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Martin boundary for homogeneous riemannian manifolds of negative curvature at the bottom of the spectrum

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0. Introduction.

Let M be a manifold and let $\mathcal{L}$ be a subelliptic second order differential operator on M. Positive $\mathcal{L}$-harmonic functions have been intensively studied for many decades. In particular, if M has negative curvature and $\mathcal{L}$ is coercive (*i.e.* there is a positive ε such that $\mathcal{L} + \varepsilon I$ admits the Green function), the Martin boundary has been described by A. Ancona [A], and earlier by M. Anderson and Schoen [AS] in the case when $\mathcal{L}$ is the Laplace-Beltrami operator. If $\mathcal{L}$ is noncoercive, the situation is much more complicated, there are no results like in [A], so various particular cases are of interest.

In this paper we treat noncoercive operators on simply connected *homogeneous* manifolds of negative curvature. J. Wolf [W] and E. Heintze [Hei] proved that such a manifold is isometric with a solvable Lie group $S = N A$, being a semi-direct product of a nilpotent Lie group N and $A = \mathbb{R}^+$ and, moreover, for a $H \in \mathcal{A}$ the Lie algebra of A the eigenvalues of $\mathrm{Ad}_H|_N$ are all greater than 0. Conversely, every such group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature.

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y ,$$

such that $Y_0, \dots, Y_m$ generate $\mathcal{S}$. Let $\pi : S \longrightarrow A = S/N$ be the canonical homomorphism. $d\pi(\mathcal{L})$ is a second order invariant operator on $\mathbb{R}^+$, hence

$$d\pi(\mathcal{L}) = (a\,\partial_a)^2 - \gamma\, a\,\partial_a ,$$

for a $\gamma \in \mathbb{R}$. $-\gamma\, a\,\partial_a$ is the $\mathcal{A}$-component of Y and $\mathcal{L} = \mathcal{L}_\gamma$ is coercive, if and only if $\gamma \neq 0$.

Let μ_t be the semigroup of measures generated by $\mathcal{L}_\gamma$. If $\gamma \geq 0$, then there is a unique (up to a constant) positive Radon measure ν_γ on N such that

$$\check{\mu}_t^\gamma * \nu_\gamma = \nu_\gamma , \qquad t > 0$$

[E]. For $\gamma > 0$ the measure ν_γ is bounded, while ν_0 is unbounded. The measures ν_γ, $\gamma > 0$ have been studied in various contexts [B], [E], [G], [Ra], see also [D1], [D2], [DH2], [DHZ]. In particular, the bounded $\mathcal{L}_\gamma$-harmonic functions, $\gamma > 0$ are described as ν_γ-Poisson integrals [Ra], [D1], [DH2] of L^∞-functions on N. If $\gamma = 0$, the only bounded $\mathcal{L}$-harmonic functions are constants but the unbounded measure ν_0 gives rise to non-trivial positive $\mathcal{L}_0$ harmonic functions.

Also ν_γ plays an essential role in description of the Martin boundary for $\mathcal{L}_\gamma$ (and $\mathcal{L}_{-\gamma}$) both in the coercive and the noncoercive case. However, while the first case can be deduced from Ancona's theory [D2], the latter requires new methods. This is the main topic of our study here.

We make use of a probabilistic method introduced in [DH1] and continued in [DHZ]. The essence of it is a decomposition of the diffusion on S generated by $a^{-2}\mathcal{L}$ into the "vertical component" generated by $(\partial_a)^2 - (\gamma/a)\,\partial_a$ (Bessel process) and the "horizontal component" for which the transition probabilities conditioned on a trajectory a_t of the "vertical component" satisfy some evolution equation (Chapter 3). The idea of this decomposition is very intuitive and goes back to [M], [MM], *cf.* also [K], [S], [Tay]. The available proofs of the properties of this decomposition are either very sketchy or quite involved. We give here a direct proof of it adapted to the situation of our interest.

The main aim of the present paper is to describe the Martin boundary for $\mathcal{L}_\gamma$, for all $\gamma \in \mathbb{R}$. In addition, we find lower and upper pointwise bounds for ν_γ. ν_γ turns out to be the main building block for all minimal positive $\mathcal{L}_\gamma$.

In the simplest two dimensional case, *i.e.* when S = "$ax + b$" the description of the Martin boundary is due to Molchanov, [Mo]. Indeed, his technique is based on properties of the Bessel process, as is ours, only in the two-dimensional case the operator in the horizontal direction can be made independent of the vertical direction which makes the decomposition mentioned above superfluous, and all the arguments are much simpler.

1. Preliminaries.

Let

$$\mathcal{S} = \mathcal{N} \oplus \mathcal{A} \tag{1.0}$$

be a solvable Lie algebra which is the sum of its nilpotent ideal $\mathcal{N}$ and a one-dimensional algebra $\mathcal{A} = \mathbb{R}^+$. We assume that

$$\begin{aligned}&\text{there exists } H \in \mathcal{A} \text{ such that the real parts}\\ &\text{of the eigenvalues of } \mathrm{ad}_H : \mathcal{N} \longmapsto \mathcal{N} \text{ are positive}.\end{aligned} \tag{1.1}$$

Let N, A, S be the connected and simply connected Lie groups whose Lie algebras are $\mathcal{N}, \mathcal{A}, \mathcal{S}$ respectively. Then $S = NA$ is a semi-direct product of N and $A = \mathbb{R}^+$.

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y\,,$$

such that $Y_0, \dots, Y_m$ generate $\mathcal{S}$. It follows from elementary linear algebra that $Y_0, \dots, Y_m$ can be chosen in the way that $Y_1(e), \dots, Y_m(e) \in \mathcal{N}$.

The decomposition (1.0) is not unique, *i.e.* there is no canonical choice of A. We put $A = \exp\{t\,Y_0 :\ t > 0\}$ and assume with no loss of generality that the real parts of the eigenvalues of ad_{Y_0} are strictly positive. Moreover, multiplying $\mathcal{L}$ by a constant we may assume that the real parts of ad_{Y_0} are large. Decomposing $s \in S$ as $s = xa$, $x \in N$,

$a = \exp{(\log a)(Y_0)}$, we write

$$\begin{aligned}\mathcal{L}f(xa) &= \mathcal{L}_\gamma f(xa)\\ &= ((a\,\partial_a)^2 - \gamma\, a\,\partial_a)\, f(xa)\\ &\quad + \Big(\sum_{j=1}^{m} \Phi_a(X_j)^2 + \Phi_a(X)\Big) f(xa)\,,\end{aligned} \tag{1.2}$$

where $\Phi_a = \mathrm{Ad}_{\exp{(\log a)Y_0}}$ and $X, X_1, \ldots, X_m$ are left-invariant vector fields on N and $X_1, \ldots, X_m$ generate $\mathcal{N}$. We shall keep the subscript γ in $\mathcal{L}$ in order to stress the role of the $\mathcal{A}$-component of Y.

(1.1) together with the assumption on the length of Y_0 imply (see *e.g.* [DHZ]) that there are $m_1, m_2 > 2$ and $C > 0$ such that

$$\|\Phi_a\|_{\mathcal{N}\to\mathcal{N}} \le C\,(a^{m_1} + a^{m_2})\,, \qquad a > 0\,. \tag{1.3}$$

In N we define a "homogeneous" norm $|\cdot|$. Let $(\cdot,\cdot)$ be an arbitrary fixed inner product in $\mathcal{N}$ and let

$$\langle X, Y\rangle = \int_0^1 (\Phi_a(X), \Phi_a(Y))\,\frac{da}{a}\,, \qquad \|X\| = \sqrt{\langle X, X\rangle}\,.$$

We put

$$|\exp X| = |X| = (\inf\{a > 0 :\ \|\Phi_a(X)\| \ge 1\})^{-1}\,.$$

Since for $X \neq 0$

$$\begin{aligned}&\lim_{a\to 0} \|\Phi_a(X)\| = 0\,,\\ &\lim_{a\to\infty} \|\Phi_a(X)\| = \infty\,,\\ &\text{and } a \longrightarrow \|\Phi_a(X)\| \text{ is increasing}\,,\end{aligned}$$

it follows that for every $Y \neq 0$ there is precisely one a such that

$$Y = \Phi_a(X)\,, \qquad |X| = 1\,, \qquad |Y| = a\,.$$

If the action of A on N is diagonal, $|\cdot|$ is the usual homogeneous norm on N. Finally, let

$$\sigma_a(\exp X) = \exp{(\log a)}\, Y_0 \exp X \exp{(-\log a)}\, Y_0$$

i.e. Φ_a is the differential of σ_a.

The space $\mathcal{H}_b$ of bounded harmonic functions for $\mathcal{L}$ is well known. If $\gamma \leq 0$, then bounded harmonic functions are constant. This is a consequence of [BR] (*cf.* also [DH2]). If $\gamma > 0$, $\mathcal{H}_b$ is in one-one correspondence with $L^\infty(N)$ via the Poisson integral

$$F(s) = \int_N f(s \cdot x)\, m_\gamma(x)\, dx\,, \tag{1.4}$$

where $x \longrightarrow s \cdot x$ denotes the action of S on $N = S/A$ ([Ra], [DH2]). m_γ is a smooth, bounded positive function with $d\nu_\gamma(x) = m_\gamma(x)\, dx$ whence $\int_N m_\gamma(x)\, dx = 1$ ([D]). Moreover [D],

$$C^{-1}\,(1+|x|)^{-Q-\gamma} \leq m_\gamma(x) \leq C\,(1+|x|)^{-Q-\gamma}\,, \qquad x \in N\,. \tag{1.5}$$

For $\gamma > 0$ the function m_γ is uniquely defined by two conditions

$$\int_N m_\gamma(x)\, dx = 1$$

and

$$P(xa) = a^{-Q}\, \breve{m}_\gamma(\sigma_{a^{-1}}(x)) \text{ is } \mathcal{L}\text{-harmonic}\,.$$

It turns out that the probability measure m_γ is also the basic ingredient in the description of positive harmonic functions for *all* $\gamma \in \mathbb{R}$.

Let

$$Q = \mathrm{Re}\ \mathrm{Tr}\ \mathrm{ad}_{Y_0} \tag{1.6}$$

and

$$P_y(xa) = a^{-Q}\, \breve{m}_\gamma(\sigma_{a^{-1}}\,(y^{-1}x))\,. \tag{1.7}$$

If $\gamma > 0$, the family $\{P_y\}_{y\in N}$ and the function a^γ are all the minimal positive $\mathcal{L}_\gamma$-harmonic functions ([A], cf also [D2]). The proofs (as well as the proof of (1.5)) are based on the Ancona's potential theory on manifolds with negative curvature. Since $\mathcal{L}_{-\gamma} f = a^{-\gamma}\mathcal{L}(a^\gamma f)$, the minimal positive $\mathcal{L}_{-\gamma}$-harmonic functions are 1 and $a^{-\gamma} P_y(xa)$.

