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Construction of functions with prescribed Hölder and chirp exponents

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Abstract. We show that the Hölder exponent and the chirp exponent of a function can be prescribed simultaneously on a set of full measure, if they are both lower limits of continuous functions. We also show that this result is optimal: In general, Hölder and chirp exponents cannot be prescribed outside a set of Hausdorff dimension less than one. The direct part of the proof consists in an explicit construction of a function determined by its orthonormal wavelet coefficients; the optimality is the direct consequence of a general method we introduce in order to obtain lower bounds on the dimension of some fractal sets.

1. Introduction and statement of results.

A bounded function f is $C^\alpha(x_0)$, $\alpha \geq 0$, if there exists a polynomial P of degree at most $[\alpha]$ and a constant C such that, if $|x-x_0| \leq 1$, $|f(x)-P(x-x_0)| \leq C\,|x-x_0|^\alpha$. The *Hölder exponent* of f at x_0 (which will be denoted by $h_f(x_0)$) is by definition the supremum of all values of α such that f is $C^\alpha(x_0)$. Note that the knowledge of $h_f(x_0)$ does not give a very sharp information about the modulus of continuity at x_0; for instance, for all $\alpha \in \mathbb{R}$, all functions $|x|^{1/2}\,(\log(1/|x|))^\alpha$ have the same Hölder exponent $1/2$ at 0. The determination of the Hölder exponent of a function at a point x_0 can be reduced to estimating its wavelet coefficients near x_0, using Proposition 3. Conversely, this proposition allows to construct explicitely functions with prescribed Hölder exponent, see [3] and [7]. The class of all admissible Hölder exponents $h_f(x)$ (if f is continuous) coincides with the class of lower limits of continuous functions, see [1], [3] and [7]. Prescribing the Hölder exponent has been proved to be an efficient technique for signal simulation, in several situations where the Hölder exponent is strongly variable, see [2], [3]; however, characterizing the regularity with the sole Hölder exponent yields a rather poor information since it does not describe the more or less oscillatory behavior of the function near the point x_0. This oscillatory behavior is properly modelled with the help of the following definition, which was introduced by Yves Meyer, [9], [11].

Definition 1. *Let f be a function in $L^\infty_{\text{loc}}(\mathbb{R})$, and denote by $f^{(-l)}$ a l-th order primitive of f; f is called a (h,β)-type chirp at x_0 if*

$$f^{(-n)} \in C^{h+n(1+\beta)}(x_0)\,, \qquad \textit{for all } n \in \mathbb{N}\,.$$

The simplest example of a (h,β)-type chirp at x_0 is supplied by the function

$$|x-x_0|^h \sin\left(\frac{1}{|x-x_0|^\beta}\right). \tag{1}$$

The interior of the set of points (h,β) such that a function f is a (h,β)-type chirp at x_0 is always a domain of the form $h < h_f(x_0)$, $\beta < \beta_f(x_0)$, see [8]. The non-negative real number $\beta_f(x_0)$ is called the *chirp exponent at x_0*.

A strong local oscillatory behavior such as in (1) is very remarkable, and it was commonly believed that it could only be found at isolated points of a function; it was therefore a great surprise when Y. Meyer showed that the Riemann function $\sum n^{-2}\sin(\pi\, n^2\, x)$ has a dense set of points which are chirps of type $(3/2,1)$. Since then, several other functions were shown to have a dense set of chirps (see [8] for instance). However the problem of determining which couples $(h(x),\beta(x))$ can be simultaneously the Hölder and chirp exponents of a function remained completely open untill recently: In sharp contrast with the problem of the prescription of the sole Hölder exponent, it was shown in [5] that the couple of functions $(h(x),\beta(x))$ must satisfy the following very strong a priori requirement.

Proposition 1. *Let f be a function whose Hölder exponent $h_f(x)$ satisfies*

$$0 < h \leq h_f(x) \leq H < +\infty\,, \qquad \text{for all } x\,.$$

Then the chirp exponent $\beta_f(x)$ vanishes on a dense set of points.

Of course, this result doesn't prevent the possibility of prescribing the Hölder and chirp exponents at "most" points, and one of our purposes is to prove that they can be prescribed on a set of full measure. We now fix a (quite arbitrary) set of points of measure 0, outside which we will prescribe h and α.

The Borel-Cantelli lemma implies that for almost every $x \in \mathbb{R}$, there exists $C > 0$ such that

$$\left| x - \frac{k}{2^j} \right| \geq \frac{C}{j^2\, 2^j}\,, \qquad \text{for all } j \in \mathbb{N}^*\,,\ k \in \mathbb{Z}\,. \tag{2}$$

We denote by E the complement of this set.

Theorem 1. *For any couple $(h(x), \beta(x))$ of bounded nonnegative functions which are lower limits of continuous functions, there exists a function f whose Hölder and chirp exponents are respectively $h(x)$ and $\beta(x)$ at every point x satisfying* (2)*. Furthermore, the restriction "at every point x satisfying* (2)*" can be dropped at the points where β vanishes.*

Remark. The set E chosen here is an explicit set of points satisfying a dyadic approximation property. However it will be clear from the proof that many other choices are possible (in particular, one can exclude from E any given countable set, or we can replace dyadic approximation by p-adic approximation...).

