Revista Matemática Iberoamericana
Vol. 14, N.° 1, 1998

Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$

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Abstract. We construct global solutions to the Navier-Stokes equations with initial data small in a Besov space. Under additional assumptions, we show that they behave asymptotically like self-similar solutions.

0. Introduction.

When studying global solutions to an evolution problem, it is natural to study their asymptotic behavior, as it is usually a simpler way to describe the long term behavior than the solution itself. Global solution of the non-linear heat equation have been showed to be asymptotically close to self-similar solutions [7]. Under certain conditions, we will show how to obtain similar results for the incompressible Navier-Stokes system.

We recall the equations

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \otimes u) - \nabla p\,, \\ \nabla \cdot u = 0\,, \\ u(x,0) = u_0(x)\,, \qquad x \in \mathbb{R}^3\,,\ t \geq 0\,. \end{cases}$$

As we are in the whole space, if $u(x,t)$ is a solution of (1), then for all $\lambda > 0$, $u_\lambda(x,t) = \lambda\, u(\lambda\, x, \lambda^2\, t)$ is also a solution.

We now note that studying the asymptotic behavior of $u(x,t)$ for large time is equivalent to studying the asymptotic behavior of $u_\lambda(x,t)$ for large λ with fixed time. Actually, we shall show that, as t goes to ∞, the natural space scale is $\sqrt{t}$ as in the heat equation. If we replace x by $x/\sqrt{t}$ and let $t \longrightarrow \infty$, we obtain the same result as if we let $\lambda \longrightarrow \infty$ in $u_\lambda(x,t)$. This new point of view is interesting for the following heuristic reason: we expect that the limit $v(x,t)$ of $u_\lambda(x,t)$ will also be a solution of (1). Furthermore, one might assume that $v(x,t)$ is the solution with initial data $v_0(x) = \lim_{\lambda\to\infty} \lambda\, u(\lambda\, x, 0)$. Of course, the limiting solution is invariant under the scaling, so

$$v(x,t) = \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,,$$

and $v_0(x)$ is an homogeneous function of degree -1.

Such self-similar solutions have been studied previously (see [4], [2]), and we shall see in the present work how to make rigorous the previous heuristic approach.

Let us define the projection operator $\mathbb{P}$ onto the divergence free vector fields

$$\mathbb{P}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} R_1\sigma \\ R_2\sigma \\ R_3\sigma \end{pmatrix}\,, \tag{2}$$

where R_j is the Riesz transform of symbol

$$\sigma_{R_j}(\xi) = \frac{\xi_i}{|\xi|}\,, \tag{3}$$

and where

$$\sigma = R_1\, u_1 + R_2\, u_2 + R_3\, u_3\,. \tag{4}$$

Therefore $\mathbb{P}$ is a pseudo-differential operator of order 0.

We transform the system (1) into an integral equation, where $S(t) = e^{t\Delta}$ denotes the heat kernel,

$$u(x,t) = S(t)\, u_0(x) - \int_0^t \mathbb{P}\, S(t-s)\, \nabla\cdot(u\otimes u)(x,s)\, ds\,. \tag{5}$$

This equation can be solved by a classical fixed point method (see [1], [5], [6]). Following the method of [1], we remark that the bilinear term in the previous equation can be reduced to a scalar operator

$$B(f,g) = \int_0^t \frac{1}{(t-s)^2}\, G\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * (fg)\, ds\,, \tag{6}$$

where G is analytic, such that

$$|G(x)| \le \frac{C}{1+|x|^4}\,, \tag{7}$$

$$|\nabla G(x)| \le \frac{C}{1+|x|^4}\,. \tag{8}$$

This comes easily from the study of the symbol of B, as we have an exact expression under the integral. The matrix of this pseudo-differential operator has components like

$$-\frac{\xi_j\, \xi_k\, \xi_l}{|\xi|^2}\, e^{-t|\xi|^2} \tag{9}$$

off the diagonal, with an additional term $\xi_j\, e^{-t|\xi|^2}$ on it. The function G is then the inverse Fourier transform of any of these functions at $t=1$. The only thing we will need is that $G \in L^1 \cap L^\infty$.

This paper is organized as follows. In a first part, we will define the functional setting which is well-suited for our study, then study global existence in this setting, and lastly the behavior of attracting solutions for large time, if they exist. Then in a second part, we will try to state a partial converse to the Theorem 3, that is a condition on the initial data in order to obtain a convergence to a self-similar solution for large time. The third part will be devoted to a better understanding of this condition, and will include reformulations of the condition and examples.

1. Global existence in Besov spaces.

A well suited functional space to study (1) is L^3 ([5]), as $\|u_\lambda\|_{L^3} = \|u\|_{L^3}$. But homogeneous functions of degree -1 are not in L^3, and we easily see that the weak limit of $u_{0,\lambda}$ is 0. We therefore have to enlarge this functional space to include homogeneous functions of degree -1. We have chosen the homogeneous Besov spaces $\dot{B}_p^{-(1-3/p),\infty}$. We will see later they arise naturally in our problem. Let us recall their definition ([9], [10]).

Definition 1. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} \equiv 1$ in $B(0,1)$ and $\widehat{\phi} \equiv 0$ in $B(0,2)^c$, $\phi_j(x) = 2^{nj}\, \phi(2^j\, x)$, $S_j = \phi_j * \cdot$, $\Delta_j = S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbb{R}^n)$.*

- *If $s < n/p$, or if $s = n/p$ and $q = 1$, f belongs to $\dot{B}_p^{s,q}$ if and only if the following two conditions are satisfied*
  - *The partial sum*
  $$\sum_{-m}^{m} \Delta_j(f)$$
  converge to f for the topology $\sigma(\mathcal{S}', \mathcal{S})$.
  - *The sequence $\varepsilon_j = 2^{js} \|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*
- *If $s > n/p$, or $s = n/p$ and $q > 1$, let us denote $m = E(s - n/p)$. Then $\dot{B}_p^{s,q}$ is the space of distributions f, modulo polynomials of degree less than $m+1$, such that*
  - *We have $f = \sum_{-\infty}^{\infty} \Delta_j(f)$ for the quotient topology.*
  - *The sequence $\varepsilon_j = 2^{js}\, \|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*