The case $\gamma = 0$ is essentially different, because Ancona's theory does not apply. To examine the Martin kernel we have to estimate the Green function $\mathcal{G}_0$ for $\mathcal{L}_0$ in another way. The final description of positive minimal $\mathcal{L}_0$-harmonic functions, however, is very similar to the case $\gamma \neq 0$.

Let μ_t be the semigroup of probability measures with the infinitesimal generator $\mathcal{L}_0$ and let $\mu = \mu_1$. The Markov chain on N with the transition probability

$$P(x, B) = \check{\mu} * \delta_x(B)\,, \qquad x \in N\,,\ B \subset N\,,$$

is a Harris chain with the unique (up to a multiplicative constant) positive Radon measure ν_0 such that $\check{\mu} * \nu_0 = \nu_0$, [E]. ν_0 has a smooth density m_0 which is not integrable in contrast to m_γ, $\gamma > 0$.

The aim of this paper is to show

Theorem. *The minimal positive $\mathcal{L}_0$-harmonic functions normalized at e are*

$$\begin{aligned} &\textit{the constant function } 1 \\ &\textit{and } P_y(xa) = \frac{1}{m_0(y)}\, a^{-Q}\, \check{m}_0(\sigma_{a^{-1}}(y^{-1}x))\,. \end{aligned} \tag{1.8}$$

Moreover, we have

$$C^{-1}(1 + |x|)^{-Q} \le m_0(x) \le C(1 + |x|)^{-Q}\,, \qquad x \in N\,. \tag{1.9}$$

To prove the theorem we proceed in the following way. For $\gamma = -2\,\alpha \le 0$ we define a new operator

$$L_\gamma = a^{-2}\mathcal{L}_\gamma$$

which is *not* left-invariant on S. We study it on the space $N \times \mathbb{R}^+$. However, it has some homogeneity with respect to the family of "dilations" D_r, $r > 0$ on $N \times \mathbb{R}^+$

$$D_r(x, a) = (\sigma_r(x), ra)\,.$$

We have

$$L_\gamma(f \cdot D_r) = r^2 L_\gamma f \cdot D_r\,. \tag{1.10}$$

Also L_γ commutes with the natural action of N on $N \times \mathbb{R}^+$ on the left.

The Green function G_γ for L_γ is given by

$$G_\gamma(x,a;y,b) = \int_0^\infty p_t(x,a;y,b)\,dt\,, \tag{1.11}$$

where

$$T_t f(xa) = \int_{N\times\mathbb{R}^+} f(y,b)\,p_t(x,a;y,b)\,b^{1+2\alpha}\,dy\,db$$

is the heat semigroup on $L^2(a^{2\alpha+1})$ generated by L_γ (see Theorem 5.6). By (1.10)

$$p_{r^2t}(x,a;y,b) = r^{-Q-2\alpha-2}\,p_t(D_{r^{-1}}(x,a);D_{r^{-1}}(y,b)) \tag{1.12}$$

and so

$$G_\gamma(x,a;y,b) = r^{-Q-2\alpha}\,G_\gamma(D_{r^{-1}}(x,a);D_{r^{-1}}(y,b))\,. \tag{1.13}$$

The operator L^*_γ conjugate to

$$L_\gamma = \partial_a^2 + (1-\gamma)\,a^{-1}\,\partial_a + a^{-2}\sum_{j=1}^m \Phi_a(X_j)^2 + a^{-2}\,\Phi_a(X)\,,$$

with respect to the measure $a^{1+2\alpha}\,dx\,da$ is

$$L^*_\gamma = \partial_a^2 + (1-\gamma)\,a^{-1}\,\partial_a + a^{-2}\sum_{j=1}^m \Phi_a(X_j)^2 - a^{-2}\,\Phi_a(X)\,.$$

Clearly,

$$p^*_t(x,a;y,b) = p_t(y,b;x,a)$$

and

$$G^*_\gamma(x,a;y,b) = G_\gamma(y,b;x,a)\,. \tag{1.14}$$

Although the case $\gamma = 0$ is the most interesting for us, we keep the assumption $\gamma \le 0$ to stress that our method works for all those cases. In particular, we obtain new proofs of (1.5) and (1.7). (Again conjugating the operator by a^γ.)

Let $\mathcal{G}_\gamma$ be the Green function for $\mathcal{L}_\gamma$, $\gamma \le 0$. $\mathcal{G}_\gamma$ is uniquely defined by the following two conditions

$$\mathcal{L}_\gamma\,\mathcal{G}_\gamma(\cdot\,;yb) = -\delta_{yb}\,, \qquad \text{as distributions}. \tag{1.15}$$

(Functions are identified with distributions via the right Haar measure $a^{-1}\,da\,dx$.)

$$\text{For every } yb \in S,\ \mathcal{G}_\gamma(\cdot, yb) \text{ is a potential for } \mathcal{L}_\gamma\,. \tag{1.16}$$

It turns out that

$$G_\gamma(x,a;y,b)\,b^{-\gamma} = \mathcal{G}_\gamma(xa;yb)\,. \tag{1.17}$$

Since the notions of potentials for L_γ and $\mathcal{L}_\gamma$ coincide, the only condition to check is (1.15). By Theorem (5.6) we have

$$\int G_\gamma(x,a;y,b)\,L_\gamma^*\phi(x,a)\,a^{2\alpha+1}\,da\,dx = -\phi(y,b)\,.$$

But

$$\begin{aligned}\int G_\gamma(x,a;y,b)\,L_\gamma^*\phi(x,a)\,a^{2\alpha+1}\,da\,dx& \\ &= \int G_\gamma(x,a;y,b)\,a^{2-\gamma}L_\gamma^*\phi(x,a)\,a^{-1}\,da\,dx \\ &= \int G_\gamma(x,a;y,b)\,a^{-\gamma}\mathcal{L}_\gamma^*\phi(x,a)\,a^{-1}\,da\,dx\,,\end{aligned}$$

which shows (1.17).

Using (1.17) we describe the Martin boundary for $\mathcal{L}_0$ (Theorem 6.3). The case $\gamma \neq 0$ was described in [D2]. For that we heavily use (1.13) to find appropriate estimates for Martin kernels.

(1.11) can be extended to $b = 0$ (see Lemma (5.2) and (5.5)) as the limit of $G_\gamma(x,a;y,b_n)$, $b_n \longrightarrow 0$. More precisely,

$$G_\gamma(x,a;y,0) = \lim_{b_n\to 0} G_\gamma(x,a;y,b_n)$$

as Radon measures. Then

$$\breve{m}_\gamma(x) = G_{-\gamma}(x,1;e,0)\,, \qquad \gamma \geq 0\,. \tag{1.18}$$

(1.18) follows from the fact that

$$G_{-\gamma}(x,a;e,0) = a^{-Q-2\alpha}G_{-\gamma}(\sigma_{a^{-1}}(x),1;e,0)$$

is $L_{-\gamma}$-harmonic. Hence $a^{-Q-2\alpha}\,\check{m}_\gamma(\sigma_{a^{-1}}(x))$ is $\mathcal{L}_{-\gamma}$-harmonic, and so $a^{-Q}\,\check{m}_\gamma(\sigma_{a^{-1}}(x))$ is $\mathcal{L}_\gamma$-harmonic. But the last condition implies that for every t

$$\check{\mu}_t * m_\gamma = m_\gamma\,, \qquad \gamma \geq 0\,,$$

which uniquely determines m_γ.

Hence, from estimates on G we conclude estimates for m_γ.

2. Bessel Process.

Let $b_\alpha(t)$ denotes the *Bessel process* with a parameter $\alpha \geq 0$, [RY], *i.e.* a continuous Markov process with state space $[0,+\infty)$ generated by $\Delta = \partial_a^2 + (2\,\alpha + 1/a)\,\partial_a$, $\alpha \geq 0$.

The transition function with respect to the measure $y^{2\alpha+1}\,dy$ is given by ([RY])

$$p_t(x,y) \tag{2.1}$$
$$= \begin{cases} c(\alpha)\,\dfrac{1}{2\,t}\exp\left(\dfrac{-x^2-y^2}{4\,t}\right) I_\alpha\left(\dfrac{x\,y}{2\,t}\right)\dfrac{1}{(x\,y)^\alpha}\,, & \text{for } x,y>0\,, \\ c(\alpha)\,(2\,t)^{-(\alpha+1)}\exp\left(\dfrac{-y^2}{4\,t}\right), & \text{for } x=0,\ y>0, \end{cases}$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+\alpha}}{k!\,\Gamma(k+\alpha+1)}$$

is the *Bessel function* [L]. Therefore, for $x \geq 0$ and $B \subset (0,+\infty)$

$$\mathbf{P}_x(b_\alpha(t)\in B) = \int_B p_t(x,y)y^{2\alpha+1}\,dy\,.$$

The Bessel process appears as the vertical component of the diffusion generated by L_γ, $\gamma = -2\,\alpha$. The aim of this chapter is to recall the basic properties of the process $b_\alpha(t)$. The proofs are rather standard, we sketch them briefly for reader's convenience.

Lemma 2.2. *Let Ω be the space of trajectories of the Bessel process $b_\alpha(t)$. For $b_\alpha \in \Omega$ and $\lambda > 0$ define $\theta_\lambda(b_\alpha)(t) = \sqrt{\lambda}\,b_\alpha(t/\lambda)$. Assume that $b_\alpha(t)$ starts from x. Then:*

i) *for every* $\lambda > 0$, $\widetilde{b}_t = \theta_\lambda(b_\alpha)(t)$ *is the Bessel process (with a parameter* α*) starting from* $\sqrt{\lambda}\, x$,

ii) *for every* $\lambda > 0$, $x \geq 0$,

$$\mathbf{E}_x f \circ \theta_\lambda = \mathbf{E}_{\sqrt{\lambda} x} f \,.$$

The Bessel process b_α on $\mathbb{R}^+$ started at $x > 0$ satisfies the following stochastic differential equation [RY, p. 416],

$$b_\alpha(t) = x + \beta(t) + (2\,\alpha + 1) \int_0^t \frac{1}{b_\alpha(s)}\, ds \,,$$

where $\beta(t)$ is the one-dimensional Brownian motion started at 0. Consequently, we have

$$\mathrm{P}_x[b_\alpha(s) \leq \lambda] \leq \mathrm{P}_0[b_\alpha(s) \leq \lambda] \quad \text{and} \quad \mathrm{P}_x[b(s) \leq \lambda] \leq \mathrm{P}_x[\beta(s) \leq \lambda] \,.$$

Also, by the comparison theorem [RY, p. 364],

$$\alpha \leq \alpha' \text{ then for all } s \geq 0\,,\ b_\alpha(s) \leq b_{\alpha'}(s)\,, \qquad \text{almost everywhere}\,,$$

whence

$$b_\alpha(s) \leq |\beta_n(s)|\,, \qquad \text{where } n = [2\,\alpha] + 3\,,$$

and β_n is the n-dimensional Brownian motion.

