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Inverse problems in the theory of analytic planar vector fields

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Abstract. In this communication we state and analyze the new inverse problems in the theory of differential equations related to the construction of an analytic planar vector field from a given, finite number of solutions, trajectories or partial integrals.

Likewise we study the problem of determining a stationary complex analytic vector field Γ from a given, finite subset of terms in the formal power series

$$V(z,w) = \lambda\,(z^2+w^2) + \sum_{k=3}^{\infty} H_k(z,w)\,, \qquad H_k(a\,z, a\,w) = a^k H_k(z,w)\,,$$

and from the subsidiary condition

$$\Gamma(V) = \sum_{k=1}^{\infty} G_{2k}\,(z^2+w^2)^{k+1}\,,$$

where G_{2k} is the Liapunov constant. The particular case when

$$V(z,w) = f_0(z,w) - f_0(0,0)$$

and $(f_0, D \subset \mathbb{C}^2)$ is a canonic element in the neigbourhood of the origin of the complex analytic first integral F is analyzed. The results are applied to the quadratic planar vector fields. In particular we constructed the all quadratic vector field tangent to the curve

$$(y - q(x))^2 - p(x) = 0\,,$$

where q and p are polynomials of degree k and $m \leq 2\,k$ respectively. We showed that the quadratic differential systems admits a limit cycle of this tipe only when the algebraic curve is of the fourth degree. For the case when $k > 5$ it proved that there exist an unique quadratic vector field tangent to the given curve and it is Darboux's integrable.

1. Introduction.

We consider analytic planar vector fields or equivalent systems of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P(x, y, t)\,, \\ \dfrac{dy}{dt} = Q(x, y, t)\,. \end{cases} \tag{1.1}$$

We shall mainly be concerned here with real systems (1.1). In order to understand such systems it is however advisable to sometimes consider the natural extension of (1.1) to the complex system

$$\begin{cases} \dfrac{dz}{dz^0} = Z(z, w, z^0)\,, \\ \dfrac{dw}{dz^0} = W(z, w, z^0)\,. \end{cases} \tag{1.2}$$

The following representation is often used instead of (1.1) and (1.2)

$$\omega \equiv P\,dy - Q\,dx = 0\,,$$

$$\Omega \equiv W\,dz - Z\,dw = \Omega_1 + i\,\Omega_2 = 0\,.$$

In the theory of differential equations (1.1) (or (1.2)) two main problems can be studied:

I) Direct problem or problem of integration (1.1) (or (1.2)).

II) Inverse problem or problem of construction (1.1) (or (1.2)) from given properties.

Before solving the direct problem, the question as to what the integration of (1.1) means must be answered. If the given equations describe the behaviour of physical phenomena, then these can be seen to change over time.

By using the theorem of existence and unicity we can determine the evolution of the phenomena in the past and future by integration. Integrating the equations without complementary information about the real situation may lead to useless results. So if integration enables us to understand the process of finding the analytical expressions for the solutions, the following question immediately arises: What character and properties must the required expression have?. It is well known that the solutions to (1.1) can be expressed though elementary functions or integrals of such functions only in some exceptional cases.

The analytical expression of linear systems is well known. However, there are few physical systems which can be described by such models. If solutions can be found for non-linear systems, the formulae for expressing them are so complicated that they are practically impossible to study. The problem of integrating (1.1) can be stated with infinite formal series. The difficulties which arise have to do with the convergence of the series which is so slow as to be useless in most cases. Finally, the problems related to the approximate calculation of the solutions to the given equations are well known. These difficulties lead the specialist to state and solve another type of problem which is that of constructing differential equations from given properties. This sort of problems are called inverse problems in the theory of differential equations. Generally speaking, by an inverse problem one usually means the problem of constructing a mathematical object from given properties. In recent years this branch of mathematics has been developing in different directions, in particular in the field of differential equations.

One of the difficulties encountered when studying such questions is that of the high degree of arbitrariness but this can be remedied by introducing subsidiary conditions inspired by the physical nature of the phenomenon.

The first inverse problem of the differential equations was stated by Newton.

Book One of Newton's Philosophiae Naturalis Mathematica is totally dominated by the idea of determining the forces capable of generating planetary orbits of the solar system.

The problem of finding the forces which generate a given motion has played a dominant role in the history of dynamics from Newton's time to the present. In fact, this problem has been studied by Bertran, Suslov, Joukovski, Darboux, Danielli, Whittaker and recently by Galiullin [1], Szebehely [2], and their followers.

Of course, this problem is essentially a problem of construction differential equations of the second order with given properties.

Another fundamental inverse problem in this theory is that of to Eruguin, who stated the problem of constructing a system of differential equations from given integral curves [3]. This idea were futher developed in [1].

The aim of this communication is to developed the Eruguin's ideas and construct the planar analytical vector field from given solutions, trajectories, partial integrals, etc. The problem posed are illustrated in a specific case. In particular, we determine all the quadratic autonomous vector fields from the given algebraic curves of the genus 2.

2. Constructing an analytic planar vector field from a given finite number of solutions.

PROBLEM 2.1. Let us specify smooth functions

$$z_j = x_j + i\,y_j : I \subset \mathbb{R} \longrightarrow \mathbb{C}$$

$$t \longmapsto z_j(t) = x_j(t) + i\,y_j(t)\,, \qquad j = 1, M\,,$$

We want to construct a differential equation

$$F\Big(z, \overline{z}, t, \frac{dz}{dt}\Big) \equiv a\,(z, \overline{z}, t)\,\frac{dz}{dt} + f(z, \overline{z}, t) = 0\,, \tag{2.1}$$

where $z = x + i\,y$, $\overline{z} = x - i\,y$, in such a way that

$$z = z_j(t)\,, \qquad j = 1, 2, \dots, M \tag{2.2}$$

be its solutions.

Evidently, the sought after equation can be represented as follows: Let us denote by $\mathcal{D}$ the matrix

$$\mathcal{D} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z & z_1(t) & \dots & z_M(t) \\ \overline{z} & \overline{z}_1(t) & \dots & \overline{z}_M(t) \\ z^2 & z_1^2(t) & \dots & z_M^2(t) \\ |z|^2 & |z_1(t)|^2 & \dots & |z_M(t)|^2 \\ \overline{z}^2 & \overline{z}_1^2(t) & \dots & \overline{z}_M^2(t) \\ \vdots & \vdots & & \vdots \\ \overline{z}^n & \overline{z}_1^n(t) & \dots & \overline{z}_M^n(t) \\ \dfrac{dz}{dt} & \dfrac{dz_1(t)}{dt} & \dots & \dfrac{dz_M(t)}{dt} \end{pmatrix}\,, \tag{2.3}$$

where $(n+1)\,(n+2)/2 = M$.

Proposition 2.1. *The differential equation admitting* (2.2) *as its solutions can be represented as follows*

$$F\Big(z, \overline{z}, t, \frac{dz}{dt}\Big) = \det \mathcal{D} - \Phi(z, \overline{z}, t) = 0\,, \tag{2.4}$$

where Φ (*which we will call Eruguin's function*) *is an arbitrary function such that*

$$\Phi(z, \overline{z}, t)|_{z=z_j(t),\, \overline{z}=\overline{z}_j(t)} \equiv 0\,, \qquad j = 1, 2, \ldots, M\,.$$

As can be seen the arbitrariness of the equations obtained is high in relation to the function Φ, but this drawback can be removed with the help of some complementary conditions. In the paper [4] we studied the problem of constructing a stationary polynomial planar vector field

$$\frac{dz}{dt} \equiv \dot{z} = \sum_{j+k=n} a_{kj}\, z^j\, \overline{z}^k\,, \qquad a_{kj} \in \mathbb{C}\,,$$

from given solutions (2.2) and with evidently subsidiary conditions which enable us to solve (2.4) with respect to $\dot{z}$.

We have proposed a method for determining the Eruguin function in [4]. In order to illustrate Proposition 2.1 and this method we shall analyze the case when the sought after vector field is quadratic. We solve the simplest problem when the given solutions are the following $z = 0$ and $z = z_0 = \text{const} \neq 0$.

