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On ovals on Riemann surfaces

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Dedicated to the memory of my father

Abstract. We prove that k ($k \geq 9$) non-conjugate symmetries of a Riemann surface of genus g have at most $2g - 2 + 2^{r-3}(9 - k)$ ovals in total, where r is the smallest positive integer for which $k \leq 2^{r-1}$. Furthermore we prove that for arbitrary $k \geq 9$ this bound is sharp for infinitely many values of g.

1. Introduction.

Let X be a compact Riemann surface of genus $g \geq 2$. By a *symmetry* of X we mean, in this paper, an antiholomorphic involution σ which has fixed points. A surface admitting a symmetry is said to be *symmetric*. The principal motivation for the study of symmetric Riemann surfaces comes from the theory of algebraic curves. A compact Riemann surface X corresponds to a smooth complex projective algebraic curve and symmetries, non-conjugate in the group $\mathrm{Aut}^{\pm}(X)$ of all automorphisms of X, give rise to non-isomorphic over the reals, real models of the curve. A classical theorem of Harnack [8] states that the set $F(\sigma)$ of fixed points of σ consists of $\|\sigma\|$ in range $1 \leq \|\sigma\| \leq g+1$ disjoint simple closed curves to which, following Hilbert's terminology, we shall refer to as the *ovals of* σ. The number of ovals of a symmetry equals the number of connected components of the corresponding real model.

In this paper we are looking for the maximal number $\omega(g, k)$ of ovals that k non-conjugate symmetries of a Riemann surface X of genus g may admit. This question was investigated at the end of seventies by S.

M. Natanzon in [11], [12] and [13] who proved many results concerning low values of k. In particular, he proved that $\omega(g,k) \leq 2\,g + 2^{k-1}$ for $2 \leq k \leq 4$ and that this bound is attained respectively for every g congruent to 1 modulo 2^{k-2}. However the problem of finding the bound for $\omega(g,k)$ for $k \geq 5$ has not been solved up to now. Results concerning surfaces of even g, which by [6] have at most 4 non-conjugate symmetries with fixed points, have been recently obtained in [7].

Recently this question was taken up by Singerman [17] who showed that for arbitrary k there exist infinitely many values of g for which there exists a Riemann surface of genus g having k non-conjugate symmetries and $M_k = 2\,g + 2^{k-3}(9-k) - 2$ ovals in total and he conjectured that this is the best bound. From the recent paper of Natanzon [14] it follows that this indeed is the case in the special situation of separable symmetries. Observe that for $k = 3$ and 4 the Singerman and Natanzon bounds coincide without this additional assumption.

Here we show that for $k \geq 9$, $\omega(g,k) \leq 2\,g - 2 + 2^{r-3}(9-k)$, where r is the smallest positive integer for which $k \leq 2^{r-1}$. Furthermore we prove that for arbitrary $k \geq 9$ this bound is sharp for infinitely many values of g. In particular there are no $k > 9$ for which Singerman's conjecture is true. It is true for $k = 9$ and probably true for $5 \leq k \leq 8$.

2. Preliminaries.

The results announced in the previous section will be proved using combinatorial techniques based on Fuchsian and NEC groups. The basic results concerning these matter can be found in [3]. However for the reader's convenience we point out some of the most important concepts and results.

The starting point in a combinatorial study of compact Riemann surfaces of genus $g \geq 2$ is the Riemann uniformization theorem by which each such surface can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore having a surface X so represented its group of automorphisms can be represented as Δ/Γ for another Fuchsian group Δ. Now the orbit space of X under the action of some symmetry σ has a structure of Klein surface and the point is that the counterpart of these results for Klein surfaces also holds (see [10] and [15]), where NEC groups play the role of Fuchsian groups.

The algebraic structure of an NEC group Λ is determined by its signature ([9], [18]) which is a symbol of the form

$$(1) \qquad (g'; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k\}),$$

where the numbers $m_i \geq 2$ are called the *proper periods*, C_i are the s_i-uples $(n_{i1}, \ldots, n_{is_i})$ called the *period cycles*, the numbers $n_{ij} \geq 2$ are the *link periods* and $g' \geq 0$ is said to be the *orbit genus* of Λ. A *surface NEC group* is an NEC group with only empty period cycles and without proper periods, *i.e*, an NEC group with signature $(g'; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\})$, a Fuchsian group can be regarded as an *NEC* group with signature $(g'; +; [m_1, \ldots, m_r]; \{-\})$ and finally a Fuchsian surface group is a Fuchsian group with signature $(g'; +; [-]; \{-\})$. A group Λ with signature (1) has a presentation with canonical generators