Lemma 2.3.

$$\mathrm{P}_a[\max_{0 \leq s \leq t} \beta_\alpha(s) \leq \lambda] \leq e^{-\varepsilon(t/\lambda^2)} \,.$$

Indeed, Let $q = \mathrm{P}_0[\beta_\alpha(1) \leq 1]$. Then $q < 1$ and

$$\begin{aligned}
\mathrm{P}_a[\max_{0 \leq s \leq t} b_\alpha(s) \leq \lambda] &\leq \mathrm{P}_{a/\lambda}[\max_{0 \leq s \leq t/\lambda^2} b_\alpha(s) \leq 1] \\
&\leq \mathrm{E}_0 \prod_{k=0}^{[t/\lambda^2]} \mathrm{P}_{b_\alpha(k)}[b_\alpha(1) \leq 1] \\
&\leq q^{[t/\lambda^2]} \\
&\leq e^{-\varepsilon(t/\lambda^2)} \,.
\end{aligned}$$

Lemma 2.4. *There exist constants c_1, c_2 such that for every $R > 0$ and for every $t > 0$,*

$$\mathbf{P}_R\Big(\inf_{s\in[0,t]} b_\alpha(s) < \frac{R}{2}\Big) \leq c_1 e^{-c_2 R^2/t}\,.$$

Indeed,

$$\mathrm{P}_R\Big[\inf_{s\in[0,t]} b_\alpha(s) < \frac{R}{2}\Big] \leq \ \mathrm{P}_R\Big[\inf_{s\in[0,t]} \beta(s) < \frac{R}{2}\Big] \leq c_1\, e^{-c_2 R^2/t}\,.$$

Lemma 2.5. *There exist constants c_1, c_2 such that for every $x \geq 0$, for every $\lambda > 0$ and for every $t > 0$,*

$$\mathbf{P}_x(\sup_{s\in[0,t]} b_\alpha(s) > x + \lambda) \leq c_1\, e^{-c_2 \lambda^2/t}\,.$$

Indeed, for $n = [2\,\alpha] + 3$

$$\mathbf{P}_x(\sup_{s\in[0,t]} b_\alpha(s) > x + \lambda) \leq \mathbf{P}_x(\sup_{s\in[0,t]} \beta_n(s) > x + \lambda) \leq c_1\, e^{-c_2 \lambda^2/t}\,.$$

Lemma 2.6. *Let $\xi > 0$. There are constants $\delta, c_1, c_2 > 0$ such that for every $a \geq 0$ and $A > 0$,*

$$\mathbf{P}_a\Big(\int_0^1 b_\alpha^\xi(s)\,ds < A\Big) \leq c_1\, e^{-c_2 A^{-\delta}}\,.$$

PROOF. Given positive δ, we have

$$\begin{aligned}\mathbf{P}_a\Big(\int_0^1 b^\xi(s)\,ds < A\Big)& \\ &\leq \mathbf{P}_a\big(\sup_{s\in[0,1]} b_\alpha(s) \leq 2A^\delta\big) \\ &\quad + \mathbf{P}_a\big(\sup_{s\in[0,1]} b_\alpha(s) > 2A^\delta, |\{s:\ b_\alpha(s) > A^\delta\}| < A^{1-\delta\xi}\big)\,.\end{aligned}$$

By Lemma 2.3,

$$\mathbf{P}_a\big(\sup_{s\in[0,1]} b_\alpha(s) \leq 2A^\delta\big) \leq c_1\, e^{-c_2 A^{-\delta}}\,.$$

To estimate the probability of

$$\Omega = \Big\{ \sup_{s\in[0,1]} b_\alpha(s) > 2A^\delta,\ |\{s:\ b_\alpha(s) > A^\delta\}| < A^{1-\delta\xi} \Big\},$$

we define the stopping time $\tau = \inf\{s : b_\alpha(s) = 2A^\delta\}$. Then by Lemma 2.4,

$$\mathbf{P}_a(\Omega) \le \mathbf{E}_a \mathbf{P}_{b_\alpha(\tau)}\Big(\inf_{s\in[0,A^{1-\delta\xi}]} b_\alpha(s) < \frac{b_\alpha(0)}{2} \Big) \le c_1\, e^{-c_2 A^{2\delta-1+\delta\xi}} .$$

We choose δ such that $2\,\delta - 1 + \delta\,\xi < 0$.

Corollary 2.7. *Let* $\xi \ge 0$. *Then*

$$\sup_{a\ge 0} \mathbf{E}_a \Big(\int_0^1 b_\alpha^\xi(s)\, ds \Big)^{-D/2} < +\infty .$$

PROOF. Since by the previous Lemma

$$\mathbf{P}_a\Big(\frac{1}{n+1} \le \int_0^1 b_\alpha^\xi(s)\, ds \le \frac{1}{n} \Big) \le c_1^{-c_2 n^\delta} ,$$

we have

$$\mathbf{E}_a \Big(\int_0^1 b_\alpha^\xi(s)\, ds \Big)^{-D/2} \le \sum_n (n+1)^{D/2} e^{-c_2 n^\delta} < +\infty .$$

3. Solution of a heat equation on the product $N \times \mathbb{R}^+$.

In this chapter we give an analytic proof of the decomposition of the diffusion on $N\times\mathbb{R}^+$ into its components. Using it we find a convenient formula for the solution of the heat equation

$$(L_\gamma - \partial_t)\, u(t,x,a) = 0 .$$

For a multi-index $\beta = (\beta_1,\ldots,\beta_k)$, $\beta_j \in \mathbb{Z}^+$ and a basis $X_1,\ldots,X_n$ of the Lie algebra $\mathcal{N}$ of the Lie group N we write

$$X^\beta = X_1^{\beta_1}\cdots X_n^{\beta_n} .$$

For $k = 0, 1, \ldots, \infty$ we define

$$C^k = \{f :\ X^\beta f \in C(N),\ \text{for } |\beta| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k :\ \lim_{x\to\infty} X^\beta f(x) \text{ exists for } |\beta| < k+1\}\,.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|\beta|\le k} \|X^\beta f\|_{C(N)}\,.$$

Let

$$L_{\sigma(t)} = \sigma(t)^{-2}\Big(\sum(\Phi_{\sigma(t)}(X_j))^2 + \Phi_{\sigma(t)}(X)\Big)\,.$$

For a continuous function $\sigma : [0,+\infty) \longrightarrow [0,+\infty) = A$ let $\{U^\sigma(s,t),\ 0 < s < t\}$ be the (unique) family of bounded operators on $C_\infty = C^0_\infty$ which satisfies

i) $U^\sigma(s,s) = I$,

ii) $U^\sigma(s,r)\,U^\sigma(r,t) = U^\sigma(s,t)$, $s < r < t$,

iii) $\partial_s U^\sigma(s,t)f = -L_{\sigma(s)}U^\sigma(s,t)f$, for every $f \in C_\infty$,

iv) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma(t)}f$ for every $f \in C_\infty$,

v) $U^\sigma(s,t) : C^2_\infty \longrightarrow C^2_\infty$.

$U^\sigma(s,t)$ is a convolution operator $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a probability measure with a smooth density. By ii) we have $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. Existence of $U^\sigma(s,t)$ follows from [T].

Let $d\mathbf{W}_a$ be the probability measure on the space $C([0,+\infty),\mathbb{R}^+)$, for the Bessel process $b_\alpha(t) = b_t$.

For $f \in C^\infty_c(N)$ we define

$$\begin{aligned} u(t,x,a) &= \int U^\sigma(0,t)f(x,\sigma(t))\,d\mathbf{W}_a(\sigma) \\ &= \mathbf{E}_a U^\sigma(0,t)f(x,\sigma(t))\,. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\gamma = -2\,\alpha$ and let $u = u(t,x,a)$ be the function on N defined by* (3.1)*. Then*

$$L_\gamma u(t,x,a) = \partial_t u(t,x,a)\,, \qquad \text{on } \mathbb{R}^+ \times N \times \mathbb{R}^+\,.$$

u is continuous and

$$u(0,x,a) = f(x,a)\,, \qquad \text{when } t \longrightarrow 0\,. \tag{3.2}$$

PROOF. First, we prove that $u = u(t,x,a)$ defined in (3.1) is a solution of the integral equation

$$u(t,x,a) = \mathbf{E}_a f(x,b_t) + \int_0^t \mathbf{E}_a L(b_{t-s})\, u(s,x,b_{t-s})\, ds\,. \tag{3.3}$$

To do this we observe that $\mathbf{E}_a L(b_{t-s})\, u(s,x,b_{t-s})$ is finite. Let Y_1, $\dots, Y_n$ be a fixed basis of $\mathcal{N}$. Then

$$\Phi_a X_j = \alpha_1^j(a)\, Y_1 + \cdots + \alpha_n^j(a)\, Y_n\,,$$

where α_i^j's are continuous functions and $|\alpha_i^j(a)| \le C\,(a^{m_1} + a^{m_2})$. Moreover,

$$Y_k \int f *_N p^\sigma(s,0)(x,\sigma_s)\, d\mathbf{W}_a(\sigma)$$

and

$$Y_k\, Y_l \int f *_N p^\sigma(s,0)(x,\sigma_s)\, d\mathbf{W}_a(\sigma)$$

are bounded for x in a compact set. We have

$$\begin{aligned} L(a)\, u(s,x,a) & \\ &= L(a) \int U^\sigma(0,s)\, f(x,\sigma_s)\, d\mathbf{W}_a(\sigma) \\ &= L(a) \int f *_N p^\sigma(s,0)(x,\sigma_s)\, d\mathbf{W}_a(\sigma) \\ &= a^{-2} \sum_{j,k,l} \alpha_k^j(a)\, \alpha_l^j(a)\, Y_k\, Y_l \int f *_N p^\sigma(s,0)(x,\sigma_s)\, d\mathbf{W}_a(\sigma) \qquad (3.4) \\ &\quad + a^{-2} \sum_{j,k} \alpha_k^j(a)\, Y_k \int f *_N p^\sigma(s,0)(x,\sigma_s)\, d\mathbf{W}_a(\sigma) \end{aligned}$$

and, by the above remarks

$$|L(a)\,u(s,x,a)| \le C\,(a^{m_3} + a^{m_4})\,, \tag{3.5}$$

where

$$m_3 = \min\{m_1, m_2, 2\,m_1, 2\,m_2, m_1 + m_2\} - 2 > 0$$

and

$$m_4 = \max\{m_1, m_2, 2\,m_1, 2\,m_2, m_1 + m_2\} - 2\,.$$

It follows that $\mathbf{E}_a\,L(b_{t-s})\,u(s,x,b_{t-s})$ is finite. Indeed, by (3.4) and (3.5), proceeding as before (*i.e.* replacing a by b_{t-s}) we obtain

$$|\mathbf{E}_a\,L(b_{t-s})\,u(s,x,b_{t-s})| \le C\,\mathbf{E}_a(b_{t-s}^{m_3} + b_{t-s}^{m_4})\,.$$