We determine the Eruguin function as linear combinations of elements of the matrix H_j which we define as follows

$$\begin{aligned} H_0(z, \overline{z}, t) &= \sum_{j+k=n} B_{jk}\, z^j\, \overline{z}^k\,, \\ H_j(z, \overline{z}, t) &= [H_{j-1}(z, \overline{z}, t), H_{j-1}(z_j(t), \overline{z}_j(t), t)]\,, \end{aligned} \tag{2.5}$$

where $j = 1, 2, \ldots, M$, B_{kj} is an arbitrary matrix of order s and $[A, B] = AB - BA$ is the Lie bracket of the matrices A and B. By introducing the vector

$$L(z, \overline{z}) = (1, z, \overline{z}, z^2, z\,\overline{z}, \overline{z}^2)\,,$$

we easily obtain, for our particular case, that the sought after quadratic vector field is such that

$$\frac{dz}{dt} = (L(z_0, \overline{z}_0), K L^T(z, \overline{z})) \,, \tag{2.6}$$

where by K we denote the antisymmetrical matrix

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & -\beta_1 & 0 & \beta_5 & \beta_6 & \beta_7 \\ 0 & -\beta_2 & -\beta_5 & 0 & \beta_8 & \beta_9 \\ 0 & -\beta_3 & -\beta_6 & -\beta_8 & 0 & \beta_{10} \\ 0 & -\beta_4 & -\beta_7 & -\beta_9 & -\beta_{10} & 0 \end{pmatrix} ,$$

where $\beta_j \in \mathbb{C}$. The equation (2.6) determines the required quadratic vector field with two critical points. The following particular case is of interest

$$\beta_j = \begin{cases} 0 \,, & \text{if } j \neq 4 \,, \\ a + i\, b \,, & \text{if } j = 4 \,. \end{cases}$$

and $z_0 = \varepsilon \in \mathbb{R}$. The above equation in this case take the form

$$\frac{dz}{dt} = -\beta_4 \, (\varepsilon^2 z - \varepsilon \, \overline{z}^2) \,,$$

or, what amounts to the same,

$$\begin{cases} \dot{x} = -\varepsilon \, (b \, \partial_y H + a \, (\varepsilon \, x + y^2 - x^2)) \,, \\ \dot{y} = \varepsilon \, (-b \, \partial_x H - a \, (\varepsilon \, y + 2 \, x \, y)) \,, \end{cases}$$

where

$$H = \frac{\varepsilon}{2} \, (x^2 + y^2) + x \, y^2 - \frac{1}{3} \, x^3 \,.$$

3. Constructing a planar vector field from a given complex analytic first integral.

In this section we shall study two problems related with the constructing of a vector field Γ such that

$$\begin{cases} \dfrac{dz}{dz^0} = -\Lambda \, w + Z(z, w) \,, \\ \dfrac{dw}{dt} = \Lambda \, z + W(z, w) \,, \end{cases}$$

where z and z^0 are complex variables, $\Lambda \in \mathbb{C}$, Z, W are polynomial functions in the variables z and w. The first problem is the following:

Problem 3.1. Let

$$V(z,w) = \frac{\Lambda}{2}\,(z^2+w^2) + \sum_{j=3}^{\infty} H_j(z,w)\,, \tag{3.2}$$

be a formal power series, where Λ is a nonzero complex parameter and H_j is a homogenous function of degree j.

The analytic vector field Γ need to be constructed in such a way that

$$\Gamma(V(z,w)) = \sum_{j=1}^{\infty} G_{2k}\,(z^2+w^2)^{k+1}\,,$$

where $G_{2k} \in \mathbb{C}$ are the Liapunov (complex) constants.

The second problem is a consequence of the Problem 3.1. Firstly we introduce the following concepts and notations [5].

Definition 3.1 *By a canonical element centered at the point* $a \in \overline{\mathbb{C}}^2$ *will be called a pair* (U_a, f_a)*, where* f_a *is the sum of a power series with its centre at* a *and* U_a *is the domain of convergence of the power series.*

Definition 3.2. *Two canonic elements* (U_a, f_a) *and* (V_a, g_a) *are said to be equivalent if* $f_a \equiv g_a$ *in the neighbourhood of* a.

Definition 3.3. *The complex analytic function* F *with domain* $\mathcal{D} \subset \mathbb{C}^2$ *will be called the set of canonic elements which can be generated from the canonic element* (U_a, f_a) *after analytic continuation along the whole path starting from the given point* $a \in U_a$.

Definition 3.4. [5] *The system* (3.1) *will be called integrable in the Liapunov sense* (*or Liapunov integrable*) *if and only if there is an analytic first integral* F *which contains the canonic element* (U_0, f_0) *with* f_0

$$f_0(z,w) = f_0(0,0) + \frac{\Lambda}{2}\,(w^2+z^2) + \sum_{j=3}^{\infty} H_j(z,w)\,,$$

where the H_j, $j = 3, 4, \ldots,$ *are homogenous functions of degree* j, *and* Λ *is a nonzero complex parameter* [5].

PROBLEM 3.2. To construct a Liapunov integrable polynomial vector field Γ of degree n such that

$$\left\{ \begin{aligned} \frac{dz}{dt} &= -\Lambda\, w + Z_n(z,w)\,, \\ \frac{dw}{dt} &= \Lambda\, z + W_n(z,w)\,, \end{aligned} \right. \tag{3.4}$$

where by Z_n and W_n we denote a polynomial function of degree $n > 1$ in of the variables z and w.

We find the solutions to these problem for $n = 2$ and $n = 3$, while for $n > 3$ solutions are found by in an analogous manner.

Proposition 3.1. *Let us suppose that the function H_3 is such that*

$$\{H_2,\, H_3\} \equiv \partial_z H_2\, \partial_w H_3 - \partial_w H_2\, \partial_z H_3 \not\equiv 0\,.$$

Then the sought after quadratic stationary vector field Γ can be represented as follows

$$\Gamma_2 = \{H,\ \} + g_1\{\ , H_2\}\,, \tag{3.5}$$

if this condition holds

$$\Gamma_2(H_{2k} + H_{2k+1}) = G_{2k}\,(z^2 + w^2)^k\,,$$

or, what amounts to the same,

$$\left\{ \begin{aligned} &\{H_2, H_{2k+1}\} + \{H_3, H_{2k}\} + g_1\{H_{2k}, H_2\} = 0\,, \\ &\{H_2, H_{2k+2}\} + \{H_3, H_{2k+1}\} + g_1\{H_{2k+1}, H_2\} \\ &\qquad\qquad\qquad = G_{2k+2}\,(z^2 + w^2)^{k+1}\,, \end{aligned} \right. \tag{3.6}$$

where H_2, H are functions such that

$$H_2(z,w) = \frac{\Lambda}{2}\,(z^2 + w^2)\,,$$
$$H(z,w) = H_2(z,w) + H_3(z,w)\,.$$

CONSEQUENCE 3.1. The Liapunov constants G_{2k} for the quadratic vector field thus constructed can be calculated by the formulas:

$$G_{2k+2} = \frac{1}{2\pi} \int_0^{2\pi} \left(- g_1(z,w) \{H_2, H_{2k+1}\} + \{H_3, H_{2k+1}\} \right)\Big|_{\substack{z=\cos t \\ w=\sin t}} dt \,,$$

where $k \in \mathbb{N}$. From the above results we can deduce the following consequence

CONSEQUENCE 3.2. Let us give the functions H_2, H_3, H_4 and the Liapunov constant G_4.

Then we can construct:

i) the quadratic vector field Γ_2,

ii) all members of the formal power series $\sum_{k=5}^{\infty} H_k(z,w)$, and

iii) the Liapunov constants G_{2k+2}, $k = 2, 3, 4, \ldots$

In order to illustrate these assertions, we shall study the following particular case. Let H_2, H_3, H_4 and G_4 be such that

$$\begin{cases} H_2 = \dfrac{\Lambda}{2} (z^2 + w^2) \,, \\ H_3 = \dfrac{1}{3} ((a_6 + a_4)\, w^3 - (a_2 + a_5)\, z^3) + a_2\, z\, w^2 - a_3\, z^2\, w \,, \\ H_4 = \dfrac{1}{4} (a_4\, (a_3 + a_4 + a_6) - a_5 (2\, a_2 + a_5)\, z^4) - a_2\, a_4\, z\, w^3 \,, \\ G_4 = \dfrac{1}{8}\, a_5\, (a_3 - a_6) \,, \end{cases}$$

where a_2, a_3, a_4, a_5, a_6 are some complex parameters.

The sought after quadratic vector field can be represented as follows

$$\begin{cases} \dfrac{dz}{dt} = -\partial_w H^* - a_5\, z\, w \,, \\ \dfrac{dw}{dt} = \partial_z H^* - a_4\, z\, w \,, \end{cases}$$

where

$$H^* = \frac{\Lambda}{2} (z^2 + w^2) + \frac{a_2}{3}\, z^3 + a_3\, z^2\, w - a_2\, z\, w^2 - \frac{a_6}{3}\, w^3 \,.$$

By using computer techniques it is easy to obtain the expression for all the terms of the power series and the Liapunov constants from the above formulas.

CONSEQUENCE 3.3. Let us suppose that the functions H_{2k} and H_{2k+1} are such that

$$\{H_2, H_{2k}\} \not\equiv 0\,, \qquad \{H_2, H_{2k+1}\} \not\equiv 0\,,$$

so we have the following relations

$$\begin{aligned} g_1(z,w) &= -\frac{\{H_2, H_{2k+1}\} + \{H_3, H_{2k}\}}{\{H_{2k}, H_2\}} \\ &= \frac{\{H_2, H_{2k+2}\} + \{H_3, H_{2k+1}\} - G_{2k+2}\,(z^2+w^2)^{k+1}}{\{H_2, H_{2k+1}\}}\,, \end{aligned} \tag{3.7}$$

where $k \in \mathbb{N}$. Likewise we can deduce the following result for cubic vector fields.