$$x_i\,, \qquad 1 \leq i \leq r\,, \qquad e_i,\ c_{ij}\,, \qquad 1 \leq i \leq k\,,\ 0 \leq j \leq s_i\,,$$

and

$$a_i,\ b_i \ \text{ or } d_i\,, \qquad 1 \leq i \leq g'\,,$$

and relators

$$x_i^{m_i}\,, \qquad 1 \leq i \leq r\,, \qquad c_{ij}^2\,,\ (c_{ij-1}\,c_{ij})^{n_{ij}},\ c_{i0}\,e_i^{-1}\,c_{is_i}\,e_i\,,$$

with $1 \leq i \leq k$, $0 \leq j \leq s_i$, and

$$x_1 \cdots x_r\, e_1 \cdots e_k\, a_1\, b_1\, a_1^{-1}\, b_1^{-1} \cdots a_{g'}\, b_{g'}\, a_{g'}^{-1}\, b_{g'}^{-1}\,,$$

or

$$x_1 \cdots x_r\, e_1 \cdots e_k\, d_1^2 \cdots d_{g'}^2\,,$$

according as the sign is $+$ or $-$.

Finally the hyperbolic area of an arbitrary fundamental region of an NEC group Λ with signature (1) equals

$$(2) \quad \mu(\Lambda) = 2\pi\Big(\varepsilon\, g' - 2 + k + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\Big)\,,$$

where $\varepsilon = 2$ if there is a "+" sign and $\varepsilon = 1$ otherwise. If Γ is a subgroup of finite index in Λ, then it is an NEC group itself and we have the Hurwitz-Riemann formula

$$(3) \qquad [\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}\,.$$

3. Centralizers, conjugacy classes and some combinatorics.

A group G is said to be *abstractly orientable* if it admits an epimorphism $\alpha : G \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$ which will be called an *abstract orientation* of G. An element g of G is said to be *orientation preserving* (respectively *orientation reversing*) subject to the orientation α if $\alpha(g) = +1$ (respectively $\alpha(g) = -1$). Examples of orientable groups are provided by proper NEC groups and groups $\mathrm{Aut}^{\pm}(X)$ of all automorphisms of symmetric Riemann surfaces X. The first lemma of this section is an immediate consequence of Sylow theorems.

Lemma 3.1. *Let 2^n be the biggest power of 2 that divides the order of an abstractly oriented finite group G. Then G has at most 2^{n-1} conjugacy classes of orientation reversing elements of order 2.*

PROOF. Indeed let S be a Sylow subgroup of G. Then each conjugacy class has a representative in S. So the lemma follows since $\mathrm{Ker}\,\alpha_{|S}$, which consists of orientation preserving elements is a subgroup of S of index 2.

Lemma 3.2. *Let G be a finite group and let y_1, y_2 be two elements of order 2 whose product has order n. Then the order of the centralizer $\mathrm{C}(G, y_i)$ of y_i in G does not exceed $2\,|G|/n$ for $i = 1, 2$.*

PROOF. Let H be the group generated by y_1 and y_2 and observe first that $\mathrm{C}(H, y_i) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ according as n is odd or even. Fix a system X of representatives for the cosets of G/H. Then each element g of G can be represented as $g = y\,x$ for some $y \in H$ and $x \in X$ uniquely determined. Now assume that both $g = y\,x$ and $g' = y'\,x \in \mathrm{C}(G, y_i)$. Then $H \ni y'y^{-1} = g'g^{-1} \in \mathrm{C}(G, y_i)$. Thus $y'y^{-1} \in \mathrm{C}(H, y_i)$ and so the lemma follows.

Finally in this section we prove the following elementary combinatorial lemma that we shall need in the sequel.

Lemma 3.3. *Assume that $k, k \geq 3$ labels are used to label s points situated on a circle in such a way that no two consecutive points have the same label. Then at least $k-1$ points have neighbours with distinct labels.*

PROOF. We shall prove the lemma by induction on s. Observe first that $s \geq k$ and that the cases $s = 3$ and $s = 4$ are trivial. So assume that $s \geq 5$. There is nothing to prove if no point has neighbours with the same label; here s points have neighbours with distinct labels. So assume that there are three consecutive points $i-1, i, i+1$, say with labels $1, k$ and 1 respectively and consider the induced configuration of $s-2$ points $1, \ldots, i-1, i+2, \ldots, s$.