Now we calculate

$$\begin{aligned}
\mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s}) &\\
&= \int L(b_{t-s}) u(s,x,b_{t-s})\,d\mathbf{W}_a(b)\\
&= \int L(b_{t-s}) \int U^\sigma(0,s)\,f(x,\sigma_s)\,d\mathbf{W}_{b_{t-s}}(\sigma)\,d\mathbf{W}_a(b)\\
&= \iint L(b_{t-s})\,U^\sigma(0,s)\,f(x,\sigma_s)\,d\mathbf{W}_{b_{t-s}}(\sigma)\,d\mathbf{W}_a(b)\\
&= \int L(b_{t-s})\,U^b(t-s,t)\,f(x,b_t)\,d\mathbf{W}_a(b)\,.
\end{aligned}$$

By (3.6), and the Fubini's theorem we obtain

$$\begin{aligned}
\int_0^t \mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s})\,ds &\\
&= \iint_0^t L(b_{t-s}) U^b(t-s,t)\,f(x,b_t)\,ds\,d\mathbf{W}_a(b)\,,
\end{aligned}$$

but

$$\int_0^t L(b_{t-s})\,U^b(t-s,t)\,f(x,b_t)\,ds = U^b(0,t)\,f(x,b_t) - f(x,b_t)\,.$$

Indeed by iii) we get

$$\begin{aligned}\frac{d}{ds}\,U^b(t-s,t)\,f(x,b_t) &= -\frac{d}{ds}\,U^b(\cdot,t)\,f(x,b_t)\Big|_{t-s}\\ &= -(-L(b_{t-s})\,U^b(t-s,t)\,f(x,b_t))\\ &= L(b_{t-s})\,U^b(t-s,t)\,f(x,b_t)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^t \mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s})\,ds\\ &\qquad = \int U^b(0,t)\,f(x,b_t)\,d\mathbf{W}_a(b) - \int f(x,b_t)\,d\mathbf{W}_a(b)\\ &\qquad = u(t,x,a) - \mathbf{E}_a f(x,b_t)\,.\end{aligned}$$

Now we are going to prove that u is a solution of the differential equation (3.2). Since u is a solution of (3.3) we have

$$\begin{aligned}&\frac{u(t+h,x,a)-u(t,x,a)}{h}\\ &= \frac{\mathbf{E}_a f(x,b_{t+h}) - \mathbf{E}_a f(x,b_t)}{h} + \frac{1}{h}\int_0^t (\mathbf{E}_a L(b_{t+h-s})\,u(s,x,b_{t+h-s})\\ &\qquad\qquad - \mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s}))\,ds\\ &\quad + \frac{1}{h}\int_t^{t+h} \mathbf{E}_a L(b_{t+h-s})\,u(s,x,b_{t+h-s})\,ds\,.\end{aligned}$$

Let Δ be the infinitesimal generator of the Bessel process *i.e.*

$$\Delta = \partial_a^2 + \frac{2\,\alpha+1}{a}\,\partial_a\,.$$

Letting h to 0 we get

$$\begin{aligned}&\partial_t u(t,x,a)\\ &\quad = \Delta\mathbf{E}_a f(x,b_t) + \Delta\int_0^t \mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s})\,ds + L(a)\,u(t,x,a)\end{aligned}$$

in a sense of distributions.

On the other hand, since u is a solution of (3.3) thus

$$\begin{aligned} Lu(t,x,a) & \\ &= (L(a)+\Delta)\,u(t,x,a) \\ &= L(a)u(t,x,a) + \Delta\Big(\mathbf{E}_a f(x,b_t) + \int_0^t \mathbf{E}_a L(b_{t-s})\,u(s,x,b_{t-s})\,ds\Big) \\ &= L(a)\,u(t,x,a) + \Delta\mathbf{E}_a f(x,b_t) + \Delta\int_0^t \mathbf{E}_a L(b_{t-s})u(s,x,b_{t-s})\,ds\,. \end{aligned}$$

So u is a solution of (3.2).

Theorem 3.2. *Let*

$$T_t f(x,a) = \int U^\sigma(0,t)\,f(x,\sigma_t)\,d\mathbf{W}_a(\sigma)\,.$$

Then $\{T_t\}$ is a semigroup.

PROOF.

$$\begin{aligned} T_s(T_t f)(x,a) &= \int U^b(0,s) T_t f(x,b_s)\,d\mathbf{W}_a(b) \\ &= \int U^b(0,s)\int U^\sigma(0,t)\,f(x,\sigma_t)\,d\mathbf{W}_{b_s}(\sigma)\,d\mathbf{W}_a(b) \\ &= \int U^b(0,s)\,U^b(s,s+t)\,f(x,b_{s+t})\,d\mathbf{W}_a(b) \\ &= \int U^b(0,s+t)\,f(x,b_{s+t})\,d\mathbf{W}_a(b) \\ &= T_{s+t}f(x,a)\,, \end{aligned}$$

where in the third equality we have used the Markov property.

4. Estimate of the evolution kernels by the Nash inequality.

Let $X, X_1, \ldots, X_m$ be as in (1.2),

$$L_a = a^{-2}\Big(\sum_{j=1}^m (\Phi_a X_j)^2 + \Phi_a(X)\Big)\,,$$

$$\Delta_0 = \sum_{j=1}^m X_j^2\,,$$

and

$$\Delta = \Delta_0 + X\,.$$

Let $\sigma : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous function such that $\sigma(t) > 0$ for $t > 0$, and $p^\sigma(t, s, x) = p^\sigma(t, s)(x)$, $s < t$ be the evolution generated by the operator $L_{\sigma(t)} + \partial_t$.

The aim of this Chapter is to prove the following estimate for $p^\sigma(t, 0, x)$:

Theorem 4.1. *For every compact set $K \subset N$, which does not contain the identity element e of N, there exist positive constants C_1, C_2, m_3, m_4 and $n \le Q$ such that for every $x \in K$ and for every t,*

$$p^\sigma(t, 0, x) \le C_1 \Big(\int_0^t \sigma^{-2(1-Q/n)}(u)\, du \Big)^{-n/2} \exp\Big(- \frac{C_2}{A(0, t)} \Big)\,,$$

where

$$A(s, t) = \int_s^t (\sigma^{m_3}(u) + \sigma^{m_4}(u))\, du\,.$$

The main tool in the proof of the above theorem is the Nash inequality (see *e.g.* [VSC])

$$\|f\|_{L^2}^{2+4/n} \le -C\,(\Delta f, f)\,\|f\|_{L^1}^{4/n} = (\Delta_0 f, f)\,\|f\|_{L^1}^{4/n}\,, \tag{4.2}$$

for all $f \in C_0^\infty(N)$, where d is the local dimension of $(N, X_1, \ldots, X_m)$ and D is the dimension at infinity of $(N, X_1, \ldots, X_m)$ n is any number satisfying $d \le n \le D$(see [VSC]). Let Q_t be the heat semi-group generated by Δ_0. Then

$$\|Q_t\|_{L^1 \to L^\infty} \le C \begin{cases} t^{-d/2}\,, & \text{if } t \le 1\,, \\ t^{-D/2}\,, & \text{if } t \ge 1\,, \end{cases}$$

(Theorem IV.4.1 in [VSC]) and so (4.1) follows by the Nash theorem (Theorem II.5.2 in [VSC]). Since we can make Q arbitrarily big (see 1.6), $\xi = -2\,(1 - Q/n)$ is positive.

Proof of Theorem 4.1. We start with some integral estimates on $f * p^\sigma(t, s)$.

Let $0 \le \varphi \in C_c^\infty(N)$, $\operatorname{supp}\varphi \subset B_r(e)$ and $\int \varphi = 1$ (r will be fixed later). Let $\eta(x) = \tau * \varphi(x)$ where τ is a left invariant Riemannian metric on N. There exists a positive constant C such that if $Y_1, \dots, Y_n$ is a fixed basis of $\mathcal{N}$ then

$$|Y_j\, \eta(x)| \le C\,, \qquad |Y_i\, Y_j\, \eta(x)| \le C\,, \qquad \text{for } i,j = 1, \dots, n \tag{4.3}$$

[H]. Moreover,

$$\tau(x) \le \int (\tau(x\, y^{-1}) + \tau(y))\, \varphi(y)\, dy \le \eta(x) + r\,, \tag{4.4}$$

and

$$\eta(e) = \int \tau(y^{-1})\, \varphi(y)\, dy \le r\,. \tag{4.5}$$

For a natural number m let $\eta_m(x) = \tau_m * \varphi(x)$, where

$$\tau_m(x) = \min\{m, \tau(x)\}\,.$$

Then there exists a positive constant C such that for every m, (4.3), (4.4) and (4.5) hold with η_m and τ_m instead of η and τ respectively.