Proposition 3.2. *Let H_4 be a function such that*

$$\{H_4, H_2\} \not\equiv 0\,. \tag{3.8}$$

Then the cubic vector field Γ admits the representation below

$$\Gamma_3 = \nu(z,w)\,\{H,\ \} + g_2\,\{\ , H_2\}\,, \tag{3.9}$$

if the following relation holds

$$\begin{cases} \{H_2, H_3\} = 0\,, \\ \nu(z,w)\,\{H_{2k+1}, H_4\} + g(z,w)\,\{H_2, H_{2k+3}\} = 0\,, \\ \nu(z,w)\,\{H_{2k}, H_4\} + g(z,w)\,\{H_2, H_{2k}\} + \{H_2, H_{2k+2}\} \\ \qquad\qquad = -G_{2k+2}\,(z^2+w^2)^{k+1}\,, \end{cases}$$

where $H = H_2 + H_4$.

As an immediate consequence we find that all functions H_{2k+1} are equal to zero. Formulas analogous to (3.7) can be deduced.

From (3.10) we easily deduce that the function ν is such that

$$\nu(z,w)\,\{H_4, H_2\} + \{H_2, H_4\} = -G_4\,(z^2+w^2)^2\,.$$

Proposition 3.3. Let us suppose that the formal power series is such that

$$V(z,w) = \sum \frac{a_k}{k}\,(z^2+w^2)^k \equiv \rho(r^2)\,.$$

So the sougth after analytic vector field can be rewritten as follows

$$\begin{cases} \dfrac{dz}{dt} = \Lambda(z,w)\,w + \mathcal{R}(r^2)\,z\,, \\ \dfrac{dw}{dt} = -\Lambda(z,w)\,z + \mathcal{R}(r^2)\,w\,, \end{cases} \tag{3.11}$$

where Λ is an arbitrary analytic function and $\mathcal{R}$ is a function

$$\mathcal{R}(r^2) = \frac{r^2 \displaystyle\sum_{k=0}^{\infty} G_{2k}\, r^{2k}}{\partial_{r^2}\rho(r^2)}\,.$$

Likewise we can study the problem of constructing a polynomial vector field of degree n. In order to illustrate these ideas we shall analyze the following specific case.

Let us give the functions H_k, $k = 2, 3, \dots, n+1$, such that

$$\begin{cases} H_2(z,w) = \dfrac{1}{2}\,(z^2+w^2)\,, \\ H_j(z,w) = 0\,, & j = 3,\dots,n\,, \\ H_{n+1}(z,w) = \dfrac{1}{2}\,(c\,(b\,w^{n+1} + a\,z^{n+1}))\,, & c,b,a \in \mathbb{C}\,, \end{cases}$$

and let us suppose that $G_{2n} = G_{2n+2} = 0$.

We wish to construct the polynomial vector field of degree n.

We obtain the solutions to this problem in the same way as in the above problem. Firstly it is easy to find that

$$\nu(z,w) = 1\,,$$
$$g_{n-1}(z,w) = \frac{2\,n\,(c-1)}{n+1}\,(a\,z^{n-1} + b\,w^{n-1})\,,$$
$$H(z,w) = H_2(z,w) + H_{n+1}\,.$$

So the sought after vector field is

$$\begin{cases} \dfrac{dz}{dt} = -w - A\,w^n + B\,w\,z^{n-1}\,, \\ \dfrac{dw}{dt} = z + A\,z^n - B\,z\,w^{n-1}\,, \end{cases}$$

where

$$A = \frac{b\,(c\,(n+1)^2 + 4\,n)}{2\,n+2}\,, \qquad B = \frac{2\,a\,n\,(c-1)}{n+1}\,.$$

For $n = 2\,m+1$ we observe that the system obtained has the symmetry $(z, w, t) \longrightarrow (-z, w, -t)$ and $(z, w, t) \longrightarrow (z, -w, -t)$, *i.e.*, it is reversible. As a consequence there is an analytic first integral.

It is interesting to observe that the complex analytic function

$$V(z, w) = z^2\,(1 + a\,z^{n-1})^c + w^2\,(1 + b\,w^{n-1})^c$$

has the canonic element (f_0, U_0) such that

$$f_0(z, w) = z^2 + w^2 + c\,(a\,z^{n+1} + b\,w^{n+1}) + c\,(c-1)\,(a^2\,z^{2n} + b^2\,w^{2n}) + \cdots$$

The solution to this Problem 3.2 can easily be obtained from the solution to Problem 3.1, by considering the complementary condition that the Liapunov constants are zero in this case.

Lunkevich and Sibirski determine the first integral for a quadratic planar vector field with its center at the origin (see [7]). It is easy to show that these quadratic systems are Liapunov integrable (see [5]).

In order to illustrate the solution to the Problem 3.2 we shall analyze the problem of constructing a quadratic vector field from a given Lunkevich-Sibirski first integral.

We shall only study the case below. The others case can be done analogously.

Firstly, we shall suppose that we have a complex analytic integral

$$V(z, w) = \exp\,(-2\,w)\,(2\,z^2 + 2\,(b-1)\,w + 2\,b\,w^2 + b - 1)\,, \qquad b \in \mathbb{C}.$$

The canonic element in the neighbourhood of the origin is the following

$$\begin{cases} U_0 = \mathbb{C}^2\,, \\ f_0(z, w) = b - 1 + 2\,(z^2 + w^2) - \dfrac{4}{3}\,(2+b)\,w^3 - 4\,w\,z^2 \\ \qquad\qquad + 4\,z^2\,w^2 + 2\,(1+b)\,w^4 + \cdots \end{cases}$$

For this case it is easy to deduce that

$$\begin{cases} g_1(z, w) = \dfrac{\{H_4, H_2\}}{\{H_2, H_3\}} = 2\,w\,, \\ H(z, w) = \dfrac{1}{2}\,(z^2 + w^2) - \dfrac{1}{3}\,(2+b)\,w^3 - w\,z^2\,. \end{cases}$$

As a consequence, we obtain the following representation for the require quadratic vector field

$$\begin{cases} \dfrac{dw}{dt} = \partial_z H(z,w) + 2\,z\,w = z\,, \\ \dfrac{dz}{dt} = -\partial_w H(z,w) - 2\,w^2 = -w + z^2 + b\,w^2\,. \end{cases}$$

We shall now analyze the specific case when the complex analytic first integral V is given by the formula

$$V(z,w) = (1+2\,a\,w)^{a-1}\big(b+3\,a-1+2\,(a-1)(2\,a-1)\,(b\,w^2-(3\,a-1)\,z^2) \\ -2\,(a-1)\,(b+3\,a-1)\,w\big)^a\,,$$

where $a, b \in \mathbb{C}$.

The canonic element of the given analytic function is such that

$$\begin{aligned} &f_0(z,w) \\ &= T\Big((b+3\,a-1)^2 - 2\,(b+3\,a-1)\,(z^2+w^2) + \frac{4}{3}\,(b+2)\,w^3 \\ &\quad - 4\,(a-1)\,w\,z^2 + 2\,(6\,a^4 - 17\,a^3 + 14\,a^2 - 9\,a - 4\,a\,b + 2 - b^2)\,w^4 \\ &\quad + 2\,(a-1)^2\,(2\,a-1)^2\,(3\,a-1)\,z^4 \\ &\quad + 4\,(a-1)\,(6\,a^3 - 11\,a^2 + 9\,a + b - 2)\,z^2\,w^2\Big) + \cdots\,, \end{aligned}$$

where

$$T \equiv (b+3\,a-1)^{a-1}\,a\,(a-1)\,(2\,a-1)\,(3\,a-1) \neq 0\,.$$

By using the proposed method we can deduce the well known quadratic vector field

$$\text{(3.12)} \qquad \begin{cases} \dfrac{dw}{dt} = z + 2\,a\,w\,z\,, \\ \dfrac{dz}{dt} = -w + b\,w^2 + (1-a)\,z^2\,. \end{cases}$$

The integrability of the case when $a\,(a-1)\,(2\,a-1)\,(3\,a-1) = 0$ was deduced in [7]. The integrability of the case when $b + 3\,a - 1 = 0$ is easy to obtain (see [5]). The analytic first integral V and its canonic element are such that

$$V(z,w) = (1+2\,a\,w)^{(a-1)/a}\,(z^2+w^2)\,,$$
$$f_0(z,w) = z^2+w^2+2\,(a-1)\,(w^3+z^2w-w^4-z^2\,w^2)+\cdots$$

4. Constructing a vector field with given trajectories.

In [4] we stated and solved the following problem

Problem 4.1 *Let*

$$\begin{aligned} w_j : \mathcal{D}\subset\mathbb{C} &\longrightarrow \mathbb{C}\\ z &\longmapsto w_j(z)\,, \qquad j=1,\dots,M\,, \end{aligned}$$

be a holomorphic function on $\mathcal{D}$ such that

$$K = \det\begin{pmatrix} 1 & 1 & \dots & 1\\ w_1(z) & w_2(z) & \dots & w_M(z)\\ {w_1}^2(z) & w_2^2(z) & \dots & w_M^2(z)\\ \vdots & \vdots & & \vdots\\ w_1^{M-1}(z) & w_2^{M-1}(z) & \dots & w_M^{M-1}(z) \end{pmatrix} \tag{4.1}$$

is identically nonvanishing on $\mathcal{D}$.