Assume first that some of these points have label k. Then by the inductive hypothesis $t \geq k-1$ points have neighbours with distinct labels. If, in the new configuration, the point $i-1$ has neighbours with the same label then in the former configuration these t points have neighbours with distinct labels whilst if $i-1$ has neighbours with distinct labels then in the former configuration $t-1$ of these points and one among $i-1$ and $i+1$ has neighbours with distinct labels.

If none of the points $1, \ldots, i-1, i+2, \ldots, s$ has label k then we have a configuration of $s-2$ points on circle labeled by $k-1$ labels. For $k = 3$, s is even and we see that $i-1$ and $i+1$ have neighbours with distinct labels. So assume that $k > 3$. Then by the inductive hypothesis, $k-2$ of these points have distinct labels. So the assertion follows since in this case these points and $i+1$ have neighbours with distinct labels in the former configuration.

4. Symmetries of Riemann surfaces and their ovals.

Let $\mathrm{Aut}^+(X)$ be the group of orientation preserving automorphisms of a compact Riemann surface X represented as $\mathcal{H}/\Gamma$. Then $\mathrm{Aut}^+(X) = \Delta/\Gamma$ for some Fuchsian group Δ which is the normalizer of Γ in $\mathrm{PSL}(2,\mathbb{R})$. Now, X is symmetric if and only if there exists an *NEC* group Λ containing Δ as a subgroup of index 2 and Γ as a normal subgroup. In such case $G = \Lambda/\Gamma = \mathrm{Aut}^{\pm}(X)$ is the group of all automorphisms of X, including those that reverse its orientation. Let $\theta : \Lambda \longrightarrow G$ be the canonical projection. A symmetry of X is an element $\sigma \in \mathrm{Aut}^{\pm}(X) \setminus \mathrm{Aut}^+(X)$ of order 2. Let us denote by $\langle\sigma\rangle$ the group generated by σ and represent it as Γ_σ/Γ for some *NEC* subgroup Γ_σ of Λ. Then the orbit space $X/\langle\sigma\rangle \cong \mathcal{H}/\Gamma_\sigma$ is a Klein surface whose boundary coincides with $\mathrm{Fix}(\sigma)$. So $\|\sigma\|$ is the number of period cycles of the signature of Γ_σ. Given a system of canonical generators of Λ, let $\{c_i : i \in I\}$ be a set of representatives for the conjugacy classes of reflections in Λ.

With these notations, a symmetry σ of X with non-empty set of fixed points is conjugate to $\theta(c_j)$ for some $j \in I$ and it was shown in [4] (see also [5]) that it has

$$\|\sigma\| = \sum [\mathrm{C}(\theta(\Lambda), \theta(c_i)) : \theta(\mathrm{C}(\Lambda, c_i))] \tag{4}$$

ovals, where the sum is taken over all elements i of I for which $\theta(c_i)$ is conjugate to σ. The index $w_i = w_i^X = [\mathrm{C}(\theta(\Lambda), \theta(c_i)) : \theta(\mathrm{C}(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\sigma\|$.

Now let $\|X\|$ be the sum of all $\|\sigma\|$, where σ is running over all conjugacy classes of symmetries of X. From (4) it follows immediately that

$$\|X\| = \sum_{i \in I} [\mathrm{C}(\theta(\Lambda), \theta(c_i)) : \theta(\mathrm{C}(\Lambda, c_i))]\,. \tag{5}$$

In this context w_i will be called a *contribution* of c_i to $\|X\|$ or we shall say simply that c_i contributes to X with w_i ovals.

Singerman [16] proved that the centralizer $\mathrm{C}(\Lambda, c_j)$ of a canonical reflection c_j in an NEC group Λ is

$$\langle c_j \rangle \times \langle e_j \rangle = \mathbb{Z}_2 \times \mathbb{Z} \tag{6}$$

if c_j corresponds to an empty period cycle and

$$\langle c_0 \rangle \times (\langle (c_0\, c_1)^{n_1/2} \rangle * \langle e^{-1} (c_{s-1}\, c_s)^{n_s/2}\, e \rangle) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) \tag{7}$$

or

$$\langle c_j \rangle \times (\langle (c_{j-1}\, c_j)^{n_j/2} \rangle * \langle (c_j\, c_{j+1})^{n_{j+1}/2} \rangle) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) \tag{8}$$

if c_j corresponds to a period cycle $(n_1, \dots, n_s)$ with even link periods, where $j = 0$ or $j \neq 0$ respectively. We are ready to state and prove the main result of the paper.