We remark that nothing in this definition restricts s from being negative. In fact, we will use $s = -(1 - 3/p)$ which is indeed negative as $p > 3$. In the particular case where $s < 0$, it is worth noting that we can replace the condition $\varepsilon_j = 2^{js}\, \|\Delta_j(f)\|_{L^p} \in \ell^q$ by the equivalent condition $\tilde{\varepsilon}_j = 2^{js}\, \|S_j(f)\|_{L^p} \in \ell^q$. This second condition implies easily the first one, and conversely, we remark that $\tilde{\varepsilon}_j$ can be seen as a convolution between ε_j and $\eta_j = 2^{sj} \in \ell^1$. We shall obtain the following theorem which extends the results of [1].

Theorem 1. *There exists a positive function $\eta(q)$, $q > 3$ such that if $u_0 \in B_p^{-(1-3/p),\infty}$, $\nabla \cdot u_0 = 0$, $p \geq 3$, satisfies*

$$\|u_0\|_{B_q^{-(1-3/q),\infty}} < \eta(q)\,, \tag{10}$$

for a fixed $q > p$, then there exists a unique solution of (1) *such that*

$$u \in C_w([0+\infty), \dot{B}_p^{-(1-3/p),\infty})\,, \tag{11}$$

where C_w denotes the weakly continuous functions, and, if $p \leq 6$ and $u = S(t)\, u_0 + w(x,t)$, then

$$w \in L^\infty([0+\infty), L^3(\mathbb{R}^3)) \tag{12}$$

and

$$\|w\|_{L^3} < \gamma(q)\,, \tag{13}$$

where $\gamma(q)$ depends only of $\eta(q)$.

We remark that the restriction $p \leq 6$ in order to obtain (12) is merely due to the linear part: the equivalent of (12) actually holds for $p > 6$ if one considers higher order terms, if u is written as a sum of multilinear operators of u. For the sake of simplicity, we restrict ourselves to the first term, which yields this restriction.

We will prove the Theorem 1, using a fixed point argument via the following abstract lemma (Picard's theorem in a Banach space).

Lemma 1. *Let $\mathcal{E}$ be a Banach space, B a continuous bilinear application, $x, y \in \mathcal{E}$*

$$\|B(x,y)\|_{\mathcal{E}} \leq \gamma\, \|x\|_{\mathcal{E}}\, \|y\|_{\mathcal{E}}\,. \tag{14}$$

Then, if $4\,\gamma\, \|x_0\|_{\mathcal{E}} < 1$, the sequence defined by

$$x_{n+1} = x_0 + B(x_n, x_n)$$

converges to $x \in \mathcal{E}$ such that

$$x = x_0 + B(x,x) \qquad \text{and} \qquad \|x\|_{\mathcal{E}} < \frac{1}{2\,\gamma}\,. \tag{15}$$

Let us define the space

$$F_q = \{f(x,t) :\ \sup_{t>0} \|f(x,t)\|_{L^q} < +\infty\}\,. \tag{16}$$

The following characterization will be very useful.

Proposition 1. *Take $\alpha > 0$, $\gamma \geq 1$, $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\|f\| = \sup_{t>0} t^{\alpha/2}\, \|S(t) f\|_{L^\gamma} \tag{17}$$

is a norm in $\dot B_\gamma^{-\alpha,\infty}$ *equivalent to the usual dyadic one.*

Therefore, using the Sobolev inclusion

$$\dot B_p^{3/p-1,\infty} \hookrightarrow \dot B_q^{3/q-1,\infty}\,,$$

for $p \le q$, we see that $u_0 \in \dot B_q^{3/q-1,\infty}$, so that

$$\sqrt{t}\,(S(t)\,u_0)\,(\sqrt{t}\,x) \in F_q\,.$$

Then, in order to apply Lemma 1 to F_q, we are left to prove that if

$$D_t f = \sqrt{t}\,f(\sqrt{t}\,x, t)\,,$$

then $D_t B(D_t^{-1}\cdot, D_t^{-1}\cdot)$ is bicontinuous on F_q. Take $\tilde f = D_t f$ and $\tilde g = D_t g$ in F_q. We denote $M = \tilde f\,\tilde g \in F_{q/2}$. We observe that the bilinear operator (renormalized with D_t) can be written as follows

$$\widetilde B(\tilde f, \tilde g) = \int_0^1 \frac{1}{(1-\lambda)^2}\,G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * M\Big(\frac{x}{\sqrt{\lambda}}, \lambda\,t\Big)\,\frac{d\lambda}{\lambda}\,.$$

Then, by Hölder and Young inequalities, we obtain

$$\|\widetilde B(\tilde f, \tilde g)\|_{F_q} \le \int_0^1 \frac{C\,d\lambda}{(1-\lambda)^{1/2+3/(2q)}\lambda^{1-3/q}}\,\|\tilde f\|_{F_q}\,\|\tilde g\|_{F_q}\,, \tag{18}$$

which gives us $\eta(q)$. Proceeding the same way, if $p \le 6$ gives

$$\|\widetilde B(\tilde f, \tilde g)\|_{F_3} \le \int_0^1 \frac{C\,d\lambda}{(1-\lambda)^{3/q}\lambda^{1-3/q}}\,\|\tilde f\|_{F_q}\,\|\tilde g\|_{F_q}\,. \tag{19}$$