We know from [5] that E has to be a dense set but one may wonder if E can be chosen "smaller". The following proposition shows on an example that the size of the set E is essentially optimal (the class $\mathcal{C}^{\log}$ will be defined below; let us just mention at this point that it is a weaker condition than assuming that $f \in \cup_{\varepsilon>0} C^{\varepsilon}(\mathbb{R})$).

Proposition 2. *Let H and B be positive real numbers, and let $\dim_H(A)$ denote the Hausdorff dimension of the set A. Any function f in $\mathcal{C}^{\log}$ satisfies*

$$\dim_H(\{x:\ h(x) \neq H \ \text{and}\ \beta(x) \neq B\}) = 1\,.$$

In other words constant exponents (H,β) cannot be prescribed outside a set of Hausdorff dimension less that one.

This proposition will be proved at the end of Section 4, as a consequence of a general technique that we will develop in Section 5 in order to obtain lower bounds for the Hausdorff dimension of a fairly general class of fractal sets. Since this technique might prove useful in other settings, Section 5 can be read independently from the rest of the paper.

Proposition 2 could have consequences in the context of multifractal analysis. Recall that the *spectrum of singularities* of a function is the function $d(h)$ which associates to each positive real number h the Hausdorff dimension of the set of points whose Hölder exponent is h, and the *spectrum of chirps* is the function $d(h,\beta)$ which associates to each couple (h,β) the Hausdorff dimension of the set of points whose Hölder and Chirp exponents are (h,β). In view of Proposition 2, one can reasonably conjecture that, in contrast with the case of the spectrum of singularities $d(h)$, the spectrum of chirps cannot be an arbitrary function, but necessarily satisfies some explicit conditions.

The main result proved in Section 5 is the following. Let λ_n be a sequence of points in $[0,1]$ and $\varepsilon_n > 0$. We consider the sets

$$E_a = \limsup_{N\to\infty} \bigcup_{n\geq N} [\lambda_n - \varepsilon_n^a, \lambda_n + \varepsilon_n^a]$$

(*i.e.*, E_a is the set of points that belong to an infinite number of intervals $[\lambda_n - \varepsilon_n^a, \lambda_n + \varepsilon_n^a]$). The function $a \longrightarrow \dim_H(E_a)$ is decreasing. Furthermore, if

$$A = \sup\left\{\alpha : \sum \varepsilon_n^\alpha = \infty\right\} = \inf\left\{\alpha : \sum \varepsilon_n^\alpha < \infty\right\},$$

using the covering by the intervals $[\lambda_n - \varepsilon_n^a, \lambda_n + \varepsilon_n^a]$, one easily obtains $\dim_H(E_a) \leq A/a$. This upper bound often turns out to be sharp in situations where the λ_n are 'equidistributed' in some sense. However this type of information is often hard to obtain or to handle; sometimes a different kind of information is easily available: For an a small enough, we may know that almost every point of $[0,1]$ belongs to E_a (it is the case in problems related to diophantine or dyadic approximation, or if the λ_n are independent equidistributed random variables). We will prove that this sole information yields a lower bound on $\dim_H(E_b)$ for $b > a$. In practice, a more precise result is often required: One needs to obtain a positive Hausdorff measure for A.

Let $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous increasing function satisfying $h(0) = 0$, and let A be a bounded subset of $\mathbb{R}^d$. If $|I|$ denotes the length of the interval I, let

$$\mathcal{H}^h_\varepsilon(A) = \inf_{\mathcal{U}} \Big\{ \sum_{(u_i)\in\mathcal{U}} h(|u_i|) \Big\},$$

where the infimum is taken on all coverings $\mathcal{U}$ by families of balls $\{u_i\}_{i\in\mathbb{N}}$ of radius at most ε. The $\mathcal{H}^h$-measure of A can be defined as

$$\mathcal{H}^h(A) = \lim_{\varepsilon\to 0} \mathcal{H}^h_\varepsilon(A).$$

Theorem 1. *Let $h_d(x) = (\log x)^2|x|^d$. If almost every x belongs to E_a,*

$$\mathcal{H}^{h_{a/b}}(E_b) > 0, \qquad \text{for all } b > a.$$

(In particular, the Hausdorff dimension of E_b is larger than a/b.)

2. Construction of the function f.

The function f with prescribed Hölder and chirp exponents will be constructed by imposing its coefficients on an orthonormal wavelet basis. Therefore, we start by recalling some properties of wavelet expansions.

If the $\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)$ form an orthonormal basis of $L^2(\mathbb{R})$, with ψ in the Schwartz class, as in [10], we define the wavelet coefficients of f by

$$C_{j,k} = 2^j \int f(x)\,\psi(2^j x - k)\,dx$$

(note that we do not use a L^2 normalization here).