Theorem 4.1. *Let $\sigma_1, \dots, \sigma_k$ be non-conjugate symmetries of a Riemann surface X of genus $g \geq 2$ for which $G = \mathrm{Aut}^{\pm}(X)$ is a 2-group. Then $\|\sigma_1\| + \cdots + \|\sigma_k\| \leq 2\,g - 2 + (9 - k)\,|G|/8$.*

Proof. Let $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Assume that Λ has signature of a general form

$$(g'; \pm; [m_1, \dots, m_r]; \{C_1, \dots, C_m, (-), \overset{l}{\dots}, (-)\})\,, \tag{9}$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$ and denote $s = s_1 + \cdots + s_m$. Observe that every link period is a power of 2. Let $\theta : \Lambda \longrightarrow G$ be the canonical epimorphism.

Assume first that none of $\sigma_1, \ldots, \sigma_k$ is central. Then $|\mathrm{C}(G, \sigma_i)| \leq |G|/2$ for $i \leq k$. So any canonical reflection c corresponding to an empty period cycle contributes to $\|X\|$ with at most $|G|/4$ ovals, by (6) and (5) whilst a reflection corresponding to a non-empty period cycle contribute to $\|X\|$ with at most $|G|/8$ ovals by (5) and (7) or (8). So $\|X\| \leq (2\,l+s)\,|G|/8$. On the other hand $g-1 \geq (4\,l+4\,m-8+s)\,|G|/8$ by the Hurwitz-Riemann formula as $\mu(\Lambda) \geq 2\pi\,(l+m-2+s/4)$. Thus since $k \leq l+s$ we obtain $6\,l+8\,m+s > 7+k$ since for $m=0$ we have $l \geq k \geq 9$. Consequently

$$\begin{aligned}\|X\| &\leq (2\,s+8\,l+8\,m-16)\,\frac{|G|}{8} + (16-6\,l-8\,m-s)\,\frac{|G|}{8} \\ &\leq 2\,g-2+(9-k)\,\frac{|G|}{8}\,.\end{aligned}$$

So we can assume that some of the symmetries in question, say z, is a central element of G. Furthermore we can assume that $l=0$ and $m=1$. Observe first that $m \neq 0$. Indeed if $m=0$ then as above we prove that $\|X\| \leq l\,|G|/2$ and $2\,g-2 \geq |G|\,(l-2)$. So

$$\begin{aligned}\|X\| &\leq l\,\frac{|G|}{2} \\ &= |G|\,(l-2)+(4-l)\,\frac{|G|}{2} \\ &\leq 2\,g-2+(16-4\,l)\,\frac{|G|}{8} \\ &< 2\,g-2+(9-k)\,\frac{|G|}{8}\end{aligned}$$

since $4\,l-k > 7$ as $l \geq k \geq 9$. Thus we can assume that $m > 0$ because otherwise the theorem holds.

We can assume that $\theta(c_{10}) \neq z$. If $l \neq 0$ consider an NEC group Λ' with signature

$$(g'; \pm; [m_1, \ldots, m_r]; \{(2,2,2,2,n_{11}, \ldots, n_{1s_1}), C_2, \ldots, C_m, (-), \overset{l-1}{\ldots}, (-)\})\,. \tag{10}$$

For the sake of technical simplicity, we denote in the same way as in the group Λ some of the canonical generators of Λ'; namely those generators which correspond to "pieces" of the signature of Λ in the signature of Λ' and for the sake of terminological convenience we shall refer to these generators of Λ' as *old generators*. To be more precise, this means here in the case of the signatures (9) and (10) that the hyperbolic generators of Λ' are $a_1, b_1, \ldots, a_{g'}, b_{g'}$ or $d_1, \ldots, d_{g'}$ according to whether the sign is $+$ or $-$, the elliptic generators are $x_1, \ldots, x_r$, generators corresponding to the first nonempty period cycle are $e_1, c'_0, c'_1, c'_2, c'_3$, $c_{10}, c_{11}, \ldots, c_{1s_1}$, the generators corresponding to the remaining nonempty period cycles are $e_i, c_{i0}, c_{i1}, \ldots, c_{is_i}$, whilst generators corresponding to empty period cycles are $e_{m+1}, c_{m+1}, \ldots, e_{m+l-1}, c_{m+l-1}$. Furthermore according to this convention c'_0, c'_1, c'_2 and c'_3, are new generators whilst the remaining are old ones. We shall consider separately two cases

$$\text{a) } \theta(c_{m+l}) \neq z\,, \qquad \text{b) } \theta(c_{m+l}) = z\,.$$