We have

$$(\partial_s(f * p^\sigma(t,s), e^{\alpha \eta_m}) = -(f * p^\sigma(t,s), L^*_{\sigma(s)} e^{\alpha \eta_m})) \tag{4.6}$$

(4.6) is obvious, if instead of $e^{\alpha\eta_m}$ we put $e^{\alpha\eta_m}\psi$, where $\psi \in C_0^\infty(N)$. So to conclude (4.6) we take the sequence $\psi_j = \psi \circ \sigma_{a_j}$ for $\psi \in C_0^\infty(N)$ such that $\psi(0) = 1$ and $a_j \longrightarrow 0$. Since $\sigma_{a_j}(x) \longrightarrow e$ for every $x \in N$ and, by (1.3), $|\Phi_{a_j}(X_j)\, \psi| \longrightarrow 0$, we obtain (4.6) as the limit of

$$\partial_s(f * p^\sigma(t,s), e^{\alpha\eta_m} \psi_j) = -(f * p^\sigma(t,s), L^*_{\sigma(s)}(e^{\alpha\eta_m}\psi_j))\,.$$

Therefore, by (1.2) and (4.3),

$$\begin{aligned} &\partial_s(f * p^\sigma(t,s), e^{\alpha\eta_m}) \\ &\qquad \le C\,(\alpha + \alpha^2)\, \sigma^{-2}(s)\, (\sigma^{m_1}(s) + \sigma^{m_2}(s))^2\, (f * p^\sigma(t,s), e^{\alpha\eta_m}) \\ &\qquad\quad + C\, \alpha\, \sigma^{-2}(s)\, (\sigma^{m_1}(s) + \sigma^{m_2}(s))\, (f * p^\sigma(t,s), e^{\alpha\eta_m})\,. \end{aligned}$$

Thus

$$\frac{\partial_s(f * p^\sigma(t,s), e^{\alpha\eta_m})}{(f * p^\sigma(t,s), e^{\alpha\eta_m})} \le C\,(\alpha + \alpha^2)\, (\sigma^{m_3}(s) + \sigma^{m_4}(s))\,,$$

and so

$$(f * p^\sigma(t,s), e^{\alpha\eta_m}) \le (f, e^{\alpha\eta_m}) \exp\left(C\,(\alpha+\alpha^2)A(s,t)\right),$$

where

$$A(s,t) = \int_s^t (\sigma^{m_3}(u) + \sigma^{m_4}(u))\,.$$

Therefore,

$$\begin{aligned}(p^\sigma(t,s), e^{\alpha\eta_m}) &\le e^{\alpha\eta_m(e)} \exp\left(C(\alpha+\alpha^2)A(s,t)\right)\\ &\le e^{\alpha r} \exp\left(C\,(\alpha+\alpha^2)A(s,t)\right).\end{aligned}$$

Now for $m \longrightarrow \infty$ (4.4) and (4.5) yield

$$\begin{aligned}(p^\sigma(t,s), e^{\alpha\tau}) &\le (p^\sigma(t,s), e^{\alpha(\eta+r)})\\ &\le e^{2\alpha r} \exp\left(C\,(\alpha+\alpha^2)A(s,t)\right).\end{aligned} \tag{4.7}$$

The next step is the Nash inequality for L_a. Applying (4.2) to $f\circ\sigma_a$ we obtain

$$\begin{aligned}a^{-Q(1+2/n)}\|f\|_{L^2}^{2(1+2/n)} &\le -C\,a^{-Q}(a^2L_af, f)\,a^{-4Q/n}\,\|f\|_{L^1}^{4/n}\\ &= -C\,a^{-Q+2-4Q/n}(L_af,f)\,\|f\|_{L^1}^{4/n}\,.\end{aligned}$$

Thus

$$\|f\|_{L^2}^{2(1+2/n)} \le -C\,a^{2(1-Q/n)}(L_af,f)\,\|f\|_{L^1}^{4/n}\,. \tag{4.8}$$

Now we proceed similarly as in the case of semigroups (*e.g.* [VSC]).

For a function $0 \le f \in C_c^\infty(N)$ such that $\int f = 1$ we define

$$f_s(x) = f * p^\sigma(t,s)(x)\,, \qquad h_s(x) = \|f_s\|_{L^2}^2\,.$$

Then

$$\begin{aligned}-\partial_s h_s &= -\partial_s(f_s, f_s)\\ &= 2\,(L_{\sigma(s)}f_s, f_s)\\ &\le -2\,C^{-1}\sigma^{-2(1-Q/n)}(s)\,\|f_s\|_{L^2}^{2(1+2/n)}\\ &= -C\,\sigma^{-2(1-Q/n)}(s)\,h_s^{1+2/n}\,.\end{aligned}$$

(By (4.7) we may exchange ∂_s with the integral.) So

$$-\partial_s h_s\, h_s^{-1-2/n} \leq -C\,\sigma^{-2(1-Q/n)}(s)\,.$$

Hence

$$-\int_s^t \partial_u h_u\, h_u^{-1-2/n}\, du = \frac{n}{2}\, h_u^{-2/n}\Big|_{u=s}^{u=t} \leq -C\int_s^t \sigma^{-2(1-Q/n)}(u)\, du\,.$$

Thus

$$\frac{n}{2}\,(h_t^{-2/n} - h_s^{-2/n}) \leq -C\int_s^t \sigma^{-2(1-Q/n)}(u)\, du\,.$$

Since $h_t^{-2/n} > 0$,

$$-\frac{n}{2}\, h_s^{-2/n} \leq -C\int_s^t \sigma^{-2(1-Q/n)}(u)\, du$$

and so

$$\|f * p^\sigma(t,s)\|_{L^2} = h_s^{1/2} \leq C\Big(\int_s^t \sigma^{-2(1-Q/n)}(u)\, du\Big)^{-n/2}\|f\|_{L^1}\,.$$

Therefore,

$$\|p^\sigma(t,s)\|_{L^2} \leq C\Big(\int_s^t \sigma^{-2(1-Q/n)}(u)\, du\Big)^{-n/4}$$

$$\begin{aligned}\|p^\sigma(t,s)\|_{L^\infty} &\leq \|p^\sigma(t,u)\|_{L^2}\|p^\sigma(u,s)\|_{L^2}\\ &\leq C\Big(\int_\xi^t \sigma^{-2(1-Q/n)}(u)\, du\Big)^{-n/4}\\ &\quad\cdot\Big(\int_s^\xi \sigma^{-2(1-Q/n)}(u)\, du\Big)^{-n/4}.\end{aligned}\tag{4.9}$$

Taking ξ such that

$$\begin{aligned}\int_s^\xi \sigma^{-2(1-Q/n)}(u)\, du &= \int_\xi^t \sigma^{-2(1-Q/n)}(u)\, du\\ &= \frac{1}{2}\int_s^t \sigma^{-2(1-Q/n)}(u)\, du\end{aligned}\tag{4.10}$$

we obtain

$$\|p^\sigma(t,s)\|_{L^\infty} \le C\Big(\int_s^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/2}.$$

By the subadditivity of the metric τ, estimates (4.7) and (4.9) we have

$$\begin{aligned}
&p^\sigma(t,0,x)\,e^{\alpha\tau(x)}\\
&\quad\le \int p^\sigma(t,s,x)\,p^\sigma(s,0,x\,y^{-1})\,e^{\alpha\tau(y)}\,e^{\alpha\tau(xy^{-1})}\,dy\\
&\quad\le \|p^\sigma(t,s)\|_{L^\infty}^{1/2}\,\|p^\sigma(s,0)\|_{L^\infty}^{1/2}\,(p^\sigma(t,s),e^{2\alpha\tau})^{1/2}\,(p^\sigma(s,0),e^{2\alpha\tau})^{1/2}\\
&\quad\le C\Big(\int_s^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/4}\Big(\int_0^s \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/4}\\
&\qquad\cdot e^{4\alpha r}\exp\left(C\,(\alpha+\alpha^2)A(s,t)\right)\exp\left(C\,(\alpha+\alpha^2)A(0,s)\right)\\
&\quad= C\Big(\int_s^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/4}\Big(\int_0^s \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/4}\\
&\qquad\cdot e^{4\alpha r}\exp\left(C\,(\alpha+\alpha^2)A(0,t)\right).
\end{aligned}$$

Now for the s such that in the last product the first two factors are equal we obtain

$$\begin{aligned}
&p^\sigma(t,0,x)\,e^{\alpha\tau(x)}\\
&\qquad\le C\Big(\int_0^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/2} e^{4\alpha r}\exp\left(C\,(\alpha+\alpha^2)A(0,t)\right).
\end{aligned}$$

If $\alpha = \varepsilon\,\tau(x)/A(0,t)$, then

$$\begin{aligned}
p^\sigma(t,0,x) &\le C\Big(\int_0^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/2}\\
&\qquad\cdot\exp\Big(\frac{4\,\varepsilon\,r\,\tau(x)}{A(0,t)} + C\,\varepsilon\,\tau(x) + \frac{C\,\varepsilon^2\,\tau^2(x)}{A(0,t)} - \frac{\varepsilon\,\tau^2(x)}{A(0,t)}\Big).
\end{aligned}$$

Now our assumptions on K imply that we may neglect $C\,\varepsilon\,\tau(x)$ and we can find r such that $r < \tau(x)/16$, $x \in K$. Moreover, we assume that $C\,\varepsilon < 1/4$. Then

$$p^\sigma(t,0,x) \le C\Big(\int_0^t \sigma^{-2(1-Q/n)}(u)\,du\Big)^{-n/2}\exp\Big(\frac{-\varepsilon\,\tau^2(x)}{2A(0,t)}\Big)$$

and the proof is completed.

Theorem 4.11. *Assume that*

$$\lambda \leq \sigma(s) \leq \Lambda\,, \qquad \textit{for } s \in [r, r+T]\,. \tag{4.12}$$

Given $0 < T_1 < T_2 < T$ *and a neighborhood* B *of* e, *we can find* $C > 0$ *independent on* r *such that*

$$p^{\sigma}(r, r+t) \geq C\,, \qquad \textit{for } z \in B\,,\ 0 < T_1 \leq t \leq T_2 < T\,, \tag{4.13}$$

and any σ *satisfying* (4.12).

Proof. Although we have an evolution here, not a semigroup, the proof of (4.12) is the same ([SS, p. 106-108]). It is based on the Poincaré inequality and upper bound estimates we have just proved. Let ρ_a be the optimal control metric defined by the vector fields $a^{-2}\,\Phi_a(X_1), \ldots,$ $a^{-2}\,\Phi_a(X_m)$ and let $B_{r,a} = \{x \in N :\ \rho_a(x) < r\}$. Then

$$\begin{aligned}\min_{z\in\mathbb{R}} \int_{B_{r,a}} |f(x) - z|^2\,dx &\leq \int_{B_{r,a}} |f(x) - f_{r,a}|^2\,dx \\ &\leq C\,r^2 \int_{B_{(3/2)r,a}} |\nabla f(x)|^2\,dx\,,\end{aligned} \tag{4.14}$$

where,

$$f_{r,a} = \frac{1}{|B_{r,a}|} \int_{B_{r,a}} f(y)\,dy \qquad \text{and} \qquad |\nabla f|^2 = \sum_{j=1}^{m} (X_j)^2\,.$$

The constant C does not depend on a, r. (4.14) implies

$$\begin{aligned}\min_{z\in\mathbb{R}} \int |f(x) - z|^2\,\Psi_{a,r}(x)\,dx &= \int |f(x) - f_{\Psi_{r,a}}|^2\,\Psi_{a,r}(x)\,dx \\ &\leq C\,r^2 \int |\nabla f(x)|^2\,\Psi_{a,2r}(x)\,dx\,,\end{aligned} \tag{4.15}$$

where

$$f_{\Psi_{a,r}} = \frac{\displaystyle\int f(y)\,\Psi_{a,r}(y)\,dy}{\displaystyle\int \Psi_{a,r}(y)\,dy}$$

and

$$\Psi_{a,r}(x) = \begin{cases} \Big(\dfrac{1-\rho_a(x)}{r}\Big)^2, & \text{if } \rho_a(x) < r\,, \\ 0\,, & \text{if } \rho_a(x) \ge r\,, \end{cases}$$

and c does not depend on a. Having (4.15) we follow the argument on [SS, p. 106-108].