We denote by $\mathcal{C}^{\log}$ the class of functions such that

$$|C_{j,k}| \le C\,2^{-j/\log j}. \tag{3}$$

It is a slightly stronger asumption than uniform continuity, but it implies no uniform Hölder regularity, see [9]. More precisely if $\mu(t) = 1/(\log\log(1/t))$,

$$|f(x) - f(y)| \le C\,|x-y|^{\mu(|x-y|)} \qquad \text{for all } x, y,$$

implies that f belongs to $\mathcal{C}^{\log}$, and conversely,

$$f \in C^{\log} \text{ implies } |f(x)-f(y)| \leq \Big(\frac{C}{\mu(|x-y|)}\Big)|x-y|^{\mu(|x-y|)}\,,$$

for all x, y. The following proposition is a slight extension of [6, Theorem 1]. For the sake of completeness, we prove it in the Appendix.

Proposition 3. *Suppose that $f \in C^{\alpha}(x_0)$; if $|k\,2^{-j} - x_0| \leq 1/2$ then*

$$|C_{j,k}| \leq C\,2^{-\alpha j}(1+|2^j x_0 - k|)^{\alpha}\,. \tag{4}$$

Conversely, if (4) holds for all j, k such that $|k\,2^{-j} - x_0| \leq 2^{-j/(\log j)^2}$, and if f belongs to $\mathcal{C}^{\log}$, there exists a polynomial P of degree at most $[\alpha]$ such that

$$|f(x) - P(x-x_0)| \leq C\,|x-x_0|^{\alpha}\,(\log\,|x-x_0|)^2\,. \tag{5}$$

The following corollary is a straightforward consequence of this proposition and will be useful in order to determine Hölder exponents.

Corollary 1. *Suppose that $f \in \mathcal{C}^{\log}$; then*

$$h_f(x) = \liminf_{|k2^{-j}-x| \leq 2^{-j/(\log j)^2}} \frac{\log\,|C^i_{j,k}|}{\log(2^{-j} + |k\,2^{-j} - x|)}\,, \tag{6}$$

where the limit is taken for $j \longrightarrow +\infty$ and $k\,2^{-j} \longrightarrow x$.

We now start the proof of Theorem 1. We thus suppose that $h(x)$ and $\beta(x)$ are respectively lower limits of the sequences of continuous functions $h_n(x)$ and $\beta_n(x)$; the prescription problem is local, so we can make the construction of the function f only on the interval $[0,1]$; thus we can suppose that each of the $h_n(x)$ and $\beta_n(x)$ are uniformly continuous. Each function h_n and β_n can itself be uniformly approximated arbitrarily well by a Lipschitz function, so that we can suppose, without losing any generality, that h_n and β_n are actually Lipschitz functions. Furthermore, since h and β are bounded, we can also suppose that

$$0 \leq h_n(x) \leq H \text{ and } 0 \leq \beta_n(x) \leq B\,, \qquad \text{for all } x, n\,. \tag{7}$$

We can also replace $h_n(x)$ by $\inf_{i=1,\dots,n} h_i(x)$, so that we can suppose that the sequence $h_n(x)$ is decreasing, and for the same reason, that the sequence $\beta_n(x)$ is also decreasing. Let

$$\widetilde{H}_n = \sup_{x\neq y} \frac{|h_n(x)-h_n(y)|}{|x-y|} \qquad \text{and} \qquad \widetilde{B}_n = \sup_{x\neq y} \frac{|\beta_n(x)-\beta_n(y)|}{|x-y|}$$

be the uniform Lipschitz constants of h_n and β_n. We define

$$A(n) = n + \widetilde{H}_n + \widetilde{B}_n \ . \tag{8}$$

Finally, we pick an increasing sequence of integers j_n such that for all n, $j_n \geq A_n$, and we replace the functions $h_n(x)$ by

$$h_n(x) + \frac{B+1}{\log j_n} \ , \tag{9}$$

where B is defined by (7).

The changes we made mean that without loss of generality, we may make the following additional asumptions: h and β are limits of decreasing sequences of nonnegative Lipschitz functions, and furthermore

$$h_n(x) \geq \frac{B+1}{\log j_n} \ , \qquad \text{for all } x\,.$$

We now define the wavelet coefficients of f. If j is not one of the numbers j_n, for all k, $C_{j,k}=0$.

Suppose now that the index j coincides with j_n. All the $C_{j_n,k}$ will vanish except for a sequence $\{k_n^i\}_{i\geq 0}$ defined as follows.

First $k_n^0 = 0$ and the corresponding wavelet coefficient is

$$C_{j_n,k_n^0} = 2^{-(h_n(0)/\beta_n(0)+1)j_n} \ .$$

We now construct the following values k_n^i. For $i \geq 0$, we denote by λ_n^i the location of the corresponding wavelet, *i.e.* $\lambda_n^i = k_n^i\, 2^{-j_n}$. The second nonvanishing wavelet coefficient is located at the distance

$$2.2^{-[(1/\beta_n(0)+1)j_n]} = \lambda_n^1 = k_n^1\, 2^{-j_n} \ ,$$

from λ_n^0 ($[x]$ denotes the integral part of x) and the corresponding wavelet coefficient is

$$C_{j_n,k_n^1} = 2^{-(h_n(\lambda_n^1)/\beta_n(\lambda_n^1)+1)j_n} \ .$$

The location λ_n^2 of the next nonvanishing wavelet coefficient is determined as follows. It is located at the **second** next integer multiple of $2^{-[(1/\beta_n(\lambda_n^1)+1)j_n]}$, and its size is

$$C_{j_n,k_n^2} = 2^{-(h_n(\lambda_n^2)/(\beta_n(\lambda_n^2)+1))j_n} .$$

We construct all the following nonvanishing wavelet coefficients the same way.