Case a). Here we define $\theta' : \Lambda' \longrightarrow G$ on all old canonical generators but e_1 by θ and we put $\theta'(e_1) = \theta(e_1 \cdots e_{m+l})\,\theta(e_2 \cdots e_{m+l-1})^{-1}$, $\theta'(c'_0) = \theta'(e_1^{-1}\,c_{1s_1}\,e_1)$, $\theta'(c'_1) = \theta'(c'_3) = z$, and $\theta'(c'_2) = \theta(c_{m+l})$. Then, using results of [3, Chapter 2], it is not difficult to see that $\Gamma' = \operatorname{Ker}\theta'$ is a Fuchsian surface group. Indeed, by Theorem 2.2.4, its signature has no proper periods, by Theorem 2.3.3, it has no link periods, and finally, by Theorem 2.1.3, its sign is $+$. Let $X' = \mathcal{H}/\Gamma'$. As $\mu(\Lambda) = \mu(\Lambda')$ we see that X and X' have the same genus. We shall show that $\|X'\| \geq \|X\|$.

As the images under θ' of all old, except c_{10}, canonical reflections corresponding to nonempty period cycles and their neighbours are the same as their images under θ we see, by (5) and (7) or (8), that each of these reflections contributes to X' with the same number of ovals as to X. Similarly, by (6) and (5), old reflections corresponding to empty period cycles contribute to X' with the same number of ovals as to X. So we have to show that c_{10}, c'_0, c'_1, c'_2 and c'_3 contribute all together to X' with at least as many ovals as c_{m+l} and c_{10} contribute to X.

Let w_{10} be the contribution of c_{10} to $\|X\|$. Then c_{10} contributes to X' with w_{10} or $w_{10}/2$ ovals according to whether $\theta(c_{10}\,c_{11})^{n_{11}/2} = z$ or not. Similarly c'_0 contributes to X' with w_{10} or $w_{10}/2$ ovals according to whether $\theta(c_{1s_1-1}\,c_{1s_1})^{n_{1s_1}/2} = z$ or not. Consequently reflections c_{10} and c'_0 contribute to $\theta'(c_{10})$ at least the same number of ovals as c_{10} to $\theta(c_{10})$.

Assume now, that c_{m+l} had contributed with k ovals to $\theta(c_{m+l})$. Then c'_2 contributes to the new surface X' also with k ovals if $\theta(e_{m+l}) \neq 1$ and in this case we are done since the new surface has at least the same number of ovals as the former one. If $\theta(e_{m+l}) = 1$ then c'_2 contribute to X' with $k/2$ ovals. Let n' and n'' be the orders of $\theta'(c'_0)\,\theta'(c'_2)$

and $\theta'(c'_2)\,\theta'(c_{10})$ respectively and let $n = \max\{n', n''\}$. Then the centralizer of $\theta(c_{m+l})$ had order not bigger than $2\,|G|/n$ by the Lemma 3.2 and so c_{m+l} had contributed to the former surface at most with $|G|/n$ ovals, *i.e.*, $k \le |G|/n$ whilst now c'_1 and c'_3 contribute to z with $|G|/4\,n' + |G|/4\,n'' \ge |G|/2\,n \ge k/2$ ovals on the new surface X'. So indeed $\|X'\| \ge \|X\|$.

Case b). If $\theta(c_{m+l}) = z$ then we define $\theta' : \Lambda' \longrightarrow G$ on all old canonical generators and on c'_0 as for the case $\theta(c_{m+l}) \ne z$ and we put $\theta'(c'_1) = \theta'(c'_3) = \theta(c_{m+l})$, and $\theta'(c'_2) = \theta(c_{10})$. Again, using results of [3, Chapter 2] one can prove that $\Gamma' = \operatorname{Ker}\theta'$ is a Fuchsian surface group and by the Hurwitz-Riemann formula $X' = \mathcal{H}/\Gamma'$ is a Riemann surface of genus g. We shall show that $\|X'\| \ge \|X\|$. Also here all old canonical reflections but c_{10} contribute to X' with the same number of ovals as to X. The new reflection c'_2 contributes to X' with no less ovals than c_{10} to X. Here c_{m+l} had contributed to $\theta(c_{m+l})$ with $|G|/4$ or $|G|/2$ ovals according as $\theta(e_{m+l}) \ne 1$ or $\theta(e_{m+l}) = 1$. In the first case we see that $\|X'\| \ge \|X\|$ as c'_3 contribute to X' with $|G|/4$ ovals also. If $\theta(e_{m+l}) = 1$, then $\theta'(e_1) = \theta(e_1)$. So in this case $\theta'(c'_0) = \theta(c_{10})$ and therefore c'_1 and c'_3 contribute to X' with $|G|/4$ ovals each. Hence again $\|X'\| \ge \|X\|$.

