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On the generalized Bernoulli numbers that belong to unequal characters

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Abstract. The study of class number invariants of absolute abelian fields, the investigation of congruences for special values of L-functions, Fourier coefficients of half-integral weight modular forms, Rubin's congruences involving the special values of L-functions of elliptic curves with complex multiplication, and many other problems require congruence properties of the generalized Bernoulli numbers (see [16]-[18], [12], [29], [3], etc.). The first steps in this direction can be found in the papers of H. W. Leopoldt (see [15]) and L. Carlitz (see [5]). For further studies, see [22], [24], [29]. This paper presents some new examples extending both old author's results and recent investigations of H. Lang (see [14]), A. Balog , H. Darmon, K. Ono (see [3]), etc.

On the whole the proved results are consequence of congruences connecting the generalized Bernoulli numbers that belong to unequal characters.

0. Notations.

Here it is listed some general notations which will be used throughout this paper

- p, a prime number greater than 3,

- $m = (p-1)\, p^{l-1}/2$, $l \in \mathbb{N}$,
- $\binom{n}{k} = n!/(k!\,(n-k)!)$, the binomial coefficient,
- $[x]$, the greatest integer at most x for a real number x, *i.e.* $[x] \le x < [x]+1$,
- B_n, the n-th Bernoulli number in the "even suffix" notation, *i.e.* $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0, \ldots$
- $B_n(x)$, the n-th Bernoulli polynomial,
- θ, the character with conductor $q > 1$, $(p, q) = 1$,
- ω, the Teichmüller character (with conductor p),
- $\chi = \theta\, \omega^s$, the character with conductor $f = q\,p$, $(p,q) = 1$, $q > 1$, $s \in \mathbb{N}$ and $1 \le s \le p-2$,
- $B_{n,\psi}$, the n-th generalized Bernoulli number belonging to a character ψ (with the corresponding conductors).

We give relevant facts about B_n, $B_n(x)$, $B_{n,\psi}$ below. All other notations will be defined as they arise.

1. Some congruences for the generalized Bernoulli numbers.

In this section, it is proved the extension of known properties concerning to the generalized Bernoulli numbers and useful in theory of modular forms of half-integer weight (see, *e.g.*, [3, Theorem 4]).

As known, the Bernoulli numbers are defined by the symbolic recurrence relation $B_{n+1} = (B+1)^{n+1}$, $n = 1, 2, \ldots, B_0 = 1$, which in expanded form becomes

$$B_n = -(n+1)^{-1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \; .$$

From this identity (or from the equivalent definition of B_n by formal power series) it is easy proved that $B_{2n+1} = 0$ for $n > 0$ and other properties of Bernoulli numbers. Also, it is well known the Staudt-Clausen theorem for denominators and the Staudt theorem for numerators of B_n (see, *e.g.*, [29], [28] or [7]).

Further, if ψ is a character with the conductor g, then $L(1-n,\psi) = -B_{n,\psi}/n$, that is, the special values of Dirichlet L-functions at negative points are represented with the help of the generalized Bernoulli numbers $B_{n,\psi}$ defined by the formal series

$$\sum_{a=1}^{g} \psi(a)\, t\, \frac{e^{at}}{e^{gt}-1} = \sum_{n=0}^{\infty} B_{n,\psi}\, \frac{t^n}{n!}\,.$$

From this identity we find that

$$B_{n,\psi} = g^{n-1} \sum_{a=1}^{g} \psi(a)\, B_n\Big(\frac{a}{g}\Big) \tag{1}$$

with Bernoulli polynomials

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j\, x^{n-j}\,. \tag{2}$$

Here $B_{n,\varepsilon} = B_n$ for $n \neq 1$, $B_{1,\varepsilon} = -B_1 = 1/2$, where ε is the identity character. Remark that properties of $B_{n,\psi}$ are also important to construct the p-adic L-functions ([13] or [29]).