Note that the substitution we made in (9) has for consequence that all wavelet coefficients satisfy $|C_{j,k}| \le 2^{-j/\log j}$, so that the function we constructed belongs to the class $\mathcal{C}^{\log}$.

This construction rule implies that for all k,

$$2^{-(1/(\beta_n(\lambda_n^k)+1))j_n} \le |\lambda_n^k - \lambda_n^{k+1}| \le 4.2^{-(1/(\beta_n(\lambda_n^k)+1))j_n} . \tag{10}$$

3. Lower bounds of the Hölder exponents of f and its primitives.

Suppose that $x \notin E$, so that (2) holds at x (we will treat the case $x \in E$ and $\beta(x) = 0$ at the end of Section 4). For each n, x will belong to one of the intervals $[\lambda_n^k, \lambda_n^{k+1}]$. By construction, λ_n^k is a multiple of $2^{-[(1/(\beta_n(\lambda_n^{k-1})+1))j_n]}$, and λ_n^{k+1} is a multiple of $2^{-[j_n/((\beta_n(\lambda_n^k)+1))]}$; thus, because of (2),

$$|x - \lambda_n^k| \ge \frac{C}{\left(\dfrac{j_n}{\beta_n(\lambda_n^{k-1})+1}\right)^2} 2^{-(1/(\beta_n(\lambda_n^{k-1})+1))j_n} , \tag{11}$$

and, because of (10),

$$|x - \lambda_n^k| \le 4.2^{-(1/(\beta_n(\lambda_n^k)+1))j_n} . \tag{12}$$

For the same reasons,

$$|x - \lambda_n^{k+1}| \ge \frac{1}{\left(\dfrac{j_n}{\beta_n(\lambda_n^k)+1}\right)^2} 2^{-(1/(\beta_n(\lambda_n^k)+1))j_n} , \tag{13}$$

and

$$|x - \lambda_n^{k+1}| \le 4.2^{-(1/(\beta_n(\lambda_n^k)+1))j_n} . \tag{14}$$

Using Corollary 1, and the particular sequence of wavelet coefficients corresponding to the locations λ_n^k, we obtain

$$(15)\qquad \begin{aligned} h_f(x) &\le \liminf \left(\frac{\dfrac{h_n(\lambda_n^k)}{\beta_n(\lambda_n^k)+1}\, j_n}{-\log_2(2^{-j_n} + |\lambda_n^k - x|)} \right) \\ &= \liminf \frac{\dfrac{h_n(\lambda_n^k)}{\beta_n(\lambda_n^k)+1}}{\dfrac{1}{\beta_n(\lambda_n^k)+1}} \end{aligned}$$

(because of (11) and (12)). Thus

$$h_f(x) \le \liminf h_n(\lambda_n^k)\,.$$

But, using the mean-value theorem and the bound on h_n' given by (8),

$$h_n(\lambda_n^k) = h_n(x) + \mathcal{O}(j_n\,|\lambda_n^k - x|) = h_n(x) + \mathcal{O}(j_n\, 2^{-j_n/(\beta_n(\lambda_n^k)+1)})$$

but, since the functions $1/(1+\beta_n(x))$ are uniformly bounded from below,

$$h_n(\lambda_n^k) = h_n(x) + \mathcal{O}(j_n\, 2^{-Cj_n})\,, \qquad \text{for a } C > 0\,.$$

Thus the Hölder exponent at x satisfies

$$h_f(x) \le \liminf h_n(x) = \lim h_n(x)\,.$$

The determination of the Hölder exponent of the iterated primitives of f is made easy by the following remark. If $(C_{j,k})$ denote the wavelet coefficients of a function f, the $(2^{-lj} C_{j,k})$ are the wavelet coefficient of $f^{(-l)}$ using the wavelets $\psi^{(l)}(2^j x - k)$, and the criterium given by Proposition 3 remains valid using this system of nonorthogonal wavelets, since it is the biorthogonal system of the $\psi^{(-l)}(2^j x - k)$, see [6]. Denote by $h_f^l(x)$ the Hölder exponent of $f^{(-l)}$. These nonvanishing biorthogonal wavelet coefficients of $f^{(-l)}$ are thus

$$\tilde{C}_{j_n, k_n^m} = 2^{-(h_n(\lambda_n^m) + l(\beta_n(\lambda_n^m)+1)/(\beta_n(\lambda_n^m)+1))j_n}\,,$$

and the same argument as above yields

$$(16)\qquad h_f^l(x) \le \lim\,(h_n(x) + l\,(\beta_n(x)+1))\,.$$

4. Upper bound of the Hölder exponents.

Let now λ_n^m be the position of a non-vanishing wavelet coefficient at the scale 2^{-j_n}. This wavelet coefficient satisfies