Thus we can assume that Λ has no empty period cycles, *i.e.*, it has signature

$$(g'; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{m1}, \dots, n_{ms_m})\}) . \tag{11}$$

Now we shall see that, actually we can assume that $m = 1$, *i.e.*, Λ has just one period cycle. For, observe that we can assume that $\theta(c_{1s_1}) \ne z$ and $\theta(c_{20}) \ne z$. Let Λ' be an NEC group with signature

$$\begin{aligned}(g'; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}, 2, 2, n_{21}, \dots, n_{2s_2}, 2, 2),\\ C_3, \dots, C_s\}) .\end{aligned} \tag{12}$$

Here the reflections corresponding to the first period cycle are

$$c_{10}, \dots, c_{1s_1}, c'_0, c_{20}, \dots, c_{2s_2}, c'_1, c'_2$$

and also here $\mu(\Lambda) = \mu(\Lambda')$. We define $\theta' : \Lambda' \longrightarrow G$ on all old canonical generators but e_1 as before *i.e.*, by θ and we put $\theta'(e_1) = \theta(e_1)\,\theta(e_2)$. Furthermore we define $\theta'(c'_0) = \theta'(c'_1) = z$ and $\theta'(c'_2) = \theta'(e_1)\,\theta(c_{10})\,\theta'(e_1^{-1})$. Once more, using results of [3, Chapter 2], we see that $\Gamma' = \operatorname{Ker}\theta'$ is a Fuchsian surface group. Then $X' = \mathcal{H}/\Gamma'$ is a Riemann surface of genus g. In a similar way, we can prove that $\|X'\| \geq \|X\|$. Indeed all old canonical reflections, but c_{10} and c_{20} contribute to X' with the same number of ovals as to X.

Let w_i^X be the contribution of c_{i0} to $\|X\|$ and let l_i be the order of the centralizer of $\theta(c_{i0})$ for $i = 1, 2$. Then $w_i^X = l_i/4\, k_i$, where k_i is the order of $\theta(c_{i0}\, c_{i1})^{n_{i1}/2}\, \theta(e_i^{-1}(c_{is_i-1}\, c_{is_i})^{n_{is_i}/2}\, e_i)$. In particular we see that $w_i^X \leq l_i/4$. On the other hand, as $\theta'(c_{10}\, c_{11}))^{n_{11}/2}\, \theta'(e_1^{-1}\, c_1'\, c_2'\, e_1)$ and $\theta'(c_{1s_1-1}\, c_{1s_1})^{n_{1s_1}/2}\, \theta'(c_{1s_1}\, c_0')$ have order 2 we see that c_{10} and c_{1s_1} contribute to X' with no less ovals than c_{10} to X. Similarly c_{20} and c_{2s_2} contribute to X' with no less ovals than c_{20} to X. So we see that indeed $\|X'\| \geq \|X\|$.

So at last we arrive at the case of an NEC group Λ with signature

$$(g'; \pm; [m_1, \ldots, m_r]; \{(n_1, \ldots, n_s)\}) . \tag{13}$$

Let $c_0, \ldots, c_s$ denote the corresponding canonical reflections. Observe that $s \leq 8\,(g-1)/|G| + 4$.

We can assume that $\theta(c_0)$ is a central symmetry of X and so in particular $\theta(c_0) = \theta(c_s)$. Consider $c_0, c_1, \ldots, c_{s-1}$ as s points on a circle labelled by $\theta(c_0), \theta(c_1), \ldots, \theta(c_{s-1})$ respectively. By the Lemma 3.3, at least for $k-1$ numbers in range $0 \leq i_1 < \cdots < i_{k-1} \leq s-1$, $\theta(c_{i_t-1}) \neq \theta(c_{i_t+1})$, where the indices are taken modulo s.