Now, let $\chi = \theta\,\omega^s$, $s = (p-1)/2$, be a representation of the character χ with the conductor f as the product of the character θ with the conductor q, $q > 1$, and ω^s, the s-th power of Teichmüller character ω with the prime conductor $p > 3$. It is known that $\omega = \omega(x)$ may be defined by the p-adic limit

$$\omega(x) = \lim_{l\to\infty} x^{p^l}\,,$$

for an integer x with $(x,p) = 1$, so that $\omega(x) \in \mathbb{Q}_p$, the field of p-adic numbers, $\omega^{p-1} = \varepsilon$ and $\omega(x) \equiv x^{p^l} \pmod{p^{l+1}}$. As usual, by setting $\omega(0) = 0$ we remark that $\omega(x)$ is a p-adic character $\pmod p$ of order $p-1$.

Note that $\omega^{(p-1)/2}(x) = (x/p)$, the Legendre symbol (see, *e.g.*, [23] or [29]), so that in our case we have $\chi = \theta(\cdot/p)$.

Theorem 1. *In the above notations, if $\theta(p) = 1$, then $B_{m+1,\chi} \equiv 0 \pmod{p^l}$, where $l \in \mathbb{N}$ and $m = (p-1)\,p^{l-1}/2$ for any odd prime $p > 3$.*

PROOF. First, with the help of (2) we conclude that the identity

$$B_{m+1,\chi} = f^m \sum_{i=1}^{f} \chi(i)\, B_{m+1}\Big(\frac{i}{f}\Big)\,,$$

may be rewritten as

$$\begin{aligned} B_{m+1,\chi} = \sum_{i=1}^{f} \chi(i) \Big(& f^{-1}\, i^{m+1} + (m+1)\, B_1\, i^m \\ & + (m+1)\, m\, f\, i^{m-1} \frac{B_2}{2} + \cdots \Big)\,, \end{aligned}$$

so that

$$B_{m+1,\chi} \equiv \sum_{i=1}^{f} \chi(i)\, f^{-1}\, i^{m+1} + (m+1)\, B_1 \sum_{i=1}^{f} \chi(i)\, i^m \pmod{p^l}\,,$$

because $m\, f \equiv 0 \pmod{p^l}$, $B_3 = 0$ and

$$\operatorname{ord}_p \left(\binom{m+1}{i} f^{i-1}\, i^{m+1-i} \right) \geq l\,, \qquad \text{for } i > 3\,.$$

Further, since $\chi(x)\,(i/p)$ is a character modulo f, we have

$$\sum_{i=1}^{f} \chi(i)\, i^m \equiv \sum_{i=1}^{f} \chi(i) \left(\frac{i}{p}\right) \equiv 0 \pmod{p^l}\,.$$

Therefore,

$$B_{m+1,\chi} \equiv \sum_{i=1}^{f} \chi(i)\, f^{-1}\, i^{m+1} \pmod{p^l}$$

or

$$f\, B_{m+1,\chi} \equiv \sum_{i=1}^{f} \chi(i)\, i^{m+1} \pmod{p^{l+1}}\,.$$

Since $f = q\,p$ with $(p,q) = 1$ and $\omega^{(p-1)/2}(x) = (x/p)$, it follows that

$$\begin{aligned} & f\, B_{m+1,\chi} \\ & \quad \equiv \sum_{j=0}^{q-1} \sum_{k=1}^{p-1} \theta(p\,j+k) \left(\frac{k}{p}\right) (p\,j+k)^{m+1} \\ & \quad \equiv \sum_{j=0}^{q-1} \sum_{k=0}^{p-1} \theta(p\,j+k) \left(\frac{k}{p}\right) \left(k^{m+1} + (m+1)\, p\, j\, k^m\right) \pmod{p^{l+1}}\,, \end{aligned}$$

because
$$\operatorname{ord}_p\left(\binom{m+1}{i}p^i\right) \geq l+1\,, \qquad \text{for } i \geq 2\,.$$