$$|C_{j_n,k_n^m}| = 2^{-(h_n(\lambda_n^m)/(\beta_n(\lambda_n^m)+1))j_n},$$

which, using (8) and the mean-value theorem, is bounded by

$$2^{-(h_n(x)/\beta_n(x)+1))j_n}\, 2^{j_n^2|x-\lambda_n^m|}.$$

Since in Corollary 1 we only have to consider the coefficients such that $|x-\lambda_n^m| \le 2^{-j/(\log j)^2}$, it follows that $j_n^2\,|x-\lambda_n^m| \le 4$ and

$$|C_{j_n,k_n^m}| \le 16.2^{-(h_n(x)/(\beta_n(x)+1))j_n}.$$

Furthermore, using (11) and (13)

$$|x-\lambda_n^m| \ge \inf\Big\{\frac{C}{j_n^2}\,2^{-(1/(\beta_n(\lambda_n^{k-1})+1))j_n}, \frac{C}{j_n^2}\,2^{-(1((\beta_n(\lambda_n^k)+1))j_n}\Big\},$$

which, using the same argument as above, is larger than

$$\frac{C}{j_n^2}\,2^{-(1/(\beta_n(x)+1))j_n}.$$

Applying Corollary 1, we obtain

$$h_f(x) \ge \lim h_n(x) = h(x).$$

We have thus obtained that, if $x \notin E$, $h_f(x) = h(x)$.

Using again that the biorthogonal wavelet coefficients of $f^{(-l)}$ are

$$\tilde C_{j_n,k_n^m} = 2^{-(h_n(\lambda_n^m)+l(\beta_n(\lambda_n^m)+1)/(\beta_n(\lambda_n^m)+1))j_n},$$

the same argument as above yields

$$h_f^l(x) \ge \lim\,(h_n(x)+l\,(\beta_n(x)+1)). \tag{17}$$

So, at every point $x \notin E$, and for every l, the Hölder coefficient of $f^{(-l)}$, a l-th iterated primitive of f, is exactly

$$h_f^l(x) = \lim\,(h_n(x)+l\,(\beta_n(x)+1)),$$

it follows that $\beta_f(x) = \lim \beta_n(x)$, and the theorem is proved.

We now consider the case where $\beta(x) = 0$ and $x \in E$. In this case we go back to (15), which is still true. The proof for the upper and lower bounds of the Hölder exponents of f and $f^{(-l)}$ remain exactly the same, except for the lower bound bound of $2^{-j_n} + |\lambda_n^k - x|$ which was obtained in (11) using the fact that $x \notin E$, and is now crudely replaced by 2^{-j_n}. The same calculations as above then yield $h_f(x) = h(x)$ and $h_f^l(x) = h(x) + l$, so that $\beta_f(x) = 0$.

Let us now show that Proposition 2 is a consequence of Theorem 2 (which will be proved in the next section). We suppose that $h(x) = H > 0$ and $\beta(x) = B > 0$ almost everywhere.

Let $A < 1/(1+B)$ and $h > H$. Using Proposition 3, applied to f and its primitives, it follows that for almost every x there exists a sequence $j_n \longrightarrow \infty$ and k_n such that

$$|x - k_n\, 2^{-j_n}| \leq 2^{-Aj_n} \qquad \text{and} \qquad |C_{j_n,k_n}| \geq 2^{-hj_n}\,.$$

Thus almost every x belongs to an infinite number of the intervals $[k\,2^{-j} - 2^{-Aj}, k\,2^{-j} + 2^{-Aj}]$, where j and k are such that $|C_{j,k}| \geq 2^{-hj}$. Let $C > A$ and denote by E_C the set of points which belong to an infinite number of intervals $[k\,2^{-j} - 2^{-Cj}, k\,2^{-j} + 2^{-Cj}]$, with $|C_{j,k}| \geq 2^{-hj}$. It follows from Theorem 2 that E_C has Hausdorff dimension at least A/C. But if $x \in E_C$, $\beta(x) \leq (1/C) - 1$. The result follows since A and B satisfy

$$A < \frac{1}{1+B} < C$$

but can be chosen arbitrarily close to each other.

5. A priori lower bounds of the dimension of "approximation-type" fractals.

The idea of the proof of Theorem 2 is to construct a generalized Cantor set K included in E_b and simultaneously a probability measure μ supported by this Cantor set, with specific scaling properties. The "mass distribution principle" will allow us to deduce from these scaling properties a lower bound for the $\mathcal{H}^{h_{a/b}}$ Hausdorff measure of E_b. The Cantor set and the measure will be constructed using an iterative procedure.

After perhaps reordering the sequence $(\lambda_n, \varepsilon_n)$, we can suppose that ε_n is non-increasing. Let $b > a$ fixed. We introduce the notations

$$I_n = [\lambda_n - \varepsilon_n^a, \lambda_n + \varepsilon_n^a]$$

and

$$\widetilde{I}_n = [\lambda_n - \varepsilon_n^b, \lambda_n + \varepsilon_n^b]\,.$$

(More generally, If I is the interval $[\lambda - e, \lambda + e]$, $\widetilde{I}$ will denote the interval $[\lambda - e^{b/a}, \lambda + e^{b/a}]$.)