We need to construct an analytic vector field on $\mathcal{D}^*\subset\mathbb{C}^2$

$$\begin{cases} \dfrac{dz}{dt} = P(z,w)\,,\\[2mm] \dfrac{dw}{dt} = Q(z,w)\,, \end{cases} \tag{4.2}$$

in such a way that

$$w = w_j(z)\,, \qquad j=1,\dots,M\,, \tag{4.3}$$

are its trajectories. We deduced the solution to this problem from the equality

$$\det\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ w & w_1(z) & w_2(z) & \dots & w_M(z) \\ w^2 & w_1^2(z) & w_2^2(z) & \dots & w_M^2(z) \\ \vdots & \vdots & \vdots & & \vdots \\ w^{M-1} & w_1^{M-1}(z) & w_2^{M-1}(z) & \dots & w_M^{M-1}(z) \\ \dfrac{dw}{dz} & \dfrac{dw_1(z)}{dz} & \dfrac{dw_2(z)}{dz} & \dots & \dfrac{dw_M(z)}{dz} \end{pmatrix} = g(z,w)\det S\,, \tag{4.4}$$

where by S we denote the following matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ w & w_1(z) & w_2(z) & \dots & w_M(z) \\ w^2 & w_1^2(z) & w_2^2(z) & \dots & w_M^2(z) \\ \vdots & \vdots & \vdots & & \vdots \\ w^{M-1} & w_1^{M-1}(z) & w_2^{M-1}(z) & \dots & w_M^{M-1}(z) \\ w^M & w_1^M(z) & w_2^M(z) & \dots & w_M^M(z) \end{pmatrix}.$$

g is an arbitrary analytic function on $\mathcal{D}^*$. From (4.4) we obtain the following expression for the most general vector field admitting the given curves as trajectories.

$$\begin{cases} \dfrac{dz}{dt} = \det A \equiv P\,, \\ \dfrac{dw}{dt} = \det B \equiv Q\,, \end{cases} \tag{4.5}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ w & w_1(z) & w_2(z) & \dots & w_M(z) \\ w^2 & w_1^2 & w_2^2 & \dots & w_M^2 \\ \vdots & \vdots & \vdots & & \vdots \\ w^{M-1} & w_1^{M-1} & w_2^{M-1} & \dots & w_M^{M-1} \\ K_1(z,w) & K_2(z,w) & K_3(z,w) & \dots & K_M(z,w) \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ w & w_1(z) & w_2(z) & \dots & w_M(z) \\ w^2 & w_1^2 & w_2^2 & \dots & w_M^2 \\ \vdots & \vdots & \vdots & & \vdots \\ w^{M-1} & w_1^{M-1} & w_2^{M-1} & \dots & w_M^{M-1} \\ g_2(z,w)\, w^M & h_1(z,w) & h_2(z,w) & \dots & h_M(z,w) \end{pmatrix}$$

and

$$h_j = -\nu(z,w)\, \frac{dw_j}{dz} + g_2(z,w)\, w_j^M\,, \qquad j = 1, 2, \dots, M\,,$$

$$K_1(z,w) = \nu(z,w) + g_1(z,w)\, w^M\,,$$

$$K_j(z,w) = g_1(z,w)\, w_j^M\,, \qquad j = 2, 3, \dots, M\,.$$

In particular for $M = 2$ and

$$\begin{cases} w_1(z) = q(z) + \sqrt{p(z)}\,, \\ w_2(z) = q(z) - \sqrt{p(z)}\,, \end{cases}$$

we easily deduce the differential equations

$$(4.6) \qquad \begin{cases} \dfrac{dz}{dt} = 2\, p(z)\, \nu^*(z,w) + \alpha(z,w)\, ((w - q(z))^2 - p(z))\,, \\ \dfrac{dw}{dt} = \nu^*(z,w) \Big(w\, \dfrac{dp(z)}{dz} + 2\, \dfrac{dq(z)}{dz}\, p(z) - q(z)\, \dfrac{dp(z)}{dz} \Big) \\ \qquad\quad + \beta(z,w)\, ((w - q(z))^2 - p(z))\,, \end{cases}$$

where $\nu = \sqrt{p(z)}\, \nu^*$, α and β are arbitrary analytic functions on $\mathcal{D}^*$.

By changing $w - q(z) \longrightarrow w$ in (4.6) we deduce the following formulas

$$(4.7) \qquad \begin{cases} \dfrac{dz}{dt} = \nu(z,w)\, \partial_w f(z,w) + \lambda_1(z,w)\, f(z,w)\,, \\ \dfrac{dw}{dt} = -\nu(z,w)\, \partial_z f(z,w) + \lambda_2(z,w)\, f(z,w)\,, \end{cases}$$

where $f(z,w) = w^2 - p(z)$ and λ_j, $j = 1, 2$, are arbitrary holomorphic functions.

The specific case when the given trajectories are conic

$$q(z) = \frac{a_1\, z + a_2}{2}\,,$$

$$p(z) = b_1\, z^2 + 2\, b_2\, z + b_3\,, \qquad a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}\,,$$

was analyzed in [4] and [8].

For the subcase when $a_j = 0$, $j = 1, 2$ and $b_1 = 1$, $b_2 = 0$, $b_3 = 1$, we obtain the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = -2\,(z^2 + 1) + \beta\,(w^2 - z^2 - 1)\,, \\ \dfrac{dw}{dt} = -2\, z\, w + \alpha\,(w^2 - z^2 - 1)\,, \qquad \alpha, \beta \in \mathbb{R}\,. \end{cases}$$

The bifurcations of the vector field on the plane (α, β) are given in [8].

Likewise, for the particular case when $a_j = 0$, $j = 1, 2$ and $b_1 = -1$, $b_2 = 0$, $b_3 = 1$ we deduce the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = 2\,(z^2 - 1) + \beta\,(w^2 + z^2 - 1)\,, \\ \dfrac{dw}{dt} = 2\, w\, z + \alpha\,(w^2 + z^2 - 1)\,, \qquad \alpha, \beta \in \mathbb{R}\,. \end{cases}$$

The bifurcations of the vector field on the plane (α, β) can be found in [8].

Finally, for the subcase when $a_1 = 2$, $a_2 = 0$ and $b_1 = 0$, $b_2 = 1$, $b_3 = 0$ we construct the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = -4\, z\,(z + w) + \beta\,((w - z)^2 - 2\, z))\,, \\ \dfrac{dw}{dt} = -2\,(z + w)^2 + \alpha\,((w - z)^2 - 2z)\,, \qquad \alpha,\, \beta \in \mathbb{R}\,. \end{cases}$$

The critical points of this system are

$$O(0,0),\ N\Big(\frac{1}{2}, -\frac{1}{2}\Big),\ M\Big(\frac{\beta^2}{K_3}, \frac{\beta\,(2\,\alpha - \beta)}{K_3}\Big)\,,$$

where $K_3 = 2\,((\alpha - \beta)^2 - 2\,\alpha)$.

The bifurcation curves are given by the formulas

$$l_1 : (\alpha - \beta)^2 - 2\,\alpha = 0\,,$$

$$l_2 : \beta^2\,(4\,\alpha + 1) + 4\,\alpha\,\beta\,(1 - 2\,\alpha) + 4\,\alpha^2\,(\alpha - 1) = 0\,,$$

$$l_3 : \beta + 2\,\alpha = 0\,,$$

$$l_4 : \beta + \alpha = 0\,,$$

$$l_5 : \beta = 0\,,$$

$$l_6 : \alpha = 0\,.$$

These curves divide the plane α, β into 17 regions in which we find a change in the behaviour of the vector field. Of special interest is the region between the curves l_3 and l_4 for $\beta < 0$, where there is a stable limit cycle. The bifurcations of the vector field are given in [8].

The problem related to studying the quadratic vector field with parabola as trajectories was analyzed in particular in [9], [10] and [11].

To conclude this section it is interesting to observe that the function $\det S$ satisfies the relations

$$P(z, w)\,\partial_z \det S + Q(z, w)\,\partial_w \det S = \mathcal{R} \det S$$

along the solutions of the equations (4.5), for some function $\mathcal{R}$.

5. Constructing the planar vector field from given algebraic partial integrals.

Darboux in [12] gives a method of integration (1.2) with $P, Q \in \mathbb{C}[z, w]$ using algebraic curves. His first idea is to search for a general integral of the form

$$F(z, w) = \prod_{j=1}^{q} f_j^{\alpha_j}(z, w)\,, \tag{5.1}$$

where $\alpha_j \in \mathbb{C}$ and $f_j \in \mathbb{C}[z, w]$. This integral is called Darboux 's first integral and the system (1.2) is called Darboux integrable.