This proves (12) and (13). We have now to prove the weak convergence when $t \to 0$. Clearly $S(t)\,u_0 \underset{t\to 0}{\rightharpoonup} u_0$ by a duality argument. As for the bilinear term, if $\phi \in C_0^\infty(\mathbb{R}^3)$ and if we denote by $Q(\theta)$ the convolution operator with $G(\,\cdot\,/\sqrt{\theta})/\theta^2$,

$$\langle Q(t-s) f g(s), \phi\rangle = \langle f g(s), S(t-s)\,\widetilde Q\,\phi\rangle,$$

where $\widetilde Q$ is defined by

$$\widehat{\widetilde Q \phi}(\xi) = \frac{\xi_j\,\xi_k\,\xi_\ell}{|\xi|^2}\,\widehat\phi(\xi)\,,$$

so that $\widetilde{Q}\phi \in L^1$, like the function G defined previously. Therefore $S(t-s)\,\widetilde{Q}\phi$ is (uniformly in $t-s$) in L^γ, with $1/\gamma + 2/q = 1$. Thus

$$\Big|\Big\langle \int_0^t Q(t-s)\, fg(s)\, ds, \phi \Big\rangle\Big|$$

$$\leq C \int_0^t \|fg(s)\|_{L^{q/2}}\, ds \tag{20}$$

$$\leq C \int_0^t \frac{ds}{s^{1-3/q}}\, \|\tilde{f}\|_{F_q}\, \|\tilde{g}\|_{F_q} \tag{21}$$

$$\leq C\, t^{3/q} \longrightarrow 0\ . \tag{22}$$

The uniqueness part of the theorem follows from the construction part, so we have proved the Theorem 1, in the case where $p = q$, with q for which (10) is verified. We next remark that the solution u actually satisfies

$$\sqrt{t}\, u(\sqrt{t}\, x, t) \in F_{q'}\ , \qquad \text{for all } q' \geq p,\ q > 3\,, \tag{23}$$

and that moreover the bilinear term w satisfies

$$\sqrt{t}\, w(\sqrt{t}\, x, t) \in F_{q'}\ , \qquad \text{for } \frac{p}{2} < q' \leq p\,. \tag{24}$$

(23) is of course true for the linear part. Then, the bilinear term is in $F_{p/2}$ and in F_q for the particular q we have fixed. And by interpolation between $F_{p/2}$ and F_q it is in all $F_{q'}$ with $p/2 < q' < q$. We are left to prove (23) for the bilinear term when $q' > q$. An easy modification of (18) takes care of this situation

$$\|\widetilde{B}(\tilde{f}, \tilde{g})\|_{F_{q'}} \leq \int_0^1 \frac{C\, d\lambda}{(1-\lambda)^{1/2+3/q-3/(2q')}\, \lambda^{1-3/q}}\, \|\tilde{f}\|_{F_q}\, \|\tilde{g}\|_{F_q} \tag{25}$$

and if $q > 6$ we get all the $q' > q$. Otherwise, we have to proceed in several steps to reach a value $q' > 6$. Note that the great amount of flexibility provided by inequalities of type (18), (25) allows us to obtain this result in many different ways. In particular, we could establish the bicontinuity of the renormalized operator from $F_q \times F_q'$ to F_q' and carry along the fixed point iterations all the properties we want, provided the different continuity constants verify inequalities in the correct way, which happens to be the case. By the way, we remark that initial data in the the space $L^{3,\infty}$ are included. In fact, we have the following embedding,

Theorem 2.

$$L^{3,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_p^{-(1-3/p),\infty}\,,$$

for all $p > 3$.

In order to prove this, we will make use of the following characterization of weak Lebesgue spaces

$$f \in L^{3,\infty} \qquad \text{if and only if} \qquad \int_E |f(x)|\,dx \leq C\,|E|^{2/3}\,,$$

for all Borel sets E. In particular, if $\varphi \in \mathcal{S}$ then $\varphi * f \in L^\infty$, and therefore is in L^p, for all $p > 3$. In fact $\varphi * f \in L^{3,\infty}$, and all bounded functions in $L^{3,\infty}$ are also in L^p, as the following estimate shows

$$\sum_{j\geq 0} 2^{-jp}\,|\{x:\ 2^{-j} \leq |g| \leq 2^{-j+1}|\} \leq C \sum_{j\geq 0} 2^{j(3-p)} < +\infty\,.$$

Thus,

$$\begin{aligned} S_j(f) &= 2^{3j} \int \varphi(2^j x - 2^j y)\, f(y)\, dy \\ &= \int \varphi(2^j x - y)\, f(2^{-j} y)\, dy \\ &= 2^j \int \varphi(2^j x - y)\, 2^{-j} f(2^{-j} y)\, dy \\ &= 2^j\, h(2^j x)\,. \end{aligned}$$

Also, as h and f have the same norm in $L^{3,\infty}$, we obtain

$$\|S_j(f)\|_{L^p} \leq 2^{1-3/p}\, \|f\|_{L^{3,\infty}}\,,$$

which achieves the proof.

Now that we have solutions in the proper functional setting, we can study the asymptotic behavior of these solutions. We begin with a definition:

Definition 2. *We say that $u(x,t)$ "converges in L^p norm" to a function $V(x)$ if and only if one of the two equivalent conditions is satisfied:*

1) *For all compact intervals $[a,b] \subset (0+\infty)$*

$$u_\lambda(x,t) \overset{L^p(dx)}{\longrightarrow} \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big), \qquad \text{as } \lambda \to \infty\,,$$

uniformly for $t \in [a,b]$

2) $\sqrt{t}\, u(\sqrt{t}\, x, t) \overset{L^p(dx)}{\longrightarrow} V(x)$, *as $t \longrightarrow \infty$.*

Then we will show the following

Theorem 3. *Let us take $3 < p < +\infty$. Let $u(x,t)$ be a solution of (1) such that*

$$\sup_{t>0} \|\sqrt{t}\, u(\sqrt{t}\, x, t)\|_{L^p} < +\infty \tag{28}$$

and

$$u(x,t) \text{ converges weakly to } u_0(x) \text{ when } t \longrightarrow 0\,. \tag{29}$$

If

$$u \text{ "converges in } L^p \text{ norm" to } V\,, \tag{30}$$

then the initial data $u_0(x)$ belongs to $B_p^{-(1-3/p),\infty}$, $V(x/\sqrt{t})/\sqrt{t}$ is a self-similar solution of (1), and

$$S(t)\, u_0 \text{ "converges in } L^p \text{ norm" to } v_1(x)\,, \tag{31}$$

where $v_1(x) = S(1)\, v_0$, and v_0 is the initial data of the self-similar solution.