Further, we can replace k^{m+1} by $(k/p)\,(m+1)\,k$ because in any case $\sum_{j=0}^{q-1}\theta(p\,j+k)=0$. Moreover, as

$$p\,k^m \equiv p\left(\frac{k}{p}\right) \pmod{p^{l+1}},$$

we have

$$\begin{aligned}
f B_{m+1,\chi} &\equiv (m+1)\sum_{j=0}^{q-1}\sum_{k=0}^{p-1}\theta(p\,j+k)\left(\frac{k}{p}\right)^2 (p\,j+k)\\
&\equiv (m+1)\sum_{\substack{i=1\\(p,i)=1}}^{f}\theta(i)\,i\\
&\equiv (m+1)\Big(\sum_{i=1}^{f}\theta(i)\,i - p\,\theta(p)\sum_{i=1}^{q}\theta(i)\,i\Big)\\
&\equiv (m+1)\Big(p\sum_{i=1}^{q}\theta(i)\,i - p\,\theta(p)\sum_{i=1}^{q}\theta(i)\,i\Big)\\
&\equiv (m+1)\,p\,(1-\theta(p))\sum_{i=1}^{q}\theta(i)\,i \pmod{p^{l+1}},
\end{aligned}$$

or

$$(3) \qquad B_{m+1,\chi} \equiv \frac{m+1}{q}\,(1-\theta(p))\sum_{i=1}^{q}\theta(i)\,i \pmod{p^{l}}.$$

The congruence implies Theorem 1.

Remarks. 1) The paper [3] considers the case $l=1$ for a real character χ only.

2) The result can be proved with the help of properties of p-adic L-functions too (see, *e.g.*, [29, Chapter 5]).

3) The values of Dirichlet L-functions $L(s|\chi)$ at negative integers s are algebraic numbers: $L(1-n|\chi) = -B_{n,\chi}/n$. We denote by $\mathbf{C}_p$ the completion of $\overline{\mathbb{Q}}_p$, the closure of $\mathbb{Q}_p$. As known, the field $\mathbf{C}_p$ is algebraicaly closed. We fix an embedding of $\overline{\mathbb{Q}}$ in $\mathbf{C}_p$ and consider$B_{n,\chi}$, as an element of $\overline{\mathbb{Q}}_p$ (for details, see also [29, Chapter 5]).

4) Note that $B_{m+1,\chi} \neq 0$ if and only if $\theta(-1) = -1$. Indeed, it is known that

$$B_{m+1,\chi} \neq 0 \qquad \text{if and only if} \qquad \theta(-1) = (-1)^{m+1}$$

(see, *e.g.*, [15] or [29, Chapter 4]). In order to finish the proof it will suffice to remark that

$$\chi(-1) = \Big(\frac{-1}{p}\Big)\theta(-1) \qquad \text{and} \qquad (-1)^m = (-1)^{(p-1)/2} = \Big(\frac{-1}{p}\Big).$$

2. Congruences for generalized Bernoulli numbers belonging to unequal characters.

In this section, we would like study some connections between the generalized Bernoulli numbers belonging to unequal characters. The first results in this direction can be found in the papers [24], [25] (see also [29]).

Theorem 2. *Let $\chi(-1) = (-1)^n$, $n > 1$, $n \in \mathbb{N}$. In the notations of Section 0, we have*

$$B_{n,\chi} \equiv B_{r,\theta} \pmod{p^{2l}}, \qquad l \in \mathbb{N},\ r = s\,p^{3l-1} + n\,, \tag{4}$$

and

$$\frac{B_{n,\chi}}{n} \equiv \frac{B_{sp^{l-1}+n,\theta}}{s\,p^{l-1}+n} \pmod{p^l}, \qquad \mathrm{ord}_p n \leq l\,. \tag{5}$$

Proof. As known (see, *e.g.*, [15]), in usual symbolic form we have

$$S := \sum_{x=1}^{fp^{l-1}} \chi(x)\,x^n = (n+1)^{-1}((B_\chi + f\,p^{l-1})^{n+1} - B_{n+1,\chi})\,.$$