5. Green function for L_γ.

Let

$$T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t)\, f(x,\sigma_t)$$

be the semigroup of operators generated by L_γ. Since

$$|\mathbf{E}_a U^\sigma(0,t) f(x,\sigma_t)| \le \|f\|_{L^\infty} \text{ and } \mathbf{E}_a U^\sigma(0,t) f(x,\sigma_t) \ge 0 \text{ for } f \ge 0\,,$$

for every $x \in N$, $a \ge 0$, $t > 0$, there exists a probability measure $p_t(x,a;\cdot,\cdot)$ such that

$$T_t f(x,a) = \int_{N\times\mathbb{R}^+} f(y,b)\, p_t(x,a;dy,db)\,.$$

Moreover, $p_t(x,a;\cdot,\cdot) \in L^2(N\times\mathbb{R}^+, dx \otimes a^{2\alpha+1}\,da)$. Indeed,

$$|U^\sigma(0,t) f(x,\sigma(t))| \le \|p^\sigma(t,0)\|_{L^2(\,dx)} \Big(\int |f(x,\sigma(t))|^2\,dx\Big)^{1/2}.$$

Therefore,

$$\begin{aligned} |T_t f(x,a)| &\le (\mathbf{E}_a \|p^\sigma(t,0)\|^2_{L^2(dx)})^{1/2} \Big(\mathbf{E}_a \int |f(x,\sigma(t))|^2\,dx\Big)^{1/2} \\ &\le c(a,t)\,(\mathbf{E}_a \|p^\sigma(t,0)\|^2_{L^2(\,dx)})^{1/2} \|f\|_{L^2(dx\otimes a^{2\alpha+1}da)} \end{aligned}$$

because for a fixed t the kernel (2.1) is bounded as a function of space variable. By (4.9), Lemma 2.2 and Corollary 2.15, $\mathbf{E}_a\|p^\sigma(t,0)\|^2_{L^2(\,dx)} < \infty$ and so, for every t, x, a,

$$p_t(x,a;\cdot,\cdot) \in L^2(N\times\mathbf{R}^+, dx\otimes da^{2\alpha+1}\,da)\,.$$

Now a standard argument shows that for fixed $x \in N$, $a > 0$,

$$(5.1) \qquad (L^* - \partial_t)\, p_\cdot(x,a;\cdot,\cdot) = 0\,.$$

We want to have (5.1) also for $a = 0$.

Lemma 5.2. *Given* $f \in C_c^\infty(N \times \mathbb{R}^+ \times \mathbb{R}^+)$, *we have*

$$\begin{aligned}\lim_{a\to 0}\int p_t(x,a;y,b)\, f(y,b,t)\, dy b^{2\alpha+1}\, db\, dt \\ = \int p_t(x,0;y,b)\, f(y,b,t)\, dy\, b^{2\alpha+1}\, db\, dt\,.\end{aligned}\tag{5.3}$$

PROOF. We rewrite (5.3) as

$$\lim_{a\to 0}\mathbf{E}_a U^\sigma(0,t)\, f(x,\sigma(t),t) = \mathbf{E}_0 U^\sigma(0,t)\, f(x,\sigma(t),t)\,.$$

Since the trajectories are continuous, it is enough to show that $U^\sigma(0,t)\, f(x,\sigma(t),t)$ is a continuous function of the trajectory σ. For an arbitrary fixed $T > 0$ let

$$d(\sigma,\sigma') = \sup_{t\in[0,T]} |\sigma(t) - \sigma'(t)|\,.$$

We have

$$\begin{aligned}U^\sigma(s,t)\, f(x,\sigma(t),t) - U^{\sigma'}(s,t)\, f(x,\sigma(t),t) \\ = U^\sigma(s,t)\, f(x,\sigma(t),t) - U^\sigma(s,t)\, f(x,\sigma'(t),t) \\ + U^\sigma(s,t)\, f(x,\sigma'(t),t) - U^{\sigma'}(s,t)\, f(x,\sigma'(t),t)\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}|U^\sigma(s,t)\, f(x,\sigma(t),t) - U^\sigma(s,t)\, f(x,\sigma'(t),t)| \\ \le \sup_{x,t} |f(x,\sigma(t),t) - f(x,\sigma'(t),t)|\,,\end{aligned}$$

which clearly tends to 0 if $d(\sigma,\sigma') \longrightarrow 0$. The second term in (5.4) can be written as

$$\begin{aligned}U^\sigma(s,t)\, f(x,\sigma'(t),t) - U^{\sigma'}(s,t)\, f(x,\sigma'(t),t) \\ = \int_s^t U^\sigma(s,r)\,(L(\sigma_r) - L(\sigma'_r))\, U^{\sigma'}(r,t)\, f(x,\sigma'(t),t)\, dr\,.\end{aligned}$$

It also tends to 0, because for $\xi \ge 0$

$$\lim_{\sigma'\to\sigma}\int_0^t |\sigma_r^\xi - \sigma_r'^\xi| = 0\,,$$

which completes the proof of Lemma 5.2.

Now we are ready to study the Green function for L_γ in greater detail. Let

$$G_\gamma(x,a;y,b)=\int_0^\infty p_t(x,a;y,b)\,dt\,. \tag{5.5}$$

The previous lemma, applied both to L_γ and L_γ^*, says that $p_t(x,a;y,b)$ is well defined also for $a\ge 0$, $b>0$ or for $a>0$, $b\ge 0$. Therefore $G_\gamma(x,a;y,b)$ is defined for arbitrary x,y in N and $a^2+b^2>0$.

Theorem 5.6. *G_γ is the Green function for L_γ. More precisely,*

$$G_\gamma(\cdot,\cdot;y,b)\in L^1_{\mathrm{loc}}(N\times\mathbb{R}^+)\,, \tag{5.7}$$

$$L_\gamma G_\gamma(\cdot,\cdot;y,b)=-\delta_{(y,b)}\,, \tag{5.8}$$

$$G_\gamma(\cdot,\cdot;y,b) \textit{ is a } L_\gamma\textit{-potential}, \tag{5.9}$$

and

$$G_\gamma(x,a;\cdot,\cdot)\in L^1_{\mathrm{loc}}(N\times\mathbb{R}^+)\,, \tag{5.10}$$

$$L_\gamma^* G_\gamma(x,a;\cdot,\cdot)=-\delta_{(x,a)}\,, \tag{5.11}$$

$$G_\gamma(x,a;\cdot,\cdot) \textit{ is a } L_\gamma^*\textit{-potential}. \tag{5.12}$$

In particular,

$$L_\gamma^* G_\gamma(x,0;\cdot,\cdot)=0 \textit{ on } N\times\mathbb{R}^+\,, \tag{5.13}$$

$$L_\gamma G_\gamma(\cdot,\cdot;y,0)=0 \textit{ on } N\times\mathbb{R}^+\,. \tag{5.14}$$

Finally, given $\varepsilon>0$, there exists $C>0$ such that

$$C^{-1}\le G_\gamma(x,a;y,b)\le C\,, \tag{5.15}$$

whenever $|x|<\varepsilon$, $0\le a<\varepsilon$, $|y|=1$, $b\le 1$ or $|y|<\varepsilon$, $0\le b<\varepsilon$, $|x|=1$, $a\le 1$, respectively.

PROOF. Since the heat semigroup $p_t^*(x,a;y,b)$ corresponding to L_γ^* is given by $p_t^*(x,a;y,b) = p_t(y,b;x,a)$ it is enough to prove (5.10)-(5.12). First we notice that

$$\int_0^\infty T_t\phi(x,a)\,dt < \infty\,, \qquad \text{for } \phi \in C_0^\infty(N\times\mathbb{R}^+)\,.$$

Indeed, if $t < 1$ then $|T_t\phi(x,a)| \le \|\phi\|_{L^\infty}$ and the beginning of the proof of Lemma 5.1 shows that

$$\int_1^\infty T_t\phi(x,a)\,dt < \infty\,.$$

To prove (5.11) we write

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_N L_\gamma^* G_\gamma(x,a;y,b)\,\phi(y,b)\,dy\,b^{2\alpha+1}\,db \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_N p_t(x,a;y,b)\,L_\gamma\phi(y,b)\,dy\,b^{2\alpha+1}\,db\,dt \qquad (5.16)\\
&= \lim_{\substack{t_1\to 0\\ t_2\to\infty}} \int_{t_1}^{t_2}\int_{\mathbb{R}^+}\int_N p_t(x,a;y,b)\,L_\gamma\phi(y,b)\,dy\,b^{2\alpha+1}\,db\,dt\,,
\end{aligned}$$

because (5.16) is absolutely convergent. But

$$\int_{\mathbb{R}^+}\int_N p_t(x,a;y,b)\,L_\gamma\phi(y,b)\,dy\,b^{2\alpha+1}\,db = \partial_t T_t\phi(x,a)\,. \qquad (5.17)$$

Moreover,

$$\lim_{t_1\to 0} T_{t_1}\phi(x,a) = -\phi(x,a)$$

and by (4.9), Corollary 2.7, Lemma 2.2

$$|T_{t_2}\phi(x,a)| \le C\,\mathbf{E}_a\Big(\int_0^{t_2} b^{\xi}(s)\,ds\Big)^{-D/2}\,,$$

which tends to 0, when $t_2\to\infty$. This proves (5.11) and (5.13). To show that $G_\gamma(x,a;\cdot,\cdot)$ is L_γ^*-potential we consider an L_γ^*-harmonic function h satisfying

$$0 \le h(y,b) \le G_\gamma(x,a;y,b)$$

and apply T_r^* to it. Then, on one hand side

$$T_r^* h(z,c) = h(z,c)\,,$$

and on the other,

$$T_r^* h(z,c) \le \int_0^\infty p_{t+r}(x,a;z,c)\,dt \longrightarrow 0\,, \qquad \text{for } (z,c) \ne (x,a)\,.$$

Hence $h = 0$. (5.15) is a direct consequence of the next Lemma.