Note that we did not make any smallness assumption on the initial data. In other respects, when $u_0 \in B_p^{-(1-3/p),\infty}$, the condition (31) implies that

$$\lambda\, u_0(\lambda\, x) \text{ converges weakly to } v_0 \text{ when } \lambda \longrightarrow 0\,, \tag{32}$$

but this is not equivalent, and we postpone the discussion on that matter to Section 3. We recall that the integral equation is

$$\sqrt{t}\, u(\sqrt{t}\, x,t) \\ = \sqrt{t}\, (S(t)\, u_0)(\sqrt{t}\, x) - \int_0^t \mathbb{P}D_t(S(t-s)\, \nabla \cdot u \otimes u(s))\, ds\,.$$

Let us denote $U(t) = \sqrt{t}\, u(\sqrt{t}\, x,t)$. Then we have

$$U(t) = \sqrt{t}\, (S(t)\, u_0)(\sqrt{t}\, x) - \widetilde{B}(U,U)(t)\,,$$

where we still use the usual notation for the bilinear operator. By hypothesis

$$M = U \otimes U \overset{L^{p/2}}{\longrightarrow} N = V \otimes V\,.$$

We consider the difference

$$(34)\quad \Delta_t(x) = \int_0^1 \frac{1}{(1-\lambda)^2}\, G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * \Big(M\Big(\frac{x}{\sqrt{\lambda}}, \lambda\, t\Big) - N\Big(\frac{x}{\sqrt{\lambda}}\Big)\Big)\frac{d\lambda}{\lambda}\,,$$

and we want to estimate the L^p-norm. Let

$$\omega(t) = \|M(x,t) - N(x)\|_{L^{p/2}}\,,$$

so

$$\Big\| M\Big(\frac{x}{\sqrt{t}}, \lambda t\Big) - N\Big(\frac{x}{\sqrt{\lambda}}\Big)\Big\|_{p/2} = \lambda^{3/p}\, \omega(\lambda\, t)\,,$$

and therefore,

$$(35)\qquad \|\Delta_t(x)\|_{L^p} \le C \int_0^1 \frac{\omega(\lambda\, t)\, d\lambda}{(1-\lambda)^{1/2+3/(2p)}\, \lambda^{1-3/p}}\,.$$

We know that $\omega(t)$ is bounded, and

$$(1-\lambda)^{-1/2-3/(2p)}\, \lambda^{3/p-1} \in L^1(0,1)\,,$$

when $p > 3$, so we can apply the Lebesgue theorem and obtain

$$\lim_{t\longrightarrow\infty} \|\Delta_t(x)\|_{L^p} = 0\,.$$

Therefore, the bilinear term becomes

$$\frac{1}{\sqrt{t}}\, W\Big(\frac{x}{\sqrt{t}}\Big) + o(1)\,,$$

with

$$W(x) = \int_0^1 \frac{1}{(1-\lambda)^2}\, G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * N\Big(\frac{x}{\sqrt{\lambda}}\Big)\, \frac{d\lambda}{\lambda}\,.$$

The equation (33) can be written as

$$V(x) = t^{1/2}\, (S(t)\, u_0)(\sqrt{t}\, x) - W(x) + o(1)\,. \tag{36}$$

We see that the Fourier transform of $\sqrt{t}\; (S(t)\, u_0)(\sqrt{t}\, x)$ is

$$\frac{1}{t}\, e^{-|\xi|^2}\, \hat{u}_0\Big(\frac{\xi}{\sqrt{t}}\Big)\,,$$

which converges in $\mathcal{F}L^p$ to a distribution. Therefore, $\hat{u}_0(\xi/\sqrt{t})/t$ converges weakly to $\tilde{v}_0(\xi)$. On other hand, by means of (35) and (36),

$$\big\|\sqrt{t}\; (S(t)\, u_0)(\sqrt{t}\, x)\big\|_{L^p} \le C < +\infty\,, \qquad \text{for all } t > 0\,. \tag{37}$$

Hence,

$$\sup_{t>0} t^{1/2-3/(2p)}\, \|S(t)\, u_0\|_{L^p} \le C\,, \tag{38}$$

which is equivalent to $u_0 \in B_p^{3/p-1,\infty}$. Then for all λ, $u_{0,\lambda} \in B_p^{3/p-1,\infty}$, and

$$\|u_{0,\lambda}\|_{\dot{B}_p^{3/p-1,\infty}} = \|u_0\|_{\dot{B}_p^{3/p-1,\infty}}\,,$$

so that we can extract a subsequence which converges to v_0 in the space of tempered distribution, actually the convergence is in the sense of the topology $\sigma(\dot{B}_p^{3/p-1,\infty}, \dot{B}_p^{1-3/p,1})$. Then because the limit is unique, we have that $\hat{v}_0 = \tilde{v}_0$, and the whole sequence converges weakly to v_0, and moreover $v_0(x)$ belongs to $\dot{B}_p^{3/p-1,\infty}$. We remark that v_0 is necessarily homogeneous of degree -1. Let us prove that V is actually a solution of (1) where u_0 has been replaced by v_0. The set $(u_\lambda)_\lambda$ satisfies the estimates (28) and (38) uniformly in λ and indeed, for fixed $t > 0$,

$$\lambda\, u(\lambda\, x, \lambda^2\, t) \underset{\lambda\to+\infty}{\overset{L^p}{\longrightarrow}} \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,.$$

Therefore, if we pass to the limit in the equation (5) which is satisfied by u_λ, we obtain

$$\frac{1}{\sqrt{t}}\,V\Big(\frac{x}{\sqrt{t}}\Big) = S(t)\,v_0 - \int_0^t \mathbb{P}S(t-s)\nabla\cdot V(s)\otimes V(s)\,ds\,. \tag{39}$$

We see that

$$\lim_{t\downarrow 0}\frac{1}{\sqrt{t}}\,V\Big(\frac{x}{\sqrt{t}}\Big) = v_0$$

weakly, which can be obtained in the same way as in the proof of Theorem 1.