Then

$$S = (n+1)^{-1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i,\chi} \, (f\,p^{l-1})^i$$

or

$$\frac{S}{f\,p^{l-1}} = B_{n,\chi} + \sum_{i=3}^{n+1} \binom{n}{i-1} B_{n+1-i,\chi}\, q^2\, f^{i-3}\, \frac{p^{(l-1)(i-1)+2}}{i}\,,$$

because $\chi(-1) = (-1)^n$ implies $B_{n,\chi} \neq 0$ and $B_{n-1,\chi} = 0$. Remark that

a) $(l-1)\,(i-1) + 2 \geq 2\,l$ for $i \geq 3$,

b) $\mathrm{ord}_p B_{n+1-i,\chi} \geq 0$ because $f = q\,p$ and $p, q > 1$, see, *e.g.*, [15] or [29],

c) $\mathrm{ord}_p(f^{i-3}/i) \geq 0$ for $i \geq 3$ and $p > 3$.

Therefore, we have the congruence

$$\frac{S}{(f\,p^{l-1})} \equiv B_{n,\chi} \pmod{p^{2l}}\,. \tag{6}$$

On the other hand, with the help of

$$\omega(x) \equiv x^{p^{3l-1}} \pmod{p^{3l}}\,,$$

we obtain that

$$S = \sum_{x=1}^{q} \sum_{y=o}^{p^l-1} \theta\,(q\,y+x)\,\omega^s\,(q\,y+x)\,(q\,y+x)^n$$

$$\equiv \sum_{x=1}^{q} \theta\,(x) \sum_{y=o}^{p^l-1} (q\,y+x)^r \pmod{p^{3l}}\,, \qquad \text{with } r = s\,p^{3l-1} + n\,.$$

Now, noting that

$$\sum_{t=0}^{N-1} (b+a\,t)^k = a^k\,(k+1)^{-1} \Big(B_{k+1}\Big(N + \frac{b}{a}\Big) - B_{k+1}\Big(\frac{b}{a}\Big)\Big)$$

or

$$\sum_{t=0}^{N-1} (b+a\,t)^k = a^k\,(k+1)^{-1} \sum_{j=1}^{k+1} \binom{k+1}{j} B_{k+1-j}\Big(\frac{b}{a}\Big) N^j\,,$$

for integers $a \neq 0$, b, $k \geq 0$, $N \geq 1$ (see, *e.g.*, [24, Lemma 2]. Remark that this identity was recently reproved in [8]), we conclude that

$$S \equiv \sum_{x=1}^{q} \theta(x)\, q^r\, (r+1)^{-1} \sum_{j=1}^{r} \binom{r+1}{j} B_{r+1-j}\Big(\frac{x}{q}\Big) p^{lj} \pmod{p^{3l}},$$

or

$$S \equiv (r+1)^{-1} \sum_{j=1}^{r} \binom{r+1}{j} p^{lj} q^j q^{r-j} \sum_{x=1}^{q} \theta(x)\, B_{r+1-j}\Big(\frac{x}{q}\Big) \pmod{p^{3l}}.$$

By the same arguments as above, we conclude that

$$S \equiv f\, p^{l-1}\, B_{r,\theta} + r\, q^2\, p^{2l}\, \frac{B_{r-1,\theta}}{2} \pmod{p^{3l}},$$

or

$$\frac{S}{f\, p^{l-1}} \equiv B_{r,\theta} + r\, q\, p^l\, \frac{B_{r-1,\theta}}{2} \pmod{p^{2l}}.$$

Further, if $B_{r,\theta} \neq 0$ then $B_{r-1,\theta} = 0$. Therefore,

$$\frac{S}{f\, p^{l-1}} \equiv B_{r,\theta} \pmod{p^{2l}}$$

and together with the congruence (6) we obtain

$$B_{n,\chi} \equiv B_{r,\theta} \pmod{p)^{2l}}, \qquad \text{for } B_{r,\theta} \neq 0.$$

