart from the applications of these functions in different fields: probabilistic norms and scalar products, multiple-valued logic, fuzzy sets thoery, etc., there is an increasing interest in their theoretic study which no doubt is going to lead to new possibilities of application. In this respect, our paper deals with the study of a certain type of non strict Archimedean t-norms closely related to Sugeno's negations.

2 Preliminaries

Definition 1 *A t-norm is a non-decreasing, associative and commutative binary operation on [0,1] with 1 as unit.*

A t-norm T is called Archimedean if it is continuous and $T(x,x) < x$ for all x in (0,1). From J. Aczél ([1]) and C.H. Ling ([3]) we have the following important characterization of this kind of functions.

Theorem 1 *A t-norm T is Archimedan if and only if there exists a continuous and strictly decreasing funciton f from [0,1] to $(0,+\infty]$ with $f(1) = 0$ such that T is representable in the form $T(x,y) = f^{(-1)}(f(x) + f(y))$ where $f^{(-1)}$ is the pseudoinverse of f, that is, $f^{(-1)}(x) = 0$ whenever $f(0) \leq x$, and $f^{(-1)}(x) = f^{-1}(x)$ whever $0 \leq x \leq f(0)$. The function f is called an additive generator of T and it is unique up to a positive multiplicative constant.*

An Archimedean t-norm T, with additive generator f, is called strict if $f(0) = +\infty$ and non-strict otherwise. If T is strict then it is strictly increasing in each place in $(0,1)^2$. If T is non-strict then the zero set of it, $Z(T) = \{(x,y) \in [0,1]^2; T(x,y) = 0\}$, is determined by $(x,y) \in Z(T)$ iff $y \leq f^{-1}(f(0) - f(x))$.

It is well known that the so-called t-norm of Lukasiewicz $W(x,y) = \text{Max}(0, x+y-1)$ (with additive generator $f(x) = 1-x$) is a fundamental t- norm in the class of non-strict Archimedean t-norms. Thus, as a consequence of theorem 1, and from an algebraic point of view, every non-strict Archimedean t-norm T is isomorphic to W (i.e. there exists a bijection $\varphi : [0,1] \to [0,1]$ such that $\varphi(T(x,y)) = W(\varphi(x), \varphi(y))$ for all $x, y \in [0,1]$).

For more information on t-norms see ([12], [6]).

Definition 2 *A strong negation N is an involutive decreasing function from [0,1] to itself.*

Let T be a non-strict Archimedean t-norm with f as additive generator, then $N_T(x) = f^{-1}(f(0) - f(x))$ is a strong negation. It is called the strong negation associated with T. From above we have that $T(x,y) = 0$ if, and only if, $y \leq N_T(x)$. The strong negation associated with W is $N_W(x) = 1-x$.

Note that any strong negation has a unique fixed point in (0,1). For more details on strong negations see ([7]).

Theorem 2 *The only strong negation of the form $N(x) = (ax+b)/(cx+d)$ are the so- called Sugeno's negations: $N(x) = (1-x)/(1+cx)$ with $c > -1$.*

3 A functional characterization of Sugeno's negations

Let us consider the functional equation

$$x(y-1)f(x)+y(1-x)f(y)=(y-x)f(x)f(y), \qquad x,y\in[0,1] \qquad (M)$$

with unknown function $f:[0,1]\to[0,1]$.

This equation admits the trivial solutions $f=0$ and $f=1$ (the only constant solutions of (M)). $f(x)=1-x$ is also solution of (M). Discontinuous solutions can be shown: $f(x)=a$ $(0<a\leq 1)$ when $x=0$ and $f(x)=0$ otherwise.