Now if $n_{i_t} > 2$ or $n_{i_t+1} > 2$ then $\theta(c_{i_t})$ is not central and so $|\mathrm{C}(G, \theta(c_{i_t}))| \leq |G|/2$. Therefore c_{i_t} contributes to the corresponding surface X with at most with $|G|/8$ ovals. If $n_{i_t} = n_{i_t+1} = 2$ then $|\theta(\mathrm{C}(\Lambda, c_{i_t}))| \geq 8$ and thus also now c_{i_t} contributes to X with at most $|G|/8$ ovals. The remaining canonical reflections contribute to X with no more than $|G|/4$ ovals. So

$$\begin{aligned}\|X\| &\leq (k-1)\,\frac{|G|}{8} + (s-k+1)\,\frac{|G|}{4}\\ &= s\,\frac{|G|}{4} + (1-k)\,\frac{|G|}{8}\\ &\leq 2\,g - 2 + |G| + (1-k)\,\frac{|G|}{8}\\ &= 2\,g - 2 + (9-k)\,\frac{|G|}{8}\,.\end{aligned}$$

This completes the proof.

Corollary 4.2. *Let $\sigma_1, \dots, \sigma_k$, where $k \geq 9$ be non-conjugate symmetries of a Riemann surface X of genus $g \geq 2$. Then $\|\sigma_1\| + \cdots + \|\sigma_k\| \leq 2\,g - 2 + 2^{r-3}\,(9-k)$, where r is the smallest positive integer for which $k \leq 2^{r-1}$.*

Proof. As we are looking for the ovals of these symmetries and conjugate symmetries have the same number of ovals we can assume, using Sylow theorem, that they generate a 2-subgroup G of $\mathrm{Aut}^{\pm}(X)$. Let $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Assume that Λ has signature (9). Then, as $s + l \geq k \geq 9$, we see, by [2] (see also [3, Theorem 2.4.7]), that its signature is maximal. So by [3, Theorem 5.1.2] there exists a maximal NEC group Λ' and algebraic isomorphism $\varphi : \Lambda \longrightarrow \Lambda'$. Let $X' = \mathcal{H}/\Gamma'$, where $\Gamma' = \varphi(\Gamma)$. Then $\mathrm{Aut}^{\pm}(X') = \Lambda'/\Gamma'$ and φ induces an isomorphism $\widetilde{\varphi} : \Lambda/\Gamma \longrightarrow \Lambda'/\Gamma'$. Now $\widetilde{\varphi}(\sigma_1), \dots, \widetilde{\varphi}(\sigma_k)$ are non-conjugate symmetries of X'. Furthermore if $\langle \sigma_i \rangle = \Lambda_i/\Gamma$, then $\|\sigma_i\|$ is the number of empty period cycles of Λ_i. So $\|\sigma_i\| = \|\widetilde{\varphi}(\sigma_i)\|$ since $\langle \widetilde{\varphi}(\sigma_i) \rangle = \varphi(\Lambda_i)/\Gamma'$. Furthermore $\|X\| \leq \|X'\|$ and $G \cong \mathrm{Aut}^{\pm}(X')$ is a 2-group. Then by Theorem 4.1, $\|X'\| \leq 2\,g - 2 + (9-k)\,|G|/8$ and by Lemma 3.1, $|G| \geq 2^r$. Hence the Corollary follows.

The next theorem shows that the bound obtained in Corollary 4.2 is sharp.

Theorem 4.3. *Let $k \geq 9$ be an arbitrary integer and let r be the smallest positive integer for which $k \leq 2^{r-1}$. Then for arbitrary $g = 2^{r-2}\,t + 1$, where $t \geq k - 3$ there exists a Riemann surface X of genus g having k non-conjugate symmetries which have $2\,g - 2 + 2^{r-3}\,(9-k)$ ovals in total.*

Proof. Let $G = \mathbb{Z}_2 \oplus \overset{r}{\cdots} \oplus \mathbb{Z}_2 = \langle z_1 \rangle \oplus \cdots \oplus \langle z_r \rangle$ and let Λ be a maximal NEC-group with signature $(0; +; [-]; \{(2, \overset{2s}{\dots}, 2)\})$, where $s = (g-1)/2^{r-2} + 2 \geq k - 1$. Let $\{a_1, \dots, a_{2^{r-1}}\}$ be all elements of order 2 in G which have odd length in $z_1, \dots, z_r$ and assume that $a_1, \dots, a_r$ generate G. Then since r is the minimal integer such that $k \leq 2^{r-1}$ we have $k \geq r$ and so the assignment

$$\theta(e) = 1\,, \text{ and } \theta(c_i) = \begin{cases} a_1\,, & \text{for } i = 2\,j,\ 0 \leq j \leq s\,, \\ a_{j+2}\,, & \text{for } i = 2\,j+1,\ 0 \leq j \leq k-2\,, \\ a_k\,, & \text{for } i = 2\,j+1,\ k-1 \leq j \leq s-1\,, \end{cases}$$

defines an epimorphism $\theta : \Lambda \longrightarrow G$ for which $\Gamma = \operatorname{Ker}\theta$ is a surface group and $X = \mathcal{H}/\Gamma$ is a Riemann surface having k non-conjugate symmetries with fixed points.