2. Initial data and asymptotic convergence.

Theorem 3 was the easy part of the study. In some sense, if we have a convergence to a function, then this function must be a self-similar solution whose initial data is obtained in a natural way from the initial data, namely the weak limit of the rescaled initial data. It would be nice if the existence of such a weak limit was enough to ensure convergence toward a self-similar solution. Unfortunately, it is untrue, and this is the purpose of Proposition 4 to explain why. Nevertheless, we can obtain a necessary and sufficient condition in order to obtain this converse to the Theorem 3. We have seen in the first theorem that it is useful to see the solution $u(x,t)$ as the sum of two terms $u(x,t) = S(t)\,u_0 + w(x,t)$, the heat term which gives a tendency, and the bilinear term which is some sort of fluctuation, more regular than the linear term. We will do the same for the self-similar solution, so that

$$v(x,t) = \frac{1}{\sqrt{t}}\,V\Big(\frac{x}{\sqrt{t}}\Big) = S(t)\,v_0 + \frac{1}{\sqrt{t}}\,W\Big(\frac{x}{\sqrt{t}}\Big)\,.$$

Theorem 4. *Let u_0 be in $\dot{B}_p^{3/p-1,\infty}$, $\nabla\cdot u_0 = 0$, $3 \le p < +\infty$, such that for some $q > p$,*

$$\|u_0\|_{\dot{B}_q^{3/q-1,\infty}} < \eta(q)\,.$$

Moreover, suppose that there exists r, $r \ge p$ and $r > 3$, such that

$$S(t)\,u_0 \text{ “converges in } L^r \text{ norm” to } v_1(x)\,. \tag{40}$$

Then $\lambda\, u_0(\lambda x)$ *converges weakly to a function* v_0 *such that* $v_1 = S(1)\, v_0$. *Further, if* $u(x,t)$ *is the solution of* (1) *with initial data* u_0, $V(x/\sqrt{t})/\sqrt{t}$ *is the solution with initial data* v_0,

$$\lim_{t\to\infty} t^{1/2-3/(2\tilde q)} \Big\| u(x,t) - \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L^{\tilde q}} = 0\,, \tag{41}$$

for all $\tilde q \geq p$, $\tilde q > 3$ *and, if* $p \leq 6$

$$\lim_{t\to\infty} \Big\| w(x,t) - \frac{1}{\sqrt{t}} W\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L^3} = 0\,. \tag{42}$$

We remark first that the case $u_0 \in L^3$ leads to $v_1 = 0$, so that $v_0 = V = 0$. In this case, (41) and (42) become the usual estimates (see [5]). Therefore, we shall assume that $r > 3$. We easily see that the convergence (40) is in $L^{\tilde q}$, $\tilde q > p$ (and even $\tilde q = p$ if $p > 3$). In fact $\sqrt{t}\, (S(t)\, u_0)(\sqrt{t}\, x)$ is bounded for the norm $\|\cdot\|_{L^{\tilde q}}$, for all $\tilde q > p$, as $\dot B_p^{3/p-1,\infty} \hookrightarrow \dot B_{\tilde q}^{3/\tilde q-1,\infty}$. Therefore, we conclude by interpolation between L^p and L^r norms or between L^r and L^∞.

We obtained

Lemma 2. *Let* $f \in \dot B_p^{3/p-1,\infty}$, $p > 3$, *such that for some* $r \geq p$,

$$\lim_{t\longrightarrow\infty} t^{1/2-3/(2r)}\, \|S(t)\, f\|_{L^r} = 0\,.$$

Then, for all $\tilde q \geq p$

$$\lim_{t\to\infty} t^{1/2-3/(2\tilde q)}\, \|S(t)\, f\|_{L^{\tilde q}} = 0\,. \tag{43}$$

From the proof of the Theorem 3, we already know that v_0, which is the weak limit of $u_{0,\lambda}$, belongs to the same Besov spaces as u_0. Therefore,

$$\|v_0\|_{\dot B_q^{3/q-1,\infty}} = \|u_0\|_{\dot B_q^{3/q-1,\infty}} < \eta(q)\,.$$

Furthermore we obtain the solutions $u(x,t)$ and $V(x/\sqrt{t})/\sqrt{t}$ by applying the Theorem 1, which used a fixed point argument. If we denote by $u^{(n)}$, respectively $V^{(n)}$, the successive approximations of u, respectively V, we remark that

$$u^{(1)}(x,t) = S(t)\, u_0\ ,$$

respectively

$$\frac{1}{\sqrt{t}} V^{(1)}\Big(\frac{x}{\sqrt{t}}\Big) = \frac{1}{\sqrt{t}} \,(S(1)\, v_0)\Big(\frac{x}{\sqrt{t}}\Big).$$

If we recall that

$$u^{(n+1)}(x,t) = S(t)\, u_0 - \int_0^t \mathbb{P} S(t-s)\, \nabla \cdot \big(u^{(n)} \otimes u^{(n)}\big)(s)\, ds\,,$$

we see from (40) that for $r = q$, we just have to prove, for a fixed n, that

$$\sqrt{t}\, u^{(n)}(\sqrt{t}\, x, t) \xrightarrow{L^q} V^{(n)}(x)\,. \tag{44}$$

This can be done using the estimates obtained in the proof of Theorem 3. Recall that we obtained a estimation on $S(t)\, u_0$ using an estimation on u and the equation. Here, the same technique applies, but we know an estimation on $S(t)\, u_0$ and u_n and deduce the estimation u_{n+1} using the equation. Then, by means of an estimates like (42) and (45) and the dominated convergence theorem, we obtain

$$\sqrt{t}\, B\big(u^{(n)}, u^{(x)}\big)(\sqrt{t}\, x, t) \xrightarrow{L^3} B\big(V^{(n)}, V^{(n)}\big)\,. \tag{45}$$

Therefore, splitting

$$u(x,t) - \frac{1}{\sqrt{t}} V\Big(\frac{x}{\sqrt{t}}\Big) = \big(u - u^{(n)}\big) + \big(V^{(n)} - V\big) + \big(u^{(n)} - V^{(n)}\big)\,,$$

we conclude with an $\varepsilon/3$ argument to obtain (41) using (44) for the fixed q we have chosen. We obtain the same result for all $\tilde{q}$ by interpolation between various L^σ norms, as in Lemma 2. We obtain (42) using (45) in the same way.