Theorem 3 *The only solutions of the equation (M), with fixed point in the open interval (0,1), are the Sugeno's negations.*

Proof. Let f be a Sugeno's negation: $f(x)=(1-x)/(1+x)$ with $c>-1$. Through an easy calculation we can prove that f satisfies (M) for all x,y in [0,1] and $c>-1$. Now, let us suppose that $f:[0,1]\to[0,1]$ is a solution of (M) and it has fixed point $s\in(0,1)$ $(f(s)=s)$. Put $y=s$ in (M), we have:

$$(s-1)xf(x)+(1-x)s`2=(s-x)sf(x) \qquad \text{for all } x \text{ in } [0,1]$$

and then we obtain

$$f(x)=\frac{(1-x)s^2}{s^2+((1-2s)x)}=\frac{1-x}{1+\frac{1-2s}{s^2}x}$$

where $(1-2s)/s^2>-1$, therefore f is a Sugeno's negation.

Corollary 1 *If N is a strong negation that verifies*

$$x(y-1)N(x)+y(1-x)N(y)=(y-x)N(x)N(y)$$

$$\textit{for all } x,y \textit{ such that } y\geq N(x) \quad (M')$$

then N is a Sugeno's negation.

Proof. Put $N(x) = u$, $N(y) = v$ in (M'). We can write:

$$N(u)(N(v) - 1)u + N(v)(1 - N(u))v = (N(v) - N(u))uv$$

$$\text{for all } u, v \text{ such that } u \leq N(v)$$

through a simple calculation we have:

$$u(v - 1)N(u) + v(1 - u)N(v) = (v - u)N(u)N(v)$$

$$\text{for all } u, v \text{ such that } v \leq N(u)$$

therefore N is a solution of equation (M) and then it is a Sugeno's negation.

4 A family of non strict Archimedean t-norms

Consider the family, $(T_p)_{p>0}$, of non strict Archimedan t-norms defined by:

$$T_p(x, y) = \text{Max } (0, (1 - p)xy + p(x + y - 1))$$

This family of t-norms has been used by several authors (see [8]).

Note that $T_1 = W$. If $p \neq 1$, then an additive generator of T_p is $f(x) = (p - 1)ln(p + (1 - p)x)$.

On the other hand, the strong negation associated with T_p is

$$N_p(x) = \frac{1 - x}{1 + \frac{1-p}{p}x}$$

which is the Sugeno's negation with parameter $c = (1 - p)/p$. Now, given T_p, let us fix $y \in [0, 1]$ and consider the y-section of T_p, T_p^y, defined by $T_p^y(x) = T_p(x, y)$, $x \in [0, 1]$. We can write:

$$T_p^y(x) = \begin{cases} 0, & \text{whenever } 0 \leq x \leq N_p(y) \\ ((1 - p)y + p)x - p(1 - y), & \text{otherwise} \end{cases}$$

thus T_p^y is a straight line on $[N_p(y), 1]$ for all y in [0,1], and all $p > 0$.

Next we shall demonstrate that T_p are the only non strict Archimedean t- norms which have the above mentioned characteristic.

Theorem 4 *Let T be a non strict Archimedean t-norm with N as associated strong negation. Let us assume that, for all y in [0,1], the y-section of T, T^y, is a striaght line on $[N(y), 1]$. In these conditions there exists $p > 0$ such that $T = T_p$.*

Proof. From our supposititiion we have for each y in [0,1]:

$$T(x,y) = T^y(x) = \begin{cases} 0, & \text{whenever } 0 \leq x \leq N(y) \\ \frac{yx - yN(y)}{1-N(y)}, & \text{whenever } x > N(y) \end{cases}$$

On the other hand, from commutativity of T, we can write $T^y(x) = T(x,y) = T(y,x) = T^x(y)$ for all x, y in [0,1], thus we have

$$\frac{yx - yN(y)}{1 - N(y)} = \frac{xy - xN(x)}{1 - N(x)} \text{ for all } x, y : y > N(x)$$

and also

$$x(y-1)N(x) + y(1-x)N(y) = (y-x)N(x)N(y)$$
$$\text{for all } x, y \text{ such that } y \geq N(x)$$

so from Corollary 1 we can say that N is a Sugeno's negation: $N(x) = (1-x)/(1+cx)$ with $c > -1$. Finally, through an easy calculation, we obtain $T(x,y) = \text{Max}(0, (1-p)xy + p(x+y-1))$ where $p = 1/(1+c)$, and the theorem in completely proved.

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