We now construct the first generation of the intervals of the cantor set K. First we will select a finite subsequence $I_{\phi(n)}$ of I_n as follows. Denote by $5\,I_n$ the interval of same center as I_n and of width $5\,|I_n|$. We first choose $\phi(1) = 1$ (*i.e.*, we select I_1); $\phi(2)$ is the first index such that $I_{\phi(2)}$ is not included in $5\,I_{\phi(1)}$; $\phi(3)$ is the first index such that $I_{\phi(3)}$ is not included in $5\,I_{\phi(1)} \cup 5\,I_{\phi(2)}, \ldots$ We stop this extraction at the first index N such that

$$\text{mes}\Big(\bigcup_{i=1}^{N} 5\,I_{\phi(i)}\Big) \geq \frac{1}{2} \tag{18}$$

(where mes$\,(A)$ denotes the Lebesgue measure of A). The index N exists because each interval I_n which has not been selected among the $I_{\phi(i)}$ is included in one of the $5\,I_{\phi(i)}$ previously selected (because ε_n is decreasing), so that

$$\bigcup_{i=1}^{\phi(N)} 5\,I_i \subset \bigcup_{i=1}^{N} 5\,I_{\phi(i)}\,. \tag{19}$$

Since almost every x belongs to E_a, mes$\,(\bigcup_{i=1}^{n} I_i) \longrightarrow 1$, and (18) follows if N is large enough.

By construction, the intervals $I_{\phi(i)}$ thus selected are disjoint, and (18) implies that

$$\text{mes}\Big(\bigcup_{i=1}^{N} I_{\phi(i)}\Big) \geq \frac{1}{10}\,. \tag{20}$$

The N intervals $\widetilde{I}_{\phi(i)}$ are the first generation intervals of our Cantor set. The measure μ will be supported by the union of these intervals, and we take

$$\mu(\widetilde{I}_{\phi(i)}) = \frac{|I_{\phi(i)}|}{\sum_{j=1}^{N} |I_{\phi(j)}|}\,, \qquad \text{for all } i\,.$$

(20) implies that

$$\mu(\widetilde{I}_{\phi(i)}) \leq 10\,|\widetilde{I}_{\phi(i)}|^{a/b}\,. \tag{21}$$

We will now construct the second generation intervals by subdivising each $\widetilde{I}_{\phi(i)}$. Let n be such that

$$\frac{1}{\varepsilon_n} \geq \exp\Big(\frac{1}{\varepsilon_{\phi(N)}}\Big)\,. \tag{22}$$

Let us consider one of the intervals $\widetilde{I}_{\phi(i)}$; since $\cup_{j\geq n} I_j$ covers almost every point of $\widetilde{I}_{\phi(i)}$, we can as above select a finite number of intervals $I_{\phi(i,1)}, \ldots, I_{\phi(i,N(i))}$ from the sequence $(I_j)_{j\geq n}$ such that

$$\operatorname{mes}\Big(\bigcup_{j=1}^{N(i)} 5\,I_{\phi(i,j)}\Big) \geq \frac{1}{2}\,|\widetilde{I}_{\phi(i)}|\,.$$

The $I_{\phi(i,j)}$ are disjoint, so that

$$\operatorname{mes}\Big(\bigcup_{j=1}^{N(i)} I_{\phi(i,j)}\Big) \geq \frac{1}{10}\,|\widetilde{I}_{\phi(i)}|\,.$$

The intervals $\widetilde{I}_{\phi(i,j)}$ are the second generation intervals in the construction of K, and we take

$$\mu(\widetilde{I}_{\phi(i,j)}) = \mu(\widetilde{I}_{\phi(i)})\,\frac{|I_{\phi(i,j)}|}{\displaystyle\sum_{j=1}^{N(i)} |I_{\phi(i,j)}|}\,. \tag{23}$$

Thus

$$\mu(\widetilde{I}_{\phi(i,j)}) \leq 10\,|\widetilde{I}_{\phi(i,j)}|^{a/b}\,\frac{\mu(\widetilde{I}_{\phi(i)})}{|\widetilde{I}_{\phi(i)}|}\,. \tag{24}$$

This construction is iterated, and we thus obtain a generalized Cantor set K, and a probability measure μ supported by K.

The intervals thus constructed at each generation are called the *fundamental intervals* of the Cantor set. Note that the fundamental intervals constructed are indexed by a tree, and the lengths of the intervals at a given depth of the tree need not be of the same order of magnitude. If I is a fundamental interval, we will denote by $\widehat{I}$ the "father" of I, *i.e.*, the fundamental interval from which I was directly obtained.

The lengths of the fundamental intervals have been chosen such that, if I is any fundamental interval of the n-th generation,

$$\frac{1}{|I|} \geq \exp\left(\sup\left(\frac{1}{|J|}\right)\right), \tag{25}$$

where the supremum is taken on all fundamental intervals J of the previous generation.