Definition 5.1 ([13]). *Let $f \in \mathbb{C}[z,w]$ and let $\gamma \subset \mathbb{C}^2 : f(z,w) = 0$ be an algebraic particular integral of* (1.2) *if and only if there exists $\lambda \in \mathbb{C}[z,w]$ such that*

$$P(z,w)\,\partial_z f(z,w) + Q(z,w)\,\partial_w f(z,w) = \lambda(z,w)\, f(z,w)\,. \tag{5.2}$$

The result below is Darboux's.

Theorem ([12]). *Consider the equation of the form* (1.2)*. Let $m = \max\{\deg P, \deg Q\}$. If $q > m\,(m+1)/2$ and*

$$f_j(z,w) = 0\,, \qquad j = 1,2,\ldots,q$$

are different algebraic solutions for which (5.2) *takes place, then there are complex numbers α_j, $j = 1,2,\ldots,q$ such that* (5.1) *is a first integral of* (1.2)*.*

In all of these papers the authors started with a system and asked what kind of invariant algebraic curves this system could have, but it seems interesting (by considering the argument given in the introduction) to analyze the inverse problem related to constructing the planar vector field tangent to the set of algebraic curves $f_j(z,w) = 0$, $j = 1,2,\ldots,q$.

This problem was first stated by Eruguin [3] and developed by Galiullin and his followers [1]. The new different approach can be found in the papers [14] and [4]. The purpose of this section is to analyze the problem from another point of view.

We will first study the case when $q \geq 2$.

Proposition 5.1. *Let us give algebraic curves*

$$f_j(z,w) = 0\,, \qquad j = 1,2 \tag{5.3}$$

such that $\{f_1, f_2\} \not\equiv 0$ in the neighbouhood of the set (5.3)*.*

So the the vector field tangent to the given curves can be represented as follows

$$\Gamma = \frac{\lambda_1 f_1\{\ ,f_2\} + \lambda_2 f_2\{f_1,\ \}}{\{f_1,\, f_2\}}\,, \tag{5.4}$$

where λ_j, $j = 1,2$ are arbitrary holomorphic functions on $\mathcal{D}^* \subset \mathbb{C}^2$.

The proof follows from the equalities

$$\begin{cases} \partial_z f_1(z,w)\,P(z\,w) + \partial_w f_1(z,w)\,Q(z,w) = \lambda_1(z,w)\,f_1(z,w)\,, \\ \partial_z f_2(z,w)\,P(z,w) + \partial_w f_2(z,w)\,Q(z,w) = \lambda_2(z,w)\,f_2(z,w)\,. \end{cases}$$

Of course if $P, Q \in \mathbb{C}[z,w]$, then λ_1 and λ_2 belong to $\mathbb{C}[z,w]$. The differential equations which generate (2.5) can be represented as follows

$$(5.5)\qquad \begin{cases} \dfrac{dz}{dt} = \dfrac{\lambda_1 f_1\{z, f_2\} + \lambda_2 f_2\{f_1, z\}}{\{f_1, f_2\}}\,, \\ \dfrac{dw}{dt} = \dfrac{\lambda_1 f_1\{w, f_2\} + \lambda_2 f_2\{f_1, w\}}{\{f_1, f_2\}}\,. \end{cases}$$

As an immediate consequence we get the following results:

Consequence 5.1. *The vector field* (5.4) *has the following algebraic curves as complementary integrals*

$$f_j(z,w) = 0\,, \qquad j = 3, 4, \ldots, q\,,$$

if and only if

$$(5.6)\qquad \lambda_1 f_1(z,w)\,\{f_j, f_2\} + \lambda_2 f_2\{f_1, f_j\} + \lambda_j f_j\{f_2, f_1\} = 0\,.$$

In fact, from the equalities

$$\Gamma(f_j) = \lambda_j f_j\,, \qquad j = 3, 4, \ldots, q\,,$$

we deduce that

$$\frac{\lambda_2 f_2\{f_1, f_j\} + \lambda_1 f_1\{f_1, f_2\}}{\{f_1, f_2\}} = \lambda_j f_j$$

and so (5.6) follows trivially.

Consequence 5.2. *Let F be function* (5.1)*. Then*

$$(5.7)\qquad \Gamma(F) = \Big(\sum_{j=1}^{q} \alpha_j\,\lambda_j\Big) F\,.$$

Consequence 5.3. *Let us suppose that*

$$\prod_{\substack{j,k=1 \\ k\neq j}}^{q} \{f_j, f_k\} \not\equiv 0 \tag{5.8}$$

in the neighbourhood of the set $\{f_j = 0,\ j = 1, 2, \ldots, q\}$*. so the vector field tangent to the given curves admits the representations*

$$\Gamma = \frac{\lambda_j f_j \{f_{j-1},\ \} + \lambda_{j-1} f_{j-1}\{\ , f_j\}}{\{f_j, f_{j-1}\}}\,, \qquad j = 1, 2, \ldots, q\,, \tag{5.9}$$

if and only if the following relations hold

$$\lambda_j f_j \{f_n, f_m\} + \lambda_m f_m \{f_j, f_n\} + \lambda_n f_n \{f_m, f_j\} = 0\,, \tag{5.10}$$

where $j, k, n, m = 1, 2, \ldots, q > 3$ *and* $n \neq k \neq j \neq m$*.*

Let us denote by A the matrix such that

$$A = \begin{pmatrix} 0 & \{f_n, f_m\} & \{f_j, f_n\} & \{f_m, f_j\} \\ \{f_m, f_n\} & 0 & \{f_k, f_n\} & \{f_m, f_k\} \\ \{f_n, f_j\} & \{f_n, f_k\} & 0 & \{f_j, f_k\} \\ \{f_j, f_m\} & \{f_k, f_m\} & \{f_k, f_j\} & 0 \end{pmatrix}.$$

Of course,

$$\begin{aligned} \det A = \big(&\{f_n,\, f_m\}\, \{f_j,\, f_k\} + \{f_k,\, f_m\}\, \{f_n,\, f_j\} \\ &+ \{f_j,\, f_m\}\, \{f_k,\, f_n\}\big)^2 \\ = 0\,. \end{aligned} \tag{5.11}$$

It is easy to prove that theses relations are an identity for all f_j, f_n, f_m and f_k. By using these identities we can easily deduce the following consequences

Consequence 5.4. *Let us suppose that the arbitrary functions* λ_j*,* $j = 1, 2, \ldots, q$ *are such that*

$$\mathcal{R}\{H, f_j\} = \lambda_j f_j\,, \tag{5.12}$$

where H and $\mathcal{R}$ are arbitrary functions. Hence the vector field Γ admits the following representation

$$\Gamma = \mathcal{R}\{H,\ \}\,. \tag{5.13}$$

From here we can observe that the function $\mathcal{R}$ is an integrant factor of the 1-form $\Omega = \Gamma(z)\,dw - \Gamma(w)\,dz$. It is clear that (5.10), in view of (5.11) holds identically.

Consequence 5.5. *Let us suppose that the following development holds*

$$\mathcal{R}\{f_j, f_n\} = \sum_{m=1}^{q} C_{jn}^m(z,w)\, f_m\,. \tag{5.14}$$

Then the functions C_{jn}^m must satisfy the relations

$$\begin{cases} C_{jn}^m + C_{nj}^m = 0\,, \\ C_{nm}^l\, C_{lk}^s + C_{mk}^l\, C_{ln}^s + C_{kn}^l\, C_{lm}^s = 0\,. \end{cases}$$

These equalities are identities in the specific case when

$$C_{jn}^l f_l = \frac{1}{\mathcal{R}}\,(\lambda_j f_j - \lambda_n f_n)\,. \tag{5.15}$$

Consequence 5.6 *Let us suppose that*

$$f_j(z,w) = w - w_j(z)\,, \qquad f'_{jz} - f'_{(j-1)z} \neq 0\,, \qquad j = 1,2,\ldots,q\,,$$

then (5.14) *holds with*

$$\mathcal{R} = \frac{\lambda_j f_j - \lambda_{j-1} f_{j-1}}{w'_{j-1}(z) - w'_j(z)}\,,$$

$$w'_j \equiv \frac{dw}{dz}\,.$$

The differential equations which generate the vector field Γ are the following

$$\begin{cases} \dfrac{dz}{dt} = \mathcal{R}\,, \\ \dfrac{dw}{dt} = \mathcal{R}\, \dfrac{\lambda_j f_j\, w'_j(z) - \lambda_{j-1} f_{j-1}\, w'_{j-1}(z)}{w'_{j-1}(z) - w'_j(z)}\,. \end{cases}$$

To conclude this section we give the solution for the stated problem when $q = 1$.