To prove the congruence (4) it remains to note that

$$\begin{aligned} &B_{n,\chi} \neq 0 \qquad \text{if and only if} \qquad B_{r,\theta} \neq 0, \\ &\text{with } r = s\, p^{3l-1} + n \text{ and } \chi = \theta\, \omega^s. \end{aligned} \tag{7}$$

A short way to do it was proposed to the author by the referee

$$\begin{aligned} B_{n,\chi} \neq 0 \qquad &\text{if and only if} \qquad \chi(-1) = (-1)^n, \\ &\text{if and only if} \qquad \theta(-1)\,(-1)^s = (-1)^n, \\ &\text{if and only if} \qquad \theta(-1) = (-1)^{n+s}, \\ &\text{if and only if} \qquad \theta(-1) = (-1)^r, \\ &\text{if and only if} \qquad B_{r,\theta} \neq 0. \end{aligned}$$

Provided that $\mathrm{ord}_p n \le l$, from the congruence (4) we conclude that

$$\frac{B_{n,\chi}}{n} \equiv \frac{B_{r,\theta}}{n} \pmod{p^l},$$

so that by Kummer's congruence

$$\frac{B_{r,\theta}}{r} \equiv \frac{B_{sp^{l-1}+n,\theta}}{s\,p^{l-1}+n} \pmod{p^l}$$

and $r/n \equiv 1 \pmod{p^l}$ the proof of Theorem 2 is finished .

Remarks. 1) A special case of Theorem 2 can be found in [24, Lemma 4].

2) The congruence (4) is Staudt's type congruence (for details, see, *e.g.*, [27] or [28]). Kummer's type congruences for the generalized Bernoulli numbers $B_{n,\psi}$ of fixed character (for example, in the case when ψ is a nonprincipal character of conductor $g \ne v^l$ with a prime v and $l \in \mathbb{N}$) show that characters ψ "smooth over" requests to congruences: they are correct for the case $n \equiv 0 \pmod{(p-1)}$ too (see [5], [29], [22] or [24]). As we see, sometimes the same situation takes place for the generalized Bernoulli numbers belonging to unequal characters.

3. Applications.

Now we will indicate some applications of the results. Firstly, let $\theta(n) = (-q/n)$ be an odd character of conductor q and $\chi(n) = \theta(n)\,(n/p)$, the real character of conductor $f = q\,p$, $(p,q) = 1$. Here $(n/p) = \omega^{(p-1)/2}(n)$ is Legendre symbol.

Then, by the congruence (3) we have

$$(8)\quad (1-\theta(p))\,h(-q) \equiv -\frac{B_{m+1,\chi}}{m+1} \pmod{p^l}, \qquad m = \frac{(p-1)\,p^{l-1}}{2},$$

where

$$h(-q) = -q^{-1}\sum_{j=1}^{q-1}\theta(j)\,j$$

is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$. Note that for $(p, h(-q)) = 1$ (in particular, for $h(-q) < p$) we have

$$\theta(p) = 1 \qquad \text{if and only if} \qquad B_{m+1,\chi} \equiv 0 \pmod{p^l}.$$

Further, it is known that $h(-q) < (1/3)\sqrt{q}\,\log q$ (see, *e.g.*, [21]). Hence, if $(1/3)\sqrt{q}\,\log q < p^{c+1}$, $c \in \mathbb{N} \cup \{0\}$ and $l > c$, the above equivalence (*i.e.*, a small sharpening of Theorem 1) is valid too.