We see that c_{2j}, for $0 \leq j \leq k-2$ contribute to a_1 with 2^{r-3} ovals whilst the remaining $2s-k+1$ non-conjugate canonical reflections of Λ contribute to the corresponding surface with 2^{r-2} ovals. As a result

$$\begin{aligned}\|\sigma_1\| + \cdots + \|\sigma_k\| &= 2^{r-3}(k-1) + 2^{r-2}(2s-k+1)\\ &= 2^{r-1}s + 2^{r-3}(1-k)\\ &= 2g-2+2^r+2^{r-3}(1-k)\\ &= 2g-2+2^{r-3}(9-k)\,.\end{aligned}$$

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References.

[1] Alling, N. L., Greenleaf, N., *Foundations of the Theory of Klein Surfaces.* Lecture Notes in Math. **219**, Springer-Verlag, 1971.

[2] Bujalance, E., Normal NEC signatures. *Illinois J. Math.* **26** (1982), 519-530.

[3] Bujalance, E., Etayo, J. J., Gamboa, J. M., Gromadzki, G., *Automorphisms Groups of Compact Bordered Klein Surfaces, A Combinatorial Approach.* Lecture Notes in Math. **1439**, Springer-Verlag, 1990.

[4] Gromadzki, G., *Groups of Automorphisms of Compact Riemann and Klein Surfaces.* University Press WSP Bydgoszcz, 1993.

[5] Gromadzki, G., On a Harnack-Natanzon theorem for the family of real forms of Rieamnn surfaces. *J. Pure Appl. Alg.* **121** (1997), 253-269.

[6] Gromadzki, G., Izquierdo, M., Real forms of a Riemann surface of even genus. *Proc. Amer. Math. Soc.* **126** (1998), 3475-3479.

[7] Gromadzki, G., Izquierdo, M., On ovals of Riemann surfaces of even genera. *Geometriae Dedicata* **78** (1999), 81-88.

[8] Harnack, A., Über die Vieltheiligkeit der ebenen algebraischen Kurven. *Math. Ann.* **10** (1876), 189-199.

[9] Macbeath, A. M., The classification of non-euclidean crystallographic groups. *Can. J. Math.* **19** (1967), 1192-1205.

[10] May, C. L., Automorphisms of compact Klein surfaces with boundary. *Pacific J. Math.* **59** (1975), 199-210.

[11] Natanzon, S. M., Automorphisms of the Riemann surface of an M-curve. *Funktsional Anal. i Priloz.* **12** (1978), 82-83. *Functional Anal. Appl.* **12** (1978), 228-229.

[12] Natanzon, S. M., On the total number of ovals of real forms of complex algebraic curves. *Uspekhi Mat. Nauk* **35** (1980), 207-208. *Russian Math. Surveys* **35** (1980), 223-224.

[13] Natanzon, S. M., Finite groups of homeomorphisms of surfaces and real forms of complex algebraic curves. *Trudy Moscov Mat. Obsch* **51** (1988), 3-53. *Trans. Moscow Math. Soc.* **51** (1989), 1-51.

[14] Natanzon, S. M., Harnack type theorem for families complexely isomorphic real algebraic curves. *Uspekhi Mat. Nauk* **52** (1997), 173-174. *Russian Math. Surveys* **52** (1997), 1314-1315.

[15] Preston, R., Projective structures and fundamental domains on compact Klein surfaces. Ph. D. Thesis. University of Texas, 1975.

[16] Singerman, D., On the structure of non-euclidean crystallographic groups. *Proc. Camb. Phil. Soc.* **76** (1974), 233-240.

[17] Singerman, D., Mirrors on Riemann surfaces. *Contemporary Mathematics* **184** (1995), 411-417.

[18] Wilkie, H. C., On non-euclidean crystallographic groups. *Math. Z.* **91** (1966), 87-102.

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