Proposition 5.2. *The planar vector field tangent to the algebraic curve* $f(z,w) = 0$ *can be represented as follows*

$$\Gamma = \mu(z,w)\,\{f,\ \} + f(z,w)\,(\lambda_1(z,w)\,\partial_z + \lambda_2(z,w)\,\partial_w)\,,$$

where μ, λ_1 *and* λ_2 *are arbitrary analytic functions, such that*

$$\Gamma(f) = \lambda(z,w)\,f(z,w)\,, \qquad \text{for all } \lambda \in \mathbb{C}[z,w]\,. \tag{5.16}$$

In order to illustrate the above assertions in the section below we shall give the solution to the stated problem for the subcase when Γ is a quadratic vector field in the variables z and w and the given algebraic curve is the following

$$f(z,w) = w^2 - 2\,w\,q(z) + v(z)\,, \qquad v(z) = q^2(z) - p(z)\,, \tag{5.17}$$

where q and p are polynomials of degree k and $m \leq 2\,k$ respectively.

6. Quadratic stationary planar vector fields with given algebraic curves (5.17).

It is well known that the domain G of a real analytic planar stationary vector field is divided into elementary regions by singular trajectories. The non singular trajectories (which are topologically equivalent) are located in these regions.

For structurally stable dynamical systems the singular trajectories can be stable simple critical points, stable limit cycles, $\alpha-\omega$ separatrices which may spread towards a node, a focus, a limit cycle. They may even leave the domain G.

From these facts we state and analyze the problem of constructing a planar vector field from a finite number of singular trajectories.

In this section we are going to construct a real quadratic vector field with a given real invariant algebraic curve (5.17). All the obtained results can be generalized (with the respective considerations) to the complex case.

The problem of constructing a quadratic planar vector field with a given algebraic curve of the type (5.17) has been studied by many specialists.

In 1966 A. I. Jablonski, published an article (see [15]) in which the author constructed a differential equation

$$w' = \frac{P(z,w)}{Q(z,w)}\,,$$

where P and Q are quadratics, which has an algebraic curve of fourth degree as a limit cycle. He also investigated the phase portraits of this equation.

In 1972 V. F. Filipsov, in the paper [16], showed that for the specific quadratic system studied by Jablonski there is an orbit of the form

$$w = b_0\,z^2 + b_1\,z + b_2 + (a_0\,z + a_1)\,\sqrt{-z^2 + l_1\,z + l_0}\,.$$

The author shows that for various values of the parameters there is no limit cycle and no separatrix going from one saddle point to another. In 1973 this author, in the article [17] is considering the quadratic system under the condition that

$$a_1 + a_0\,z + b_0\,w^2 + b_1\,z\,w + c_0\,z\,w^2 + c_1\,z^2\,w + c_2\,z^3 + z^4 = 0\,,$$

is a solution. The author shows that in this case a global analysis of the topology of integral curves is possible.

Later in the paper "Algebraic limit cycles" the author finds conditions under which the quadratic differential systems

$$\begin{cases} \dot z = P(z,w)\,, \\ \dot w = Q(z,w)\,, \end{cases}$$

have a limit cycle that is an algebraic curve of the fourth degree.

In 1991 Shen Boian, in the paper [18], proves that a quadratic system possesses a quartic curve solution

$$\text{(A)} \qquad (w + c\,z^2)^2 + z^2\,(z-a)\,(z-b) = 0\,, \qquad (a-b)\,a\,b\,c \neq 0\,,$$

if and only if the quadratic system can be written in the form

$$\begin{cases} \dot z = -4\,a\,b\,c\,z - (a+b)\,z + 3\,(a+b)\,c\,z^2 + 4\,z\,w\,, \\ \dot w = -(a+b)\,a\,b\,z - 4\,a\,b\,c\,w + (4\,a\,b\,c^2 - \dfrac{3}{2}\,(a+b)^2 + 4\,a\,b\,)\,z^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdot 8\,(a+b)\,c\,z\,w + 8\,w^2\,. \end{cases}$$

For this system a necessary and sufficient condition for the existence of a type of quartic curve limit cycle (A) and a separatrix cycle are given.

The aim of the present section is to state and solve the following

PROBLEM 6.1. Let us give the algebraic curve (5.17). We require to construct a real quadratic planar vector field which admits it as a particular integral.

Firstly we give the following aspects related to the plane curve (5.17).

Let us suppose that the algebraic curve (5.17) is found on the plane. The critical points (z_0, w_0) of this curve are the points such that

$$\begin{cases} p(z_0) = 0\,, \\ w_0 - q(z_0) = 0\,, \\ \left.\dfrac{dp(z)}{dz}\right|_{z=z_0} = 0\,. \end{cases} \tag{6.1}$$

Proposition 6.1 *The following type of critical points can be obtained for the curve* (5.17):

i) *Isolated point. The point with coordinates* $(z_0, q(z_0))$ *where* z_0 *is the maximum of the function* p.

ii) *Knot (saddle) point. The point* $(z_0, q(z_0))$ *where* z_0 *is a minimum of* p.

iii) *If* $p''(z)|_{z=z_0} = 0$ *then the well known* 4 *configurations are possible.*

Proposition 6.2. *The relation* (5.16) *holds for the quadratic planar vector field*

$$\begin{cases} \Gamma = (\alpha(z) + \beta(z)\, w + \gamma\, w^2)\, \partial_z + (a(z) + b(z)\, w + c\, w^2)\, \partial_w\ , \\ \alpha(z) = \alpha_2\, z^2 + \alpha_1\, z + \alpha_0\ , \\ \beta(z) = \beta_1\, z + \beta_0\ , \\ a(z) = a_2\, z^2 + a_1\, z + a_0\ , \\ b(z) = b_1\, z + b_0\ , \end{cases} \tag{6.2}$$

if and only if the following equality holds

$$-P(z,w)\,((2\,(w-q(z))\,q'(z)+p'(z))+Q(z,w)\,(2\,w-2\,q(z))$$
$$= (A\,z+B\,w+C)\,((w-q(z))^2-p(z))\,, \tag{6.3}$$

or, what amounts to the same,

$$\begin{cases} \gamma\,q'(z)=c-\dfrac{B}{2}\,, \\ 2\,(B-c)\,q(z)-2\,\beta(z)\,q'(z)+\gamma\,v'(z)=A\,z+C-2\,b(z)\,, \\ 2\,(A\,z+C-b(z))\,q(z)-2\,\alpha(z)\,q'(z)-B\,v(z)+\beta(z)\,v'(z) \\ \qquad\qquad = -2\,a(z)\,, \\ -2\,a(z)\,q(z)-(A\,z+c)\,v(z)+\alpha(z)\,v'(z)=0\,, \end{cases} \tag{6.4}$$

where $v=q^2(z)-p(z)$.

In order to solve this system we first introduce the following notations

$$S(z)=((A\,z+C-b(z))\,(A\,z+C)+B\,a(z))\,q(z)$$
$$-\,\alpha(z)\,(A\,z+C)\,\frac{dq}{dz}+a(z)\,(A\,z+C)\,,$$
$$D(z)=(A\,z-B\,\alpha_2)\,z^2+(A\,\beta_0+C\,\beta_1-\alpha_1\,B)\,z+C\,\beta_0-\alpha_0\,B\,,$$
$$R(z)=((A\,z+C-b(z))\,\alpha(z)+a(z)\,b(z))\,q(z)-\alpha^2(z)\,\frac{dq}{dz}+a(z)\,\alpha(z)\,.$$

Then for v and dv/dz from (3.1) we obtain the following relations

$$\begin{cases} D(z)\,v(z)=R(z)\,, \\ D(z)\,\dfrac{dv}{dz}=S(z)\,. \end{cases}$$

As a consequence the compatibility conditions gives us the relations

$$\frac{dD(z)}{dz}\,R(z)=\Big(\frac{dR(z)}{dz}-S(z)\Big)D(z)\,, \tag{6.5}$$

where q is a polynomial such that

$$(6.6) \quad q(z) = \begin{cases} k\,z + k_0\,, & \text{if } \gamma \neq 0\,, \\ k\,(\beta_1\,z + \beta_0)^n + k_1\,z + k_0\,, & \text{if } \gamma \neq 0,\ \beta_1 \neq 0\,, \\ k\,z^2 + k_1\,z + k_0\,, & \text{if } \gamma = B = c = \beta_1 = 0\,. \end{cases}$$

By using computer techniques the solutions to (6.4) can be obtained.

The first case in (6.6) enables us to obtain all quadratic vector fields admitting the conics as trajectories. For the second case, we deduce that it is important when $n = 2, 3, 4, 5$. For $n > 5$ we deduce that there is only one quadratic vector field tangent to the given curve.

As Poincaré observed (see [19]) in order to recognize when the stationary planar vector field is algebraically integrable it is sufficient to find a bound for the degrees of the invariant algebraic curves which the system could have. In [14] the following problem is stated: find a bound for the degrees of the invariant algebraic curves which a system (1.1) could have.

In the development of some aspects of this problem, the results below about the construction of a quadratic vector field from given algebraic curves for $n > 5$ seems to be interesting.