Remark. The congruence (8) was proved by A. A. Kiselev ([10], $l = 1$) and I. Sh. Slavutskii ([12] or [22], $l \in \mathbb{N}$) in the form

$$(1 - \theta(p))\, h(-q) \equiv -\frac{B_{2m+1,\theta}}{2\,m+1} \pmod{p^l}.$$

But the right sides of the congruences coincide $\pmod{p^l}$. Indeed, by the congruence (5) with $n = m+1 = (p-1)\,p^{l-1}/2+1$ and $s = (p-1)/2$ we obtain

$$\frac{B_{m+1,\chi}}{m+1} \equiv \frac{B_{2m+1,\theta}}{2\,m+1} \quad (\mathrm{mod}\ p)^l. \tag{9}$$

Now by the congruence (9), Theorem 1 implies

Corollary 1. *In the above notations, if* $\theta(p) = 1$ *then* $B_{2m+1,\theta} \equiv 0 \pmod{p^l}$, *where* $l \in \mathbb{N}$ *and* $m = (p-1)\,p^{l-1}/2$ *for any odd prime* $p > 3$.

Among other things, the last congruence has an equivalent form.

Corollary 2.

$$B_{2m+1,\theta} \equiv 0 \pmod{p^l} \ \textit{if and only if} \ \sum_{\substack{x=1\\(x,p)=1}}^{fp^l} \theta(x)\left[\frac{x}{p^l}\right] \equiv 0 \pmod{p^l},$$

$$(p,q) = 1\,,\ q > 1\,,\ \theta \pmod{q}.$$

Indeed, by Voronoi's congruence for the generalized Bernoulli numbers ([22])

$$2\,\frac{B_{n,\theta}}{n} \equiv \frac{2}{q}\sum_{x=1}^{qN} \theta(x)\, x^{n-1}\left[\frac{x}{N}\right] \pmod{N},$$

$$n, N \in \mathbb{N},\ q > 1,\ (N,q) = 1\,,$$

(for more general congruences, see [24, Lemma 1]) we can conclude that

$$\frac{B_{n,\theta}}{n} \equiv \frac{1}{q}\sum_{x=1}^{qp^l} \theta\,(x)\, x^{n-1}\left[\frac{x}{p^l}\right] \quad (\mathrm{mod}\ p^l)\,, \qquad (p,q)=1 \text{ and } q>1\,.$$

Now, let $n = 2\,m+1 = (p-1)\,p^{l-1}+1$. If we note that $x^{n-1} \equiv 0$ or 1 (mod p^l) respectively when $p|x$ or $(p,x)=1$, then Corollary 2 is proved.

Further, we can supplement the congruence (9) by the similar ones

$$\frac{B_{m,\chi}}{m} \equiv \frac{B_{2m,\theta}}{2\,m} \ (\mathrm{mod}\ p^l)\,, \tag{10}$$

$$\frac{B_{2m,\chi}}{2\,m} \equiv \frac{B_{m,\theta}}{m} \ (\mathrm{mod}\ p^l)\,, \tag{11}$$

$$\frac{B_{2m+1,\chi}}{2\,m+1} \equiv \frac{B_{m+1,\theta}}{m+1} \ (\mathrm{mod}\ p^l)\,, \tag{12}$$

if in the congruence (5) we assume $n = m$ (respectively $m+1$ and $2\,m+1$).

With the help of these congruences it is possible to rewrite the known system of the congruences for class numbers of quadratic fields in the universal form, *i.e.* to obtain an approximation of Dirichlet's class number formula of quadratic fields (or the p-adic L-functions) in the universal form.

In 1948-64 by efforts of a group of the authors [9], [10], [1], [2], [4], [20], [12] (see also the survey [27]) it was proved the system of the congruences for class numbers of quadratic fields

$$\begin{gathered} h(d)\,\frac{U_l}{p^{l-1}} \equiv -T_l\,\frac{B_{m,\chi}}{2\,m} \ (\mathrm{mod}\ p^l)\,, \\ d = q\,p>0\,,\ q\geq 1\,,\ \chi \bmod q\,, \end{gathered} \tag{13}$$

$$\begin{gathered} h(d)\,\frac{\overline{U}_l}{p^l} \equiv -\overline{T}_l\,\frac{B_{2m,\chi}}{4\,m} \ (\mathrm{mod}\ p^l)\,, \\ d>0\,,\ (d,p)=1\,,\ q\geq 1\,,\ \chi \bmod d\,, \end{gathered} \tag{14}$$