3. Understanding the condition on the initial data.

We might ask about the meaning of condition (40) and the relationship with the remark we made previously. Let us first introduce an equivalent definition of our Besov spaces $B_p^{-(1-3/p),\infty}$, $p > 3$.

Proposition 2. *Let* $\{\psi_{\varepsilon_j}\}_\varepsilon$ *be a set of 7 wavelets such that the set* $\{\psi_\varepsilon(2^j x - k)\}_{\varepsilon;j,k\in\mathbb{Z}}$ *is an orthogonal basis of* $L^2(\mathbb{R}^3)$*. Then if*

$$f(x) = \sum_{\varepsilon,j,k} \alpha_\varepsilon(j,k)\, 2^j\, \psi_\varepsilon(2^j x - k)\,,$$

$f \in \dot{B}_p^{-(1-3/p),\infty}$ *is equivalent to*

$$\sup_j \Big(\sum_k |\alpha_\varepsilon(j,k)|^p \Big)^{1/p} < +\infty\,.$$

Then we have

Proposition 3. *The following two conditions are equivalent,*

If $\alpha_\varepsilon(j,k)$ *are the wavelets coefficients of* f *under the previous normalization, and*

$$f \in B_p^{-(1-3/p),\infty}\,, \qquad 3 < p < +\infty\,, \tag{46}$$

1) f *satisfies*

$$\lambda\, f(\lambda\, x) \ \textit{converges weakly to}\ 0\,, \tag{47}$$

and

$$\lim_{j\to-\infty} \Big(\sum_k |\alpha_\varepsilon(j,k)|^p \Big)^{1/p} = 0\,. \tag{48}$$

2) *The function* f *satisfies*

$$\lim_{t\to+\infty} t^{1/2-3/(2p)}\, \|S(t) f\|_{L^p} = 0\,. \tag{49}$$

Using the previous propositions, we will later prove the promised Proposition 4, which explains why the condition (40) is necessary and sufficient in order to obtain Theorem 4. It is in fact deeply linked to the nature of the functional space we are using, rather than to the equation itself. On the other hand, no other pathological examples are known to the author other than those constructed in the proof of this proposition. On simple practical examples, where we start with a rather regular initial data, the condition will be fulfilled. Let us give an example, where we forget about the divergence free vectors and deal with a scalar function for sake of simplicity. Take

$$u_0(x) = \frac{\varepsilon}{1+|x|} \,,$$

then, by rescaling it converges weakly to

$$v_0(x) = \frac{\varepsilon}{|x|} \,.$$

We put an ε in order to comply with the smallness assumption. Then the condition (40) is verified, because the difference $\delta = u_0 - v_0$ belongs to L^3 outside of the unit ball, so that the solution of the heat equation with initial data $\delta(x)$ has its $L^3(\mathbb{R}^3 \backslash B(0,1))$ norm going to zero as time goes to infinity, and by Sobolev's embedding we get (40). In other words, what matters is the behavior of the initial data for low frequencies.

Proposition 4. *There exists a function $f \in B_3^{0,\infty}(\mathbb{R}^3)$, such that $\lambda\, f(\lambda\, x)$ converges weakly to 0 when $\lambda \to +\infty$, but such that, if $p > 3$*

$$\lim_{\lambda\to\infty} \|S(1)\,(\lambda\, f(\lambda\, x))\|_{L^p} \neq 0 \,.$$

We will now prove Proposition 3. Proposition 2 is nothing else than the usual characterization of Besov spaces with wavelets coefficients ([8]). We only changed the normalization. We restrict ourselves to Littlewood-Paley wavelets, as defined in [8], because they are closely related to Littlewood-Paley decomposition. But the same results hold for any wavelets basis, provided it has sufficient regularity. Let us recall a few useful properties of these particular wavelets basis, as they will be used later. The so-called scaling function of the wavelet basis is a function $\phi \in \mathcal{S}$, such that $\hat{\phi}(\xi) = 1$ if $-2\pi/3 < \xi < 2\pi/3$, $\hat{\phi}(\xi) = 0$ if $4\pi/3 < \xi$, $\hat{\phi}(\xi)$ is even, positive and such that $\hat{\phi}^2(\xi) + \hat{\phi}^2(2\pi - \xi) = 1$ if $0 < \xi < 2\pi$. Then the equivalent of operator S_j in the Littlewood-Paley analysis is an operator E_j, defined as follow:

Definition 3. *The operator E_j is a sum of three terms,*

$$E_j = \Sigma_j + M_j \Delta_j^- + M_j^{-1} \Delta_j^+ \,,$$

where the three terms Σ_j, Δ_j^- and Δ_j^+ are the Fourier multipliers by $\hat\phi^2(2^{-j}\xi)$, $\hat\phi(2^{-j}\xi)\,\hat\phi(2^{-j}\,(2\pi+\xi))$, and $\hat\phi(2^{-j}\xi)\,\hat\phi(2^{-j}(2\pi-\xi))$. M_- is the multiplication by $\exp(2\pi i\, 2^j x)$. We then define $D_j = E_{j+1} - E_j$, which is very close to the usual Δ_j from Definition 1.