6.1. Quadratic vector field with given conics.

For the case when the given algebraic curve is the following

$$(6.7) \qquad f(z, w) = (w - k\,z - k_0)^2 - p_2\,z^2 - p_1\,z - p_0 = 0\,,$$

we obtain all the quadratic vector fields tangent to it.

In particular, for the case when $p_1 = p_0 = 0$ and $p_2 \neq 0$ we get the following result:

Proposition 6.3 *The quadratic vector field tangent to the curve* (6.7) *with* $p_1 = p_0 = 0$ *and* $p_2 \neq 0$ *is the following*

$$
(6.8) \qquad \begin{cases}
\dfrac{dz}{dt} = -k_0\,(k_0\,\gamma + \beta_0) \\
\qquad + (\Omega\,(\beta_0 + 2\,k_0\,\gamma) + k_0\,\gamma\,(2\,\beta_1 - B) - C\,\gamma)\,\dfrac{z}{-2\,\gamma} \\
\qquad + \beta_0\,w\,\beta_1\,z\,w + \gamma\,w^2 \\
\qquad + (\Omega^2 + \Omega\,(2\,\beta_1 - B) - 2\,A\,\gamma)\,\dfrac{z^2}{-4\,\gamma}\ , \\[2ex]
\dfrac{dw}{dt} = \dfrac{k_0}{-2\,\gamma}\,(\Omega\,(\beta_0 + k_0\,\gamma) + C\,\gamma) \\
\qquad + (\Omega^2\,(\beta_0 + 2\,k_0\,\gamma) \\
\qquad\quad + 2\,\Omega\,(\beta_0\,\beta_1 - 2\,B\,k_0\,\gamma - B\,\beta_0 + 4\,k_0\,\gamma\,\beta_1) \\
\qquad\quad + 2\,\beta_0\,(A - 2\,b_1) - 4\,k_0\,\gamma^2\,(b_1 - A)) \\
\qquad\quad \cdot \dfrac{z}{-4\,\gamma^2}\,\dfrac{(\Omega\,(\beta_0 - k_0\,\gamma) + C\,\gamma)}{2\,\gamma}\,w \\
\qquad + (\Omega^2 + \Omega\,(2\,\beta_1 - B) - 2\,A\,\gamma)\,\dfrac{z^2}{-4\,\gamma}\,b_1\,z\,w + c\,w^2\,,
\end{cases}
$$

where $\Omega, B, \beta_1, A, \gamma, b_1$ are parameters such that

$$
(6.9) \qquad \begin{cases}
\Omega = 2\,\gamma\,k\,, \\
B = 2\,c + 2\,\gamma\,k\,, \\
\Omega\,(B - 2\,\beta_1) + 2\,\gamma\,(2\,b_1 - A) = 4\,\gamma^2\,p_2\ , \qquad p_2 \neq 0\,.
\end{cases}
$$

Of course if $p_2 > 0$ then the quadratic vector field has two invariant straight lines

$$
w = (k + \sqrt{p_2}\,)\,z + k_0\ ,
$$
$$
w = (k - \sqrt{p_2}\,)\,z + k_0\ .
$$

We can deduce the important subcase when

$$
(6.10) \qquad \begin{cases}
\beta_0 = -k_0\,\gamma\,, \\
c = 0\,, \\
A = -2\,\gamma\,, \\
B = \beta_1\ .
\end{cases}
$$

Under these restrictions we obtain the well known Darboux integrable quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = \beta_0\, w + \beta_1\, z\, w + \gamma\, w^2 - \gamma\, z^2\,, \\ \dfrac{dw}{dt} = -\beta_0\, z + \beta_1\, z\, w\,. \end{cases}$$

In this case the relations (6.10) and (6.9) take the form

$$\begin{cases} \beta_1^2 + 4\,\gamma\,(b_1 + \gamma) = 4\,\gamma^2\, p_2\,, \qquad p_2 > 0\,, \\ \Omega = 2\,\gamma\, k\,, \\ \beta_1 = B = -\Omega\,. \end{cases}$$

Likewise we deduce all the quadratic planar vector fields with given trajectories (6.7).

6.2. Quadratic planar vector fields, with a given curve of fourth degree.

We now shall analyze the above stated problem when the given curve is an algebraic curve of fourth degree

$$f(z, w) = (w - k_0\, z^2 - k_1\, z - k_2)^2 - p(z) = 0\,,$$

where p is a polynomial of degree four. This case was analyzed, in particular, in the papers refered to in the section above.

Proposition 6.4. *Let*

$$(w - k_0\, z^2 - k_1\, z - k_2)^2 + z^4 - 4\, h_3\, z^3 - 4\, h_2\, z^2 - 4\, h_0 = 0\,, \tag{6.11}$$

be a curve such that

$$\begin{cases} h_2 < 0\,, \\ 9\, h_3^2 > -8\, h_2\,. \end{cases}$$

Then the curve (6.11) *has an oval.*

Proposition 6.5. *The curve* (6.11) *with* $h_0 = 0$ *is a trajectory of the following quadratic system*

$$\begin{cases} \dot z = -k_0\,\beta_1\Big(\dfrac{3}{4}\,p_1\,z^2 + p_0\,z\Big) + \beta_1\Big(z - \dfrac{1}{4}\,p_1\Big)w\,, \\ \dot w = -\beta_1\Big(\Big(p_0 + \dfrac{3}{8}\,p_1^2 + k_0^2\,p_0\Big)z^2 + \dfrac{1}{4}\,p_0\,p_1\,z\Big) - \beta_1\,k_0\,(2\,p_1\,z + p_0)\,w\,. \end{cases}$$

The parameters A, B and C are determined as follows

$$A = -3\,k_0\,\beta_1\,b_1\,, \qquad B = 4\,\beta_1\,, \qquad C = -2\,q_0\,p_0\,\beta_1\,.$$

The existence of limit cycles can be deduced by analyzing the Liapunov function V

$$V(z,w) = w^2 + p_0\,z^2 - k_0\,w\,z^2 - p_1\,z^3 + (1 + k_0^2)\,z^4\,.$$

Of course, this function is definitively strictly positive for $p_0 > 0$. By considering that its derivative is such that

$$\dot V = -2\,q_0\,p_0\,\beta_1\,V + (4\,\beta_1\,w - 2\,q_0\,\beta_1\,b_1\,z)\,V\,,$$

we deduce that the origin is asymptotically stable if $q_0\,\beta_1\,p_0 > 0$ and unstable if $q_0\,\beta_1\,p_0 < 0$. On the other hand, the curve $V(z,w) = 0$ has an oval around the origin, which is evidently a limit cycle of the system.

Likewise we can analyze the problem of the construction of a quadratic vector field with algebraic with $n = 3, 4, 5$.

It should be pointed out that from the solution of the stated problem it follows that if the quadratic differential system has an algebraic limit cycle, this must be an algebraic curve of the fourth degree.

7. Quadratic vector fields with algebraic curves with $n > 5$.

With no loss of generality we shall suppose that $\beta_1 = 1$ and $\beta_0 = 0$.

By using computer techniques the following results can be easily deduced:

Proposition 7.1. *Let us suppose that* $\deg\,(q(z)) = n > 5$. *Then the only solutions to* (6.4) *are the following*:

i)

$$(w - K_0\,z^n - K_1\,z - K_2)^2 - (p_0\,z^n + p_1\,z + p_2)^2 = 0\,,$$

$$P(z,w) = z\,(\alpha_2\,z + w + \alpha_1)\,,$$

$$Q(z,w) = -(\alpha_2\,z + w + \alpha_1)\,((n\,\alpha_2 - b_1)z - n\,w + \alpha_1\,n)\,,$$

where $K_0, K_1, K_2, p_0, p_1, p_2,$ *are parameters such that*

$$\begin{cases} K_1 = \dfrac{b_1 - \alpha_2}{2\,n\,(n-1)}\;, \\ K_2 = \dfrac{b_0}{2\,n}\;, \\ p_0 = K_0\;, \\ p_1 = \dfrac{n\,(2\,\alpha_2\,n - b_1 - \alpha_2)}{4\,n\,(n-1)}\;, \\ p_2 = \dfrac{(n-1)(2\,\alpha_1\,n - b_0)}{2\,n\,(n-1)}\;, \\ A = \alpha_2 + b_1\;, \\ B = 2\,n\,, \\ C = b_0\;, \end{cases}$$

and

ii)

$$(w - K_0\,z^n - K_1\,z - K_2)^2 - z^n\,(p_0\,z^n + p_1\,z + p_2) = 0\,,$$

$$P(z,w) = z\Big(\alpha_2\,z + w - \frac{4}{3}\,\alpha_1 + \frac{2\,b_0}{3\,n}\Big)\,,$$

$$\begin{aligned} Q(z,w) = \frac{z}{9\,n\,(n-1)}\,&(n\,(b_1 + (n-2)\,\alpha_2)\,(2\,b_1 - (n+2)\,\alpha_2)\,z \\ &+ (n-2)\,((n+2)\,\alpha_2 - 2\,b_1)\,(b_0 - 2\,n\,\alpha_1)) \\ &+ b_1\,z\,w + n\,w^2 - \frac{w}{3}\,(b_0 - 2\,n\,\alpha_1)\,, \end{aligned}$$

where $K_0, K_1, K_2, p_0, p_1, p_2$ *are parameters such that*

$$\begin{cases} K_1 = \dfrac{(n+1)\,\alpha_2 - 2\,b_1}{3\,(n-1)}\,, \\ K_2 = \dfrac{n\,\alpha_1 - 2\,b_0}{3\,n}\,, \\ p_0 = K_0^2\,, \\ p_1 = \dfrac{2\,K_0\,((2\,n-1)\,\alpha_2 - b_1)}{3\,(n-1)}\,, \\ p_2 = -\dfrac{2\,K_0\,(b_0 - 2\,n\,\alpha_1)}{3\,n}\,, \\ A = \dfrac{2}{3}\,((n+1)\,\alpha_2 + b_1)\,, \\ B = 2\,n\,, \\ C = \dfrac{2}{3}\,(n\,\alpha_1 + b_0)\,. \end{cases}$$