Lemma 5.18. *Given $\xi > 0$, $\alpha \ge 0$, $D > 0$, $a_1 > 0$, there is C such that if $a \le a_1$, $0 < b < 1$, $0 < \eta < 1$, then*

$$\int_0^\infty \mathbf{E}_a\Big(\int_0^t b_\alpha^\xi(s)\,ds\Big)^{-D/2} e^{-c/A(0,t)} \cdot \mu([b-\eta,b+\eta])^{-1}\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(t)\in[b-\eta,b+\eta]\}}\,dt < C\,,$$

where $A(0,t)$ is defined in Theorem 4.1 and $\mu(A) = \int_A r^{2\alpha+1}\,dr$.

PROOF. Assume first that $t \ge 1$. Then, by the Markov property, it is enough to estimate

$$\int_1^\infty \mathbf{E}_a\Big(\int_0^{t/2} b_\alpha^\xi(s)\,ds\Big)^{-D/2} \cdot \mu([b-\eta,b+\eta])^{-1}\,\mathbf{E}_{b_\alpha(t/2)}\,\mathbf{1}_{\{\sigma_\alpha\,:\,\sigma_\alpha(t/2)\in[b-\eta,b+\eta]\}}(\sigma_\alpha)\,. \tag{5.19}$$

But by (2.1) and Lemma 2.3

$$\mathbf{E}_{b_\alpha(t/2)}\,\mathbf{1}_{\{\sigma_\alpha\,:\,\sigma_\alpha(t/2)\in[b-\eta,b+\eta]\}}(\sigma_\alpha) \le C\,t^{-1-\alpha}\,\mu([b-\eta,b+\eta])\,.$$

On the other hand by Lemma 2.2

$$\mathbf{E}_a\Big(\int_0^{t/2} b_\alpha^\xi(s)\,ds\Big)^{-D/2} = 2^{(1+\xi/2)D/2}\,t^{-(1+\xi/2)D/2}\,\mathbf{E}_{a/\sqrt{t}}\Big(\int_0^1 b_\alpha^\xi(s)\,ds\Big)^{-D/2}.$$

Now, Corollary 2.7 implies that (5.19) is dominated by a constant for every a, b, η.

Let $t<1$. First we notice that for every $M, c>0$ there is C such that $e^{-c/x}\le C\,x^M$ for every $x>0$. Therefore, it suffices to estimate

$$\int_0^1 \mathbf{E}_a\Big(\int_0^t b_\alpha^\xi(s)\,ds\Big)^{-D/2} A(0,t)\,\mu([b-\eta,b+\eta])^{-1}\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(t)\in[b-\eta,b+\eta]\}}\ ,$$

where

$$A(0,t)=\int_0^t (b_\alpha^{m_3}(s)+b_\alpha^{m_4}(s))\,ds\,,$$

Since

$$A(0,t)^M\le C\Big(\Big(\int_0^t b_\alpha^{m_3}(s)\,ds\Big)^M+\Big(\int_0^t b_\alpha^{m_4}(s)\,ds\Big)^M\Big),$$

we are left with

$$\begin{aligned} I=\int_0^1 \mathbf{E}_a\Big(\int_0^t b_\alpha^\xi(s)\,ds\Big)^{-D/2}\Big(\int_0^t b_\alpha^{m_j}(s)\,ds\Big)^M&\\ \cdot\,\mu([b-\eta,b+\eta])^{-1}\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)\,,&\qquad \xi, m_j>0\,,\end{aligned}$$

and so, in view of the Schwartz inequality, we are to estimate

$$I_1=\int_0^1 \mathbf{E}_a\Big(\int_0^t b_\alpha^\xi(s)\,ds\Big)^{-D}\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)\,,$$

and

$$I_2=\int_0^1 \mathbf{E}_a\Big(\int_0^t b_\alpha^{m_j}(s)\,ds\Big)^{2M}\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)\,.$$

By Lemma 2.2 and Corollary (2.15),

$$\begin{aligned} I_1&=t^{-(1+\xi/2)D}\,\mathbf{E}_{a/\sqrt{t}}\Big(\int_0^1 b_\alpha^\xi(s)\,ds\Big)^{-D}\\ &\qquad\cdot\,\mathbf{1}_{\{b_\alpha\,:\,b_\alpha(1)\in[(b-\eta)/\sqrt{t},(b+\eta)/\sqrt{t}]\}}(b_\alpha)\\ &\le t^{-(1+\xi/2)D}\,\mathbf{E}_{a/\sqrt{t}}\Big(\int_0^{1/2} b_\alpha^\xi(s)\,ds\Big)^{-D}\\ &\qquad\cdot\,\mathbf{E}_{b_\alpha(1/2)}\,\mathbf{1}_{\{\sigma_\alpha\,:\,\sigma_\alpha(1/2)\in[(b-\eta)/\sqrt{t},(b+\eta)/\sqrt{t}]\}}(\sigma_\alpha)\\ &\le C\,t^{(1+\xi/2)D-1-\alpha}\,\mu([b-\eta,b+\eta])\,.\end{aligned}$$

Let $\Omega_{-1} = \{b_\alpha :\ \sup_{s\in[0,1]} b_\alpha(s) \le a_1\}$ and

$$\Omega_m = \Big\{b_\alpha :\ a_1 + m < \sup_{s\in[0,1]} b_\alpha(s) \le a_1 + m + 1\Big\}, \qquad m = 0, 1, 2, \ldots$$

Then

$$I_2 = \sum_{m=-1}^{\infty} \mathbf{E}_a\Big(\int_0^t b_\alpha^{m_j}(s)\,ds\Big)^{2M}\, \mathbf{1}_{\Omega_m}(b_\alpha)\,\mathbf{1}_{\{b_\alpha : b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)\,.$$

We treat the cases $m = -1, 0, 1$ and $m \ge 2$ separately. For $m = -1, 0, 1$ we have

$$\mathbf{E}_a\Big(\int_0^t b_\alpha^{m_j}(s)\,ds\Big)^{2M}\, \mathbf{1}_{\Omega_{-1}\cup\Omega_0\cup\Omega_1}(b_\alpha)\,\mathbf{1}_{\{b_\alpha : b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)$$
$$\le C\,t^{2M-1-\alpha}\,\mu([b-\eta, b+\eta])\,.$$

Let $0 < \sigma_1 < 1/2$, $A = (\sum_{n=1}^{\infty} 2^{-n\sigma_1})^{-1}$ Then

$$\Omega_m \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1} \Omega_{m,n,k}\ ,$$

where

$$\Omega_{m,n,k} = \Big\{b_\alpha :\ b_\alpha\Big(\frac{k\,t}{2^n}\Big) - b_\alpha\Big(\frac{(k-1)\,t}{2^n}\Big) > \frac{mA}{2^{n\sigma_1}}\Big\}\,.$$

Indeed, since $b_\alpha(t) \le 2$ and $\sup_{s\in[0,t]} b_\alpha(s) > 2$, we can always find n and $k < 2^n$ such that $b_\alpha \in \Omega_{m,n,k}$. Therefore, by Lemma (2.6),

$$\mathbf{E}_a\Big(\int_0^t b_\alpha^{m_j}(s)\,ds\Big)^{2M} \mathbf{1}_{\Omega_{m,n,k}}(b_\alpha)\,\mathbf{1}_{\{b_\alpha : b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)$$
$$\cdot\, t^{2M}(a_1+m+1)^{2Mm_j}\,\mathbf{E}_a \mathbf{1}_{\Omega_{m,n,k}}(b_\alpha)\,\mathbf{E}_{b_\alpha(kt/2^n)}$$
$$\cdot\, \mathbf{1}_{\{\sigma_\alpha : s_\alpha(t-kt/2^n)\in[b-\eta,b+\eta]\}}(\sigma_\alpha)$$
$$\le C\,t^{2M-1-\alpha}(a_1+m+1)^{2Mm_j}\,2^{n(1+\alpha)}\,\mu([b-\eta, b+\eta]$$
$$\cdot\, \mathbf{E}_a\,\mathbf{E}_{b_\alpha(((k-1)t)/2^n)}\,\mathbf{1}_{\{\sigma_\alpha : \sigma_\alpha(t/2^n) > mA/2^{n\sigma_1}+\sigma_\alpha(0)\}}(\sigma_\alpha)$$
$$\le C\,t^{2M-1-\alpha}(a_1+m+1)^{2Mm_j}\,2^{n(1+\alpha)}\,\mu([b-\eta, b+\eta])$$
$$\cdot\, \exp\Big(-\,\frac{c_2\,m^2\,A^2\,2^{n(1-2\sigma_1)}}{t}\Big)\,.$$

Hence,

$$I_2 \leq C\, t^{M-\alpha-1}\, \mu([b-\eta, b+\eta])$$

and finally,

$$I \leq C \int_0^1 t^{-(1+\xi/2)(D/2)+M-\alpha-1}\, dt < +\infty\,.$$

Now we pass to the lower estimate for the Green function. Let $|y| = 1$, $\eta > 0$ and let ϕ_η be a family of smooth functions with the properties: $\operatorname{supp}\phi_\eta \subset \{z \in N:\ |y^{-1}z| < \eta\}$, $\phi_\eta \geq 0$, $\int \phi_\eta(z)\, dz = 1$. Finally, let $\psi_\eta(\,\cdot\,) = \mu([b-\eta, b+\eta])^{-1}\, \mathbf{1}_{[b-\eta,b+\eta]}(\,\cdot\,)$.