$$\begin{gathered} h(d) \equiv -\frac{B_{m+1,\chi}}{m+1} \ (\mathrm{mod}\ p^l)\,, \\ d = -q\,p<-4\,,\ q\geq 1\,,\ \chi \bmod q\,, \end{gathered} \tag{15}$$

$$(1-\chi(p))\,h(d) \equiv -\frac{B_{2m+1,\chi}}{2\,m+1} \pmod{p^l}\,, \tag{16}$$
$$d < -4\,,\ (d,p)=1\,,\ \chi \bmod |d|\,,$$

where $E_1 = T_1 + U_1\sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$, $d > 0$, $E_l = T_l + U_l\sqrt{d} = E_l^{p^{l-1}}$, $\overline{E}_l = \overline{T}_l + \overline{U}_l\sqrt{d} = E_1^{(p-\chi(p))p^{l-1}}$, $m = (p-1)\,p^{l-1}/2$, $l \in \mathbb{N}$. Here p is an odd prime and χ is Kronecker character (of the corresponding conductor).

Now it is easy to see that the congruences (13)-(16) may be arranged into groups. Firstly, we will consider the case of the imaginary field, that is, the congruences (15) and (16). We can combine them in the form

$$(1-\chi(p))\,h(d) \equiv -\frac{B_{2m+1,\chi}}{2\,m+1} \pmod{p^l}\,, \quad d < -4\,,\ \chi \bmod |d|\,. \tag{17}$$

Indeed, if $(d,p)=1$, then the congruence coincides with (16). If $p|d$ and $d=-p\,q$ then $\chi(p)=0$. Hence, by the condition (12) the congruence implies (15).

Let $d > 0$. We claim that the congruences (13) and (14) may be grouped together in the form

$$h(d)\,\frac{\overline{U}_l}{p^l} \equiv -\overline{T}_l\,\frac{B_{2m,\chi}}{4\,m} \pmod{p^l}\,, \qquad \chi \bmod d\,. \tag{18}$$

Indeed, if $(d,p)=1$, then the congruences (14) and (18) coincide. But if $d = q\,p > 0$ then the congruence (18) implies

$$h(p\,q)\,\frac{U_{l+1}}{p^l} \equiv -T_{l+1}\,\frac{B_{2m,\chi}}{4\,m} \pmod{p^l}\,, \qquad \chi \bmod d\,. \tag{19}$$

To finish we must prove that

$$\frac{U_{l+1}}{p^l} \equiv \frac{U_l}{p^{l-1}} \qquad \text{and} \qquad T_{l+1} \equiv T_l \pmod{p^l}\,. \tag{20}$$

First of all it should be noted that $U_l \equiv 0 \pmod{p^{l-1}}$ (see [21, Lemma]). Then

$$(T_l + U_l\sqrt{d})^{p-1} = E_1^{(p-1)p^{l-1}} \qquad \text{implies} \qquad T_l^{p-1} \equiv 1 \pmod{p^l}$$

and

$$T_{l+1} + U_{l+1}\sqrt{d} = (T_l + U_l\sqrt{d}\,)^p$$

implies

$$T_{l+1} = T_l^p + p\,(p-1)\,T_l^{p-2}\,U_l^2\,\frac{p\,q}{2} + \cdots$$

and

$$U_{l+1} = p\,T_l^{p-1}\,U_l + \cdots$$

Hence, the conditions (20) is proved. Finally, with the help of the congruences (11), (20) and (19) we have

$$h(p\,q)\,\frac{U_l}{p^{l-1}} \equiv -T_l\,\frac{B_{m,\theta}}{2\,m} \pmod{p^l}\,, \qquad \theta \bmod q\,.$$