We see that (49) can be written as

$$\lim_{\lambda\to\infty} \|S(1)\,(\lambda\, f(\lambda\, x))\|_{L^p} = 0\,. \tag{50}$$

Then, if $\phi \in \mathcal{S}$ and supp $\hat\phi$ is compact,

$$\lim_{\lambda\to\infty} \|\phi * (\lambda\, f(\lambda\, x))\|_{L^p} = 0\,. \tag{51}$$

We remark then that

$$\Big(\sum_{\varepsilon,k} |\alpha_{j,k,\varepsilon}|^p\Big)^{1/p} = \|D_0(\lambda\, f(\lambda\, x))\|_{L^p}\ , \tag{52}$$

with $\lambda = 2^{-j}$, and D_0 defined as in [8, p. 45]. Then we know from (3) that D_0 is a sum of operators like $M\Delta$, where M is a multiplication by an imaginary exponential, and Δ is a convolution by a function whose Fourier transform is compactly supported. We deduce our result by using (51). Conversely, if we suppose that (46) is true, we first prove that for ϕ as defined above,

$$\lim_{\lambda\to\infty} \|\phi * f_\lambda\|_{L^p} = 0\,.$$

Doing a rescaling and taking λ of the order of 2^N, we are left to prove that

$$\lim_{N\to\infty} 2^{N(1-3/p)} \Big\| \sum_{j<-N} \sum_{\varepsilon,k} \alpha_{f,k,\varepsilon}\, 2^j\, \psi_\varepsilon(2^j x - k)\Big\|_{L^p} = 0\,.$$

The sum on $j < -N$ being the convolution with ϕ, if we assume the support of $\hat\phi$ to be contained in the unit ball. However, for a fixed j

$$\Big\| \sum_{\varepsilon,k} \alpha_{\varepsilon,j,k}\, 2^j\, \psi_\varepsilon(2^j x - k)\Big\|_{L^p} \le C\, 2^{j(1-3/p)} \Big(\sum_{\varepsilon,k} |\alpha_{j,k,\varepsilon}|^p\Big)^{1/p}\,.$$

Then, by means of (43)

$$\Big\| \sum_{k,\varepsilon} \alpha_{\varepsilon,j,k}\, 2^j\, \psi_\varepsilon\, (2^j x - k) \Big\|_{L^p} \leq 2^{j(1-3/p)} \varepsilon_j \ ,$$

with $\lim_{j\to-\infty} \varepsilon_j = 0$.

Then

$$2^{(1-3/p)N} \sum_{j\leq -N} \varepsilon_j\, 2^{(1-3/p)j} \longrightarrow 0\,,$$

as it it a convolution between ℓ^∞ and ℓ^1. Equation (49) follows by splitting $S(1)$ into a sum of dyadic blocks.

Let us go back to Proposition 4. It helps to understand why (31) is a necessary and sufficient condition, unlike (32). In fact, let us forget for a while the proposition and suppose only (32); in the opinion of the author, the following gives a good heuristic of the situation, and could be made rigorous except that in our case, and unlike [3], it doesn't produce any useful results. With the help of the Theorem 1, we can construct a set $(u_\lambda)_\lambda$ of solutions of (1) with initial data $u_{0,\lambda}$. All the estimates do not change by rescaling, which means they are independent of λ. Therefore, we can extract a subsequence which converges in $C([t_1,t_2], \times B(0,R))$, where $t_1 > 0$, for exactly the same reasons as in [3]: by bootstrap we obtain $u_\lambda \in C([t_1,t_2], W^{1,\infty})$, with a bound independent of λ, and then we know that $W^{1,p}(B(0,R)) \hookrightarrow C(B(0,R))$. We also obtain easily that $v(x,t)$ is actually the (self-similar) solution of (1) with an initial condition v_0, which is the weak limit of $(u_{0,\lambda})_\lambda$. But to prove (41), we just have to prove

$$\lim_{\lambda\to\infty} \|u_\lambda(x,1) - v(x,1)\|_{L^q} = 0\,. \tag{53}$$

This last sentence is true if we replace L^q by $L^q(B(0,R))$, and in order to prove (53), we should prove something like

$$\lim_{R\to\infty} \|\chi_R u_\lambda(x,1)\|_{L^q} = 0\,,$$

uniformly with regards to λ, where $\chi_R(x) = \chi(x/R)$ has value zero on $B(0,1)$, and one outside $B(0,2)$. Let us deal with the linear part: suppose that $u_0 \in L^3$, $\|\chi_R u_{0,\lambda}\|_{L^3} \leq \|u_0\|_{L^3(|x|>\lambda R)}$, we obtain easily

$$\lim_{R\to\infty} \|\chi_R S(1)\, u_{0,\lambda}\|_{L^q} = 0\,, \qquad \text{uniformly in } \lambda \geq 1\,.$$

We conclude with such a proof for the two dimensional case, as in ([3]). However, if $u_0 \in \dot{B}_3^{0,\infty}$ but $u_0 \notin L^3$, then $\|\chi_R u_0\|_{\dot{B}_3^{0,\infty}} \longrightarrow 0$ is not always true when $R \longrightarrow \infty$. For instance, if we take $f = 1/|x|$, then

$$\|\chi_R f\|_{\dot{B}_3^{0,\infty}} = \|\chi f\|_{\dot{B}_3^{0,\infty}} = \text{constant}\,.$$

We could hope to have a property like

$$\lim_{R\to\infty} \|\chi_R S(1)\, u_{0,\lambda}\|_{L^q} = 0\,,$$

uniformly if $\lambda \geq 1$. In fact, it is not possible, as we will see.

Proposition 5. *There exists $f \in \dot{B}_3^{0,\infty}$ such that for all R, there exists $\lambda \geq 1$ such that*

$$\|\chi_R S(1)\, f_\lambda\|_{L^4} = 1\,.$$

Here, we have chosen $p = 3$, $q = 4$, but we could have chosen any other values.