We will now check that, if I is an arbitrary open interval,

$$\mu(I) \leq C\,|I|^{a/b}\,(\log|I|)^2\,, \tag{26}$$

following [4, Principle 4.2], the Hausdorff measure of E_b constructed with the dimension function $h_{a/b}$ will then be positive.

We first check that (26) holds for the fundamental intervals, by induction on the generation of the interval; (21) asserts that it is true for the first generation. Suppose now that I is any interval of the n-th generation. The analogue of (24) at the n-th generation states that

$$\mu(I) \leq 10\,|I|^{a/b}\,\frac{\mu(\widehat{I})}{|\widehat{I}|}\,,$$

which, using the induction hypothesis, is bounded by

$$10\,|I|^{a/b}\,|\widehat{I}|^{(a/b)-1}\,(\log|\widehat{I}|)^2\,,$$

which, because of (25), is bounded by $10\,|I|^{a/b}\,|\log|I|\,|\,|\log(\log(|I|))^2$. Thus (26) holds for the intervals of generation n.

Let now I be an arbitrary open interval. If I does not intersect the Cantor set, $\mu(I) = 0$. Else, I contains fundamental intervals. Denote by $\widetilde{L}_1, \ldots, \widetilde{L}_p$ the fundamental intervals of smallest generation included in I; I intersects at most two more fundamental intervals of the same generation, which we denote by $\widetilde{L}_0$ and $\widetilde{L}_{p+1}$. All these fundamental intervals share either one or two fathers.

First case. We suppose that they share two fathers; for instance $\widetilde{L}_0, \ldots,$ $\widetilde{L}_k$ are the sons of $\widetilde{M}_1$ and $\widetilde{L}_{k+1}, \ldots, \widetilde{L}_{p+1}$ are the sons of $\widetilde{M}_2$. Denote by J the interval between $\widetilde{M}_1$ and $\widetilde{M}_2$; the definition of $\widetilde{I}_n$ implies that the gap between two fundamental intervals is much wider than these intervals, so that

$$|I| \geq |J| \geq |\widetilde{M}_1| + |\widetilde{M}_2|\,,$$

and thus, since (26) holds for fundamental intervals,

$$\begin{aligned} \mu(I) &\leq \mu(\widetilde{M}_1) + \mu(\widetilde{M}_2) \\ &\leq C\,|\widetilde{M}_1|^{a/b}\,(\log|\widetilde{M}_1|)^2 + C\,|\widetilde{M}_2|^{a/b}\,(\log|\widetilde{M}_2|)^2 \\ &\leq 2\,C\,|I|^{a/b}\,(\log|I|)^2\,. \end{aligned}$$

Second case. We suppose that $\widetilde{L}_0, \ldots, \widetilde{L}_{p+1}$ share a common father $\widetilde{M}$. If $\widetilde{L}_0$ and $\widetilde{L}_{p+1}$ do exist, we will write I as a union of three intervals I_1, I_2 and I_3. Suppose that $\widetilde{L}_0 = [a_0, b_0], \ldots, \widetilde{L}_{p+1} = [a_{p+1}, b_{p+1}]$. We take

$$I_1 = I \bigcap \Big[a_0, \frac{b_0 + a_1}{2}\Big]\,,$$

$$I_2 = I \bigcap \Big[\frac{b_0 + a_1}{2}, \frac{b_p + a_{p+1}}{2}\Big]\,,$$

$$I_3 = I \bigcap \Big[\frac{b_p + a_{p+1}}{2}, b_{p+1}\Big]\,.$$

$|I_1| \geq |\widetilde{L}_0|$ (we use again the fact that the gap between two fundamental intervals is much wider than these intervals), and $\mu(I_1) \leq \mu(\widetilde{L}_0)$; thus (26) holds for I_1 because it holds for $\widetilde{L}_0$. For the same reason, (26) holds for I_3. The conclusion will follow if we check that (26) holds for I_2. In the following, the only assumption we make on I_2 is that it includes $\widetilde{L}_1, \ldots, \widetilde{L}_p$, in order to cover the cases where $\widetilde{L}_0$ or $\widetilde{L}_{p+1}$ do not exist. We separate two cases:

If $p = 1$. $\widetilde{L}_1 \subset I_2$ and $\mu(\widetilde{L}_1) = \mu(I_2)$; thus (26) holds for I_2 because it holds for $\widetilde{L}_1$.