The first case is trivial. A qualitative analysis of the second case gives us the following: denoting

$$\nabla = b_0 - 2\,n\,\alpha_1 \equiv \frac{3\,n\,p_2}{K_0}$$

and

$$\tau = (2\,n-1)\,\alpha_2 - b_1 \equiv \frac{3\,(n-1)\,p_1}{2\,K_0}\,,$$

the critical points are

$$(z_1, w_1) = (0, 0)\,,$$

$$(z_2, w_2) = \left(0, \frac{\nabla}{3\,n}\right),$$

$$(z_3, w_3) = \left(\frac{\nabla}{\tau}, \frac{((n-2)\,\alpha_2 - 2\,b_1)\,\nabla}{3\,n\,\tau}\right),$$

$$(z_4, w_4) = \left(\frac{(n-1)\,\nabla}{n\,\tau}, \frac{((n+2)\,\alpha_2 - 2\,b_1)\,\nabla}{3\,n\,\tau}\right).$$

The quantity

$$\delta(z,w) = \partial_z P(z,w)\,\partial_w Q(z,w) - \partial_w P(z,w)\,\partial_z Q(z,w)\,,$$
$$\sigma(z,w) = \partial_z P(z,w) + \partial_w Q(z,w)\,,$$

calculated at the above points give us the following results

$$\delta(z_1,w_1) = \frac{2\,\nabla^2}{9\,n}\,, \qquad \sigma(z_1,w_1) = -\frac{(n+2)\,\nabla}{3\,n}\,,$$
$$\delta(z_2,w_2) = -\frac{\nabla^2}{9\,n}\,, \qquad \sigma(z_2,w_2) = \frac{(n-1)\,\nabla}{3\,n}\,,$$
$$\delta(z_3,w_3) = \frac{2\,\nabla^2}{9\,n^2}\,, \qquad \sigma(z_3,w_3) = 0\,,$$
$$\delta(z_4,w_4) = \frac{2\,\nabla^2}{9\,n^2}\,, \qquad \sigma(z_4,w_4) = \frac{\nabla}{n}\,.$$

Of course, we obtain the bifurcation curves from the equalities: i) $\nabla = 0$, ii) $\tau = 0$. The behaviour of the constructed planar vector field is easily obtained.

In fact, with no loss of generality we shall suppose that $K_0 = 1$ and under the change

$$\begin{cases} \alpha_1 = p_2 - K_2\,, \\ \alpha_2 = p_1 - K_1\,, \\ b_0 = \dfrac{n}{2}\,p_2 - 2\,n\,K_2\,, \\ b_1 = (1-2\,n)\,K_1 + \dfrac{(n+1)\,p_1}{2}\,, \\ z = X\,, \\ w = Y + K_1 X + K_2\,, \end{cases}$$

we deduce that the constructed differential equations coincide with the two dimensional logistic system

$$\begin{cases} \dot X = X\,(p_2 + p_1\,X + Y)\,, \\ \dot Y = Y\left(\dfrac{n}{2}\,p_2 + \dfrac{(n+1)\,p_1}{2}\,X + n\,Y\right). \end{cases} \tag{7.1}$$

The function (5.17) and the equation (6.3) in the coordinates X, Y take the form respectively

$$(7.2)\qquad \begin{cases} f(X,Y) = Y^2 - 2\,Y\,X^n - p_1\,X^{n+1} - p_2\,X^n\,, \\ \dfrac{df(X,Y)}{dt} = 2\left(\dfrac{n}{2}\,p_2 + \dfrac{(n+1)\,p_1}{2}\,X + n\,Y\right) f(X,Y)\,. \end{cases}$$

The critical points of (7.1) are the following

$$(0,0),\quad \left(0, -\frac{p_2}{2}\right),\quad \left(\frac{-p_2}{p_1}, 0\right),\quad \left(\frac{-n\,p_2}{(n-1)\,p_1}, \frac{p_2}{n-1}\right).$$

Proposition 7.2. *If $p_2 \neq 0$ then the equations (7.1) do not admit the first integral which can be developed in a formal power series with respect to X and Y.*

By making a linear approximation of (7.1) we find for arbitrary set of $m_1, m_2 \in \mathbb{N}$, $m_1 + m_2 \geq 0$ and for $p_2 \neq 0$ that

$$\left(m_1 + \frac{m_2\,n}{2}\right) p_2 \neq 1\,.$$

Hence, using Liapunov's results, we can prove 7.2.

To study the case when $p_2 = 0$ we can apply the results obtained in [20] and [21], which are related with the arithmetic properties of the Kovalevski exponents.

For the equations (7.1) it is easy to calculate the Kovalevski exponent $\rho_1 = -1$, $\rho_2 = 1 - n$ when $p_1 \neq 0$.

Proposition 7.3. *The equations*

$$\begin{cases} \dot{X} = X(p_1\,X + Y)\,, \\ \dot{Y} = Y\left(\dfrac{(n+1)\,p_1}{2}\,X + n\,Y\right), \end{cases}$$

do not admits polynomials first integral.

The proof follows from the fact that for this case, and for an arbitrary set of natural numbers m_1, m_2 such that $m_1 + m_2 \geq 1$ we deduce that

$$m_1\,\rho_1 + m_2\,\rho_2 = m_1 + (n-1)\,m_2 \neq 0\,.$$

By applying the results given in [20] we deduce the veracity of the above assertion.

It is important to observe that autonomous analytic vector field on the plane cannot have chaotic behaviour and so in some sense they are integrable. But under some conditions the first integral is a "bad integral". One of these integrals are Darboux's integrals.

Proposition 7.4. *The system* (7.1) *is Darboux integrable.*

In fact, in view of (7.1), (7.2) we easily get that the function

$$F(X,Y) = f(X,Y)\,Y^{-2}$$

is the Darboux's first integral. It is easy to deduce the following representation for the system (7.1)

$$\begin{cases} \dot{X} = \mu(X,Y)\,\dfrac{\partial F}{\partial Y}\,, \\ \dot{Y} = -\mu(X,Y)\,\dfrac{\partial F}{\partial X}\,, \end{cases}$$

where $\mu(X,Y) = 2\,Y^{-3}\,X^n$.

When $n=5$, as well as the vector field constructed above, there are two complementary vector fields tangent respectively to the following curves (we suppose that $\beta_1 = 1, \beta_0 = 0$)

$$\begin{cases} (w - K_0\,x^5 + K_1\,x + K_2)^2 \\ \qquad\qquad -\dfrac{1}{6718464\,p_1^4}\,(p_0^2\,x^2 + 24\,p_0\,p_1)\,(p_0\,x^2 - 6\,p_1)^4 = 0\,, \\ p_0 = b_1 - 9\,\alpha_2\,, \\ p_1 = \alpha_0\,, \\ K_j \in \mathbb{R}, \qquad j = 0,1,2 \end{cases}$$

and

$$\begin{cases} (w - K_0\,z^5 - K_1\,z - K_2)^2 \\ \qquad\qquad -\dfrac{3}{2379293284\,p_1^4}\,(p_0^2 + 29\,p_0\,p_1)^2\,(3\,p_0\,z^2 - 58\,p_1)^3 = 0\,, \\ p_0 = b_1 - 9\,\alpha_2\,, \\ p_1 = \alpha_0\,, \\ K_j \in \mathbb{R}, \qquad j = 0,1,2\,. \end{cases}$$

The critical points are easy to find. The quantity δ and σ for these vector fields are, respectively, the following

$$\delta = -\frac{25}{162}\, p_0\, p_1 < 0\,,$$

$$\sigma = -\frac{5}{18}\, \sqrt{p_0\, p_1}\,,$$

and

$$\delta = -25\, p_0\, p_1 < 0\,,$$

$$\sigma - \frac{25}{29}\sqrt{p_0\, p_1}\,.$$

For the polynomial vector field of degree $n > 2$ we can study the problem stated above analogously.

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