Lemma 5.21. *Given $a_1 > 0$ and a compact set $K \subset N$, there is $c > 0$ such that for every $a \leq a_1$, $0 < b < 1$, $0 < \eta < 1$,*

$$\int_1^2 \mathbf{E}_a U^b(0,t)\, \varphi_\eta(x)\, \psi_\eta(b_\alpha(t))\, dt \geq c\,, \qquad x \in K\,.$$

Proof. Let d, D be positive numbers which will be chosen later. We consider the set

$$\Omega = \big\{b_\alpha:\ \sup_{s\in[0,t]} b_\alpha(s) \leq D,\ \inf_{s\in[t/4,3t/4]} b_\alpha(s) \geq d\big\}\,,$$

and we estimate

$$\int_1^2 \mathbf{E}_a \varphi_\eta * p^b(t,0)(x)\, \mathbf{1}_\Omega(b_\alpha)\, \mu([b-\eta,b+\eta])^{-1}\, \mathbf{1}_{\{b_\alpha:\, b_\alpha(t)\in[b-\eta,b+\eta]\}}(b_\alpha)$$

from below. We have

$$\begin{aligned}&\varphi_\eta * p^b(t,0)(x)\\ &\quad = \iint \varphi_\eta * p^b\Big(t, \frac{2\,t}{3}\Big)(z)\, p^b\Big(\frac{2\,t}{3}, \frac{t}{3}\Big)(z^{-1}\,x\,y^{-1})\, p^b\Big(\frac{t}{3}, 0\Big)(y)\, dz\, dy\,.\end{aligned}$$

In view of (4.7), we choose a compact set K_1 such that for $b \in \Omega$ and $1 \leq t \leq 2$,

$$\int_{K_1} \varphi_\eta * p^b\Big(t, \frac{2\,t}{3}\Big)(z)\, dz \geq \varepsilon > 0\,, \qquad \int_{K_1} p^b\Big(\frac{t}{3}, 0\Big)(y)\, dy \geq \varepsilon > 0\,,$$

where $\varepsilon=\varepsilon(A)$. Then, by Theorem (4.11) there is $C=C(D,d,K,K_1)$ such that
$$p^b\Big(\frac{2\,t}{3},\frac{t}{3}\Big)(z^{-1}\,x\,y^{-1})\geq C\,,$$
for $z,y\in K_1$, $x\in K$, $b_\alpha\in\Omega$, $1\leq t\leq 2$. Therefore we are left with
$$\begin{aligned}I&=\mu([b-\eta,b+\eta])^{-1}\,\mathbf{P}_a(b_\alpha:\ b_\alpha\in\Omega,b_\alpha(t)\in[b-\eta,b+\eta])\\&\geq\mathbf{E}_a\mathbf{1}_{\{\sup_{s\in[0,2t/3]}b_\alpha(s)\leq D_2,\inf_{s\in[t/3,2t/3]}b_\alpha(s)\geq d\}}(b_\alpha)\,\mu([b-\eta,b+\eta])^{-1}\\&\qquad\cdot\mathbf{P}_{b_\alpha(2t/3)}\Big(\sup_{s\in[0,t/3]}\sigma_\alpha(s)\leq D,\ \sigma_\alpha\Big(\frac{t}{3}\Big)\in[b-\eta,b+\eta]\Big)\end{aligned}$$
provided $D_2<D$. Notice that if $d\leq b_\alpha(2\,t/3)\leq D_2$,
$$\mu([b-\eta,b+\eta])^{-1}\,\mathbf{P}_{b_\alpha(2t/3)}\Big(\sigma_\alpha\Big(\frac{t}{3}\Big)\in[b-\eta,b+\eta]\Big)\geq C=C(d,D_2)\,.$$
But, proceeding as in the proof of the previous theorem we see that
$$\begin{aligned}\mu([b-\eta,b+\eta])^{-1}\,\mathbf{P}_{b_\alpha(2t/3)}\Big(\sup_{s\in[0,t/3]}\sigma_\alpha(s)\geq D,\ \sigma\Big(\frac{t}{3}\Big)\in[b-\eta,b+\eta]\Big)&\\\leq c_1\,e^{-c_2(D-D_2)^2}\,.&\end{aligned}$$
Therefore choosing D and D_2 appropriately we have
$$\begin{aligned}\mu([b-\eta,b+\eta])^{-1}\,\mathbf{P}_{b_\alpha(2t/3)}\Big(\sup_{s\in[0,t/3]}\sigma_\alpha(s)\leq D,\ \sigma_\alpha\Big(\frac{t}{3}\Big)\in[b-\eta,b+\eta]\Big)&\\\geq C(d,D,D_2)\,,&\end{aligned}$$
for $1\leq t\leq 2$. Hence for $D_1<D_2$,
$$\begin{aligned}I\geq C(d,D,D_2)\,&\mathbf{E}_a\mathbf{1}_{\{b_\alpha:\sup_{s\in[0,t/3]}b_\alpha(s)\leq D_1,b_\alpha(t/3)>2d\}}\\&\cdot\mathbf{P}_{b_\alpha(t/3)}\big(\inf_{s\in[0,t/3]}\sigma_\alpha(s)\geq d,\ \sup_{s\in[0,t/3]}\sigma_\alpha(s)\leq D_2\big)\,.\end{aligned}$$
By Lemmas 2.12 and 2.13
$$\begin{aligned}&\mathbf{P}_{b_\alpha(t/3)}\big(\inf_{s\in[0,t/3]}\sigma_\alpha(s)\geq d,\ \sup_{s\in[0,t/3]}\leq D_2\big)\\&\geq 1-\mathbf{P}_{b_\alpha(t/3)}\big(\inf_{s\in[0,t/3]}\sigma_\alpha(s)<d\big)-\mathbf{P}_{b_\alpha(t/3)}\big(\sup_{s\in[0,t/3]}\sigma_\alpha(s)>D_2\big)\\&\geq 1-c_1\,e^{-c_2d^2}-c_1\,e^{-c_2(D_2-D_1)^2}\\&\geq C>0\end{aligned}$$

provided d and $D_2 - D_1$ are large enough. Finally,

$$\begin{aligned}\mathbf{P}_a\Big(\sup_{s\in[0,t/3]} b_\alpha(s) \le D_1,\ b_\alpha\Big(\frac{t}{3}\Big) > 2\,d\Big)&\\ &\ge 1 - \mathbf{P}_a\big(\sup_{s\in[0,t/2]} b_\alpha(s) > D_1\big) - \mathbf{P}_a\Big(b_\alpha\Big(\frac{t}{3}\Big) < 2\,d\Big)\\ &\ge c_1\, e^{-c_2 d^2} - c_1\, e^{-c_2 D_1^2} \ge C > 0\,,\end{aligned}$$

for sufficiently large D_1.

6. Estimates of the Poisson kernels and the Martin boundary.

(5.15) and (1.13) imply immediately the following estimates for m_γ.

Theorem 6.1. *Let m_γ be the Poisson kernel of $\mathcal{L}_\gamma$, $\gamma > 0$. Then there exists a constant C_γ such that*

$$C_\gamma^{-1}\,(|x|+1)^{-Q-\gamma} \le m_\gamma(x) \le C_\gamma\,(|x|+1)^{-Q-\gamma}\,,$$

for $x \in N$. In particular,

$$C^{-1}\,(|x|+1)^{-Q} \le m_0(x) \le C\,(|x|+1)^{-Q}\,,$$

for $x \in N$.

PROOF. Theorem 5.6 says that there is a positive constant C_γ such that

$$C_\gamma^{-1} \le G_{-\gamma}(x,a;e,0) \le C_\gamma \tag{6.2}$$

if $|x| = 1, a \le 1$. Let $x = \sigma_a(y)$, $|x| = a \ge 1$, $|y| = 1$. By (1.18), we have

$$\begin{aligned} m_\gamma(x) &= G_{-\gamma}(x^{-1},1;e,0)\\ &= G_{-\gamma}(\sigma_a(y),1;e,0)\\ &= a^{-Q-\gamma} G_{-\gamma}(y,a^{-1};e,0)\\ &= |x|^{-Q-\gamma} G_{-\gamma}(y,a^{1};e,0)\,,\end{aligned}$$

and the proof is completed.

Now we consider the case $\gamma = 0$, *i.e.* we look at the operator $\mathcal{L}_0$. The next theorem gives description of the Martin boundary for $\mathcal{L}_0$.

Theorem 6.3. *The Martin boundary for $\mathcal{L} = \mathcal{L}_0$ consists of the following functions*:

a) *the constant function* 1,

b) $P_y(xa) = \dfrac{1}{m_0(e)}\, a^{-Q}\, \breve{m}_0(\sigma_{a^{-1}}(y^{-1}x))\,.$

All of them are minimal.

PROOF. By (1.17) we may use G to write the Martin kernels. Assume that
$$\lim_{n\to\infty} \frac{G(x,a;y_n,b_n)}{G(e,1;y_n,b_n)} = K(x,a)$$
and $|y_n| \longrightarrow \infty$ or $b_n \longrightarrow \infty$.

Let $r_n = \max\{|y_n|, b_n\}$. Then
$$G(x,a;y_n,b_n) = r_n^{-Q}\, G(\sigma_{r_n^{-1}}(x), r_n^{-1}a; \sigma_{r_n^{-1}}(y_n), r_n^{-1}\, b_n)\,.$$

We take n such that
$$|\sigma_{r_n^{-1}}(x)| < \frac{1}{4}\,, \qquad r_n^{-1}\, a < \frac{1}{4}\,.$$

Since $|\sigma_{r_n^{-1}}(y_n)| = 1$ and $r_n^{-1}\, b_n \le 1$ or $\sigma_{r_n^{-1}}(y_n) \le 1$ and $r_n^{-1} b_n = 1$, by Theorem 5.4 and the Harnack inequality for L^*, there is a constant c independent of x, a such that
$$c^{-1} \le G(\sigma_{r_n^{-1}}(x), r_n^{-1}\, a; \sigma_{r_n^{-1}}(y_n), r_n^{-1}\, b_n) \le c\,,$$
$$c^{-1} \le G(e, r_n^{-1}; \sigma_{r_n^{-1}}(y_n), r_n^{-1}\, b_n) \le c\,.$$

Therefore $K(x,a)$ is bounded and so must be constant (see [BR]).

Now we assume that $y_n \longrightarrow y_0$ and $b_n \longrightarrow 0$. First we prove that
$$\lim_{n\to\infty} \frac{G(x,a;y_n,b_n)}{G(e,1;y_n,b_n)} = \lim_{n\to\infty} \frac{G(y_0^{-1}x,a;e,b_n)}{G(e,1;e,b_n)}\,, \tag{6.4}$$

i.e. that

$$\lim_{n\to\infty} \frac{G(y_n^{-1}x, a; e, b_n)}{G(y_0^{-1}x, a; e, b_n)} = 1\,. \tag{6.5}$$

Notice that for n sufficiently large (depending on x, a), $\tau(y_n^{-1}x, a; y_0^{-1}x, a) < 1$. Hence by the Harnack inequality

$$|G(y_n^{-1}x, a; e, b_n) - G(y_0^{-1}x, a; e, b_n)| \le G(y_0^{-1}x, a; e, b_n)\, \tau(y_n^{-1}x, a; y_0^{-1}x, a)\,.$$

and (6.5) follows. We have

$$G(x, a; e, b_n) = a^{-Q}\, G(\sigma_{a^{-1}}(x), 1; e, a^{-1}b_n)\,.$$

Therefore when $b_n \longrightarrow 0$,

$$\lim_{b_n\to 0} G(x, a; e, b_n) = a^{-Q}\, G(\sigma_{a^{-1}}(x), 1; e, 0) = a^{-Q}\, \check{m}(\sigma_{a^{-1}}(x))$$

and so

$$\lim_{b_n\to 0} \frac{G(x, a; e, b_n)}{G(e, 1; e, b_n)} = \frac{1}{m_0(e)}\, a^{-Q}\, \check{m}_0(\sigma_{a^{-1}}(x)) = P_e(xa)\,.$$

1 is minimal because the only bounded $\mathcal{L}$-harmonic functions are constants, P_e is minimal if and only if P_y is minimal. Hence all of them are minimal.

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