Therefore, it is proved

Theorem 3. *The system of congruences* (13)-(16) *is equivalent to*

$$\begin{cases} (1-\chi(p))\,h(d) \equiv -\dfrac{B_{2m+1,\chi}}{2\,m+1} \pmod{p^l}\,, & d<-4\,,\ \chi \bmod |d|\,, \\[2ex] h(d)\,\dfrac{\overline{U}_l}{p^l} \equiv -\overline{T}_l\,\dfrac{B_{2m,\chi}}{2\,m} \pmod{p^l}\,, & d>0\,,\ q\geq 1\,,\ \chi \bmod d\,, \end{cases}$$

where $\overline{E}_l = E_1^{(p-\chi(p))p^{l-1}}$ *and* $E_1 = T_1 + U_1\sqrt{d}$ *is the fundamental unit of the real quadratic field* $\mathbb{Q}(\sqrt{d})$.

As we earlier indicated, the system is an approximated form of the class number formula for quadratic fields (for details, see [22], [29]). Therefore, it was solved the old problem which was set up by H. Hasse (see his review Zbl **43**.40 of the paper [1]). Namely, it was given the universal form of the cited congruences for class numbers of quadratic fields, belonging to A. Kiselev, N. C. Ankeny, E. Artin, S. Chowla, L. Carlitz, etc.

REMARK. Consider one special situation. Let $p=3$ and $l=1$. Combining the congruences of the above system (13)-(16), we obtain the relations between $h(3\,q)$ and $h(-q)$ (or between $h(q)$ and $h(-3\,q)$), the main case of Scholz theorem (see [19]). A. Scholz used in his proof methods of class field theory. By elementary methods this theorem was proved by A. A. Kiselev ([10], [11]), and the result has been reproved by several authors ([2], [26], [14], etc., see also [29]).

Now we want to rewrite the proved above Theorem 1 in the terms of the theory of modular forms. In the cited paper [3] the authors were interested in the congruences of the type $a(p\,N) \equiv 0 \pmod{p}$ for every $N \in \mathbb{Z}$ with $(-N/p) = 1$, where $a(n)$ are integer coefficients of holomorphic half-integer weight form. Following H. Cohen (see [6]), they explicitly constructed holomorphic modular forms of half-integer weight whose Fourier coefficients are explicit expressions involving the special values at negative integers of Dirichlet L-functions of quadratic characters.

Let $k \geq 2$, $k \in \mathbb{N}$. We consider holomorphic modular forms of half-integer weight $k + 1/2$ with Fourier coefficients $a(n) = H(k, n) = -B_{k,\psi}/k$ where $\psi = ((-1)^k\, n/\cdot)$ when $(-1)^k\, n$ is a fundamental discriminant.

Definition ([3]). *Let $F(n)$ be an integer valued arithmetic function, M a positive integer, and p a prime. If $F(p\,q) \equiv 0 \pmod{M}$ for every positive integer q that is a quadratic residue (respectively nonresidue)* modulo p, *then we say that F has a quadratic congruence* modulo M *of type $(p, +1)$ (respectively $(p, -1)$).*

With these notations we prove

Theorem 4. *Let $\chi = \theta\,\omega^{(p-1)/2}$ be the character of a quadratic field with the fundamentail discriminant*

$$D = (-1)^{(p+1)/2} p\,q = ((-1)^{(p-1)/2} p)\,(-q)$$

and $\theta = (-q/\cdot)$ where positive integer q is prime to p. Then Fourier coefficients $H(m + 1, n)$ of the weight $m + 3/2$ modular form satisfy a quadratic congruence modulo p^l *of type $(p, (-1/p))$.*

Proof. As above, let $m = (p-1)\,p^{l-1}/2$, so that $m \equiv (p-1)/2 \pmod{2}$, and χ, θ, $\omega^{(p-1)/2}$ are the Kronecker characters for the corresponding conductors (in particular, $\omega^{(p-1)/2}$ is the Legendre symbol). It is obvious that $(q/p) = (-1/p)$ implies $\theta\,(p) = (-q/p) = 1$. Hence, by Theorem 1 we have

$$H(m+1, p\,q) \equiv -\frac{B_{m+1,\chi}}{m+1} \equiv 0 \pmod{p^l}\,.$$

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