If $p \geq 2$. Since I_2 contains the intervals between $\widetilde{L}_i$ and $\widetilde{L}_{i+1}$ for $i = 1, \ldots, p-1$, it follows that

$$|I_2| \geq \frac{1}{4} \sum_{i=1}^{p} |L_i|\,. \tag{27}$$

We denote by $\widetilde{L}_1, \dots, \widetilde{L}_n$ $(n \geq p)$ all the intervals sons of $\widetilde{M}$. Since

$$\sum_{i=1}^{n} |L_i| \geq \frac{1}{10}\, |\widetilde{M}|\,,$$

(23), rewritten for $\widetilde{M}$, implies that

$$\mu(\widetilde{L}_i) \leq 10\, \frac{|L_i|\, \mu(\widetilde{M})}{|\widetilde{M}|}\,, \qquad \text{for all } i\,.$$

Thus

$$\mu(I_2) = \mu(\widetilde{L}_1) + \cdots + \mu(\widetilde{L}_p) \leq 10\, \frac{|L_1| + \cdots + |L_p|}{|\widetilde{M}|}\, \mu(\widetilde{M}) \leq 40\, \frac{|I_2|}{|\widetilde{M}|}\, \mu(\widetilde{M})\,,$$

using (27). Since $\mu(\widetilde{M}) \leq C\, |\widetilde{M}|^{a/b} (\log |\widetilde{M}|)^2$, we obtain

$$\mu(I_2) \leq C\, |I_2|\, |\widetilde{M}|^{(a/b)-1} (\log |\widetilde{M}|)^2 \leq C\, |I_2|\, |I_2|^{(a/b)-1} (\log |I_2|)^2\,,$$

because $|I_2| \leq |\widetilde{M}|$, and $(a/b) - 1 < 0$.

It follows that the measure μ thus constructed is a probability measure supported by a subset of E_b and satisfies, for any interval I,

$$\mu(I) \leq C\, (\log |I|)^2\, |I|^{a/b}\,,$$

so that, following [4, Principle 4.2], the Hausdorff measure of E_b constructed with the dimension function $h_{a/b}$ is positive.

Appendix. Proof of Proposition 3.

Suppose that f belongs to $C^\alpha(x_0)$. Then

$$\begin{aligned}
|C_{j,k}| &= \Big| \int f(x)\, 2^j\, \psi(2^j x - k)\, dx \Big| \\
&= \Big| \int (f(x) - P(x - x_0))\, 2^j\, \psi(2^j x - k)\, dx \Big| \\
&\leq C \int |x - x_0|^\alpha\, \frac{2^j}{(1 + 2^j\, |x - k\, 2^{-j}|)^N}\, dx \\
&\leq C\, 2^j \int \frac{|x - k\, 2^{-j}|^\alpha + |k\, 2^{-j} - x_0|^s}{(1 + 2^j\, |x - k\, 2^{-j}|)^N}\, dx \\
&\leq C\, 2^{-\alpha j}\, (1 + |2^j\, x_0 - k|^\alpha)\,, \qquad \text{if } N \geq [\alpha] + 2
\end{aligned}$$

(the second inequality is true because the wavelets have vanishing moments.) Let us now prove the converse result.

Let j_0 denote the integer such that

$$2^{-j_0-1} \le |x - x_0| < 2^{-j_0}\,,$$

let $j_1 = j_0^2$ and

$$f_j(x) = \sum_k c_{j,k}\,\psi(2^j x - k)\,.$$

From (4), using the localization of the wavelets, we deduce

$$|f_j(x)| \le C\,2^{-\alpha j}\,(1 + 2^j\,|x - x_0|)^\alpha\,, \tag{28}$$

and, since $f \in \mathcal{C}^{\log}$,

$$|f_j(x)| \le C\,2^{-j/\log j}\,. \tag{29}$$

Similarly, for any l, using the localization of the derivatives of the wavelets,

$$|f_j^{(l)}(x)| \le C\,2^{(l-s)j}\,(1 + 2^j\,|x - x_0|)^s\,. \tag{30}$$

If g is a smooth function, let $T(g)(x_0)$ be the Taylor expansion of g at the order $[\alpha]$ at x_0. Then

$$\begin{aligned} &|f(x) - T(f)(x_0)| \\ &\qquad \le \sum_{j \le j_0} |f_j(x) - T(f_j)(x_0)| + \sum_{j \ge j_0} |f_j(x)| + \sum_{j \ge j_0} |T(f_j)(x_0)|\,. \end{aligned}$$

Let $l = [\alpha] + 1$. Using (30), the first term is bounded by

$$C\,|x - x_0|^l \sum_{j \le j_0} \sup_{[x,x_0]} |f_j^l(x_0)| \le C\,|x - x_0|^l \sum_{j \le j_0} 2^{(l-\alpha)j} \le C\,|x - x_0|^\alpha\,.$$

As regards the second term, using (28),

$$\sum_{j_0 \le j < j_1} |f_j(x)| \le \sum_{j_0 \le j < j_1} |x - x_0|^\alpha \le C\,(j_1 - j_0)\,|x - x_0|^\alpha\,,$$

and using (29),

$$\sum_{j \ge j_1} |f_j(x)| \le \sum_{j \ge j_1} 2^{-j/\log j} \le C\,j_1\,2^{-j_1/\log j_1}\,.$$

By (30), the third term is bounded by

$$C \sum_{j \geq j_0} \sum_{m=0}^{[\alpha]} |x - x_0|^m \, 2^{(m-\alpha)j} \leq C \, |x - x_0|^{\alpha} \, .$$

Hence the converse part of the proposition, since

$$j_1 \leq C \Big(\log \Big(\frac{2}{|x - x_0|} \Big) \Big)^2 \, .$$

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