We remark that, if λ is fixed, $S(1)f_\lambda \in L^4$ and

$$\lim_{R\to\infty} \|\chi_R S(1) f_\lambda\|_{L^4} = 0\,.$$

We will need the following lemma:

Lemma 3. *If $f \in L^4, g \in L^1$, then*

$$\Big(\int_{|x|>R} |f * g|^4\, dx\Big)^{1/4} \leq \|g\|_{L^1}\Big(\int_{|x|>R/2} |f|^4\, dx\Big)^{1/4} + \|f\|_{L^4}\int_{|x|>R/2} |g|\, dx\,.$$

Therefore, in order to prove that $\|\chi_R S(1) f_\lambda\|_{L^4}$ is large enough, we just need to find a function $g \in L^1$ such that $\|\chi_R (g * S(1) f_\lambda)\|_{L^4}$ is large. Let $\phi \in \mathcal{S}$ be a function such that $\operatorname{supp} \hat{\phi} \subset \{9/10 \leq |\xi| \leq 10/9\}$, and

$$f(x) = \sum_0^\infty 2^{-j}\,\phi(2^{-j}x - x_j)\,, \qquad \text{where } |x_j| \longrightarrow \infty\,,$$

$$\begin{aligned} 2^m f(2^m x) &= \sum_0^{m-1} 2^{m-j}\phi(2^{m-j}x - x_j) + \phi(x - x_m) \\ &\quad + \sum_{m+1}^\infty 2^{m-j}\phi(2^{m-j}x - x_j) \\ &= u_m(x) + \phi(x - x_m) + v_m(x)\,. \end{aligned}$$

We observe that the frequencies of u_m are in $\{|\xi| \geq 9/5\}$ and the ones of v_m in $\{|\xi| \leq 5/9 \leq 9/10\}$. Thus there exists $g \in \mathcal{S}$ such that

$$\operatorname{supp} \hat{g} \subset \left\{\frac{10}{18} \leq |\xi| \leq \frac{18}{10}\right\}$$

and

$$\hat{g}(\xi) = e^{|\xi|^2}\,, \qquad \text{for } \frac{9}{10} \leq |\xi| \leq \frac{10}{9}\,.$$

We take $\lambda = 2^m$, $g * S(1) f_\lambda = \phi(x - x_m)$, and

$$\lim_{m\to\infty} \int_{|x|>R} |\phi(x - x_m)|^4\,dx = \|\phi\|_{L^4}^4\,.$$

We can go further in our study of f.

If $2^m < \lambda < 2^{m+1}$, we split f as

$$\lambda\, f(\lambda\, x) = u_m(x) + v_m(x)\,,$$

where u_m is the part of frequencies $2^{-j}\,\lambda$ with $|m - j| < N$ and v_m the one where $|m - j| \geq N$.

Then, we take a test function ψ such that $0 \notin \operatorname{supp} \hat{\psi}$, and N such that $\operatorname{supp} \hat{\psi} \subset [2^{-N}, 2^N]$. Then $\int \lambda\, f(\lambda\, x)\, \psi(x)\, dx$ contains only terms with $|j - m| \leq N$, which are in finite number and go to 0 when $|x_j| \longrightarrow \infty$.

We have proved the following proposition:

Proposition 6. *There exists* $f \in \dot{B}_3^{0,\infty}$ *such that* $f_\lambda \rightharpoonup 0$ *for the weak topology* $\sigma(\dot{B}_3^{0,\infty}, \dot{B}_{3/2}^{0,1})$, *but nevertheless,* $\|\chi_R S(1) f_\lambda\|_{L^4}$ *does not go to* 0 *when* $R \longrightarrow \infty$, *uniformly in* $\lambda \geq 1$.

The reader should consult [10] to see why the test functions ψ we used are dense into $\dot{B}_{3/2}^{0,1}$.

We have now to link the Proposition 6 and the condition (31).

Proposition 7. *Let*

$$f \in \dot{B}_3^{0,\infty}, \qquad f_\lambda(x) = \lambda f(\lambda x).$$

The two following properties are equivalent:

1) *The function* f *satisfies*

$$\lim_{t\to\infty} t^{1/8} \|S(t)f\|_{L^4} = 0. \tag{55}$$

2) $f_\lambda \rightharpoonup 0$ *for the topology* $\sigma(\dot{B}_3^{0,\infty}, \dot{B}_{3/2}^{0,1})$, *and*

$$\Big(\int_{|x|>R} |S(1)f_\lambda|^4\Big)^{1/4} \leq \varepsilon_R, \tag{56}$$

with $\lim_{R\to\infty} \varepsilon_R = 0$ *independently of* $\lambda \geq 1$.

Let us prove that the first condition implies the second one. The weak convergence has already been proved. Knowing that (55) is equivalent to

$$\lim_{\lambda\to\infty} \|S(1)f_\lambda\|_{L^4} = 0,$$

this proves

$$\Big(\int_{|x|\geq R} |S(1)f_\lambda|^4\,dx\Big)^{1/4} \leq \varepsilon,$$

for $\lambda \geq \lambda_0$. It remains the case where $\lambda \in [1, \lambda_0)$. As

$$S(1)f_\lambda(x) = \lambda\,(S(\lambda^2)f)\,(\lambda\, x),$$

we remark that the functions $S(\lambda^2)f$ are in a compact set of L^4. Then there exists R_ε such that, for $\lambda \in [1, \lambda_0)$ and $R > R_\varepsilon$,

$$\Big(\int_{|x|>R} |S(1) f_\lambda|^4 \, dx \Big)^{1/4} \le \varepsilon \,,$$

the converse statement can be easily proved. In fact, if

$$f_\lambda \rightharpoonup 0 \,,$$

we obtain that

$$S(1) f_\lambda(x) \longrightarrow 0$$

uniformly on any compact set. We can therefore estimate

$$\| S(1) f_\lambda \|_{L^4}$$

by splitting for $|x| \le R$ and $|x| > R$, which ends the proof.

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Recibido: 21 de octubre de 1.996
Revisado: 5 de marzo de 1.997

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