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Initial traces of solutions to a one-phase Stefan problem in an infinite strip

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Introduction.

The main result of this paper is an integral estimate valid for non-negative solutions (with no reference to initial data) $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ to

$$(0.1) \qquad u_t - \Delta(u-1)_+ = 0 \,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0,T)) \,,$$

for $T > 0$, $n \geq 1$. Equation (0.1) is a formulation of a one-phase Stefan problem: in this connection u is the enthalpy, $(u-1)_+$ the temperature, and $u = 1$ the critical temperature of change of phase. Our estimate may be written in the form

$$(0.2) \qquad \int_{\mathbb{R}^n} u(x,t) e^{-|x|^2/(2(T-t))} \, dx \leq C \,, \qquad 0 < t < T \,,$$

where C depends on n, T, t, u but it stays bounded as $t \to 0$.

Inequalities of this kind are well known in the case of diffusion equations

$$(0.3) \qquad u_t - \Delta u^m = 0 \,, \qquad m \geq 1 \,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0,T)) \,.$$

When $m = 1$ and (0.3) reduces to the standard heat equation, it goes back to Tichonov [18], Täcklind [17] and Widder [19] that the representation formula

$$(0.4)\quad u(x,t) = (4\pi)^{-n/2}(t-t_0)^{-n/2}\int_{\mathbb{R}^n} u(\xi,t_0)\, e^{-|\xi-x|^2/(4(t-t_0))}\, d\xi\,,$$

$0 < t_0 < t < T$, $x \in \mathbb{R}^n$ actually holds for all nonnegative solutions to (0.3) with $m = 1$ (see also [14] and [3] for extensions to more general equations).

In the case of the porous media equation, *i.e.* (0.3) with $m > 1$, it was proved in [4] that

$$(0.5)\quad \begin{aligned} &\rho^{-(n+2/(m-1))}\int_{B_\rho(0)} u(x,t)dx \\ &\qquad \le c\left[T^{-1/(m-1)} + T^{n/2}\,\frac{u(0,3T/4)^{1+(m-1)n/2}}{\rho^{n+2/(m-1)}}\right], \end{aligned}$$

$0 < t < T/4$, is satisfied by all nonnegative solutions (see also [9] and [2] for extensions to more general equations).

The first consequence of estimates like (0.2), (0.4)-(0.5) is the fact that the growth of u as $|x| \to \infty$ cannot be arbitrary: indeed it must satisfy the restriction imposed by the corresponding inequality. We remark that the growth allowed by (0.2) is the same as the one given in (0.4) (*i.e.* roughly speaking, $u(\cdot,t) \sim e^{C(t)|x|^2}$ as $|x| \to \infty$) though the representation formula (0.4) obviously cannot hold for solutions to (0.1), and though the property of infinite speed of propagation does not hold for (0.1), contrarily to (0.3) for $m = 1$.

It can be easily shown that the growth predicted by (0.2) is actually optimal (see Section 2); in Section 3 we prove that solutions to (0.1) exist corresponding to arbitrary nonnegative locally integrable initial data satisfying (0.2). A by-product of this existence result is that the growth condition $u(x,t) \sim e^{C(t)|x|^2}$, $|x| \to \infty$, $t > 0$, is fulfilled in a pointwise sense (rather than in an integral sense as in (0.2): see Section 3).

A second consequence of (0.2) is the existence of a trace of u for $t = 0$ (the "initial trace"). This trace is -in general- a Radon measure and it is taken in the appropriate sense. It follows from the results proved here that the initial trace to a solution u to (0.1) belongs to $\mathcal{G}_{1/2T}$, where for $C > 0$ we define

$$(0.6)\qquad \mathcal{G}_C = \left\{\mu \text{ Radon measure in } \mathbb{R} : \int_{\mathbb{R}^n} e^{-C|x|^2}d\mu < +\infty\right\}.$$

The initial trace is actually unique (see Section 3). The technique of proof of (0.2) relies on a suitable procedure of approximation of u by compactly supported (in the space variables) solutions to (0.1) and on the use of the fundamental solution to the heat equation.

Another essential ingredient is the continuity of $(u-1)_+$: this follows from the results in [8], which in turn, may be applied since $u \in L^\infty_{\text{loc}}(\mathbb{R}^n \times (0,T))$ (*cf.* [13]) and $(u-1)_+ \in W^{1,1}_{2,\text{loc}}(\mathbb{R}^n \times (0,T))$ (see Section 2).

Also, a comparison result valid for solutions to (0.1) belonging to $\mathcal{G}_C$ is employed extensively; this follows from a generalisation of the results in [7] (see Remark 2.2).

We remark that the results found here carry over to more general equations of the type

$$u_t - L(u-1)_+ = 0 \ ,$$

where L is a linear elliptic operator, $\left(\frac{\partial}{\partial t} - L\right) z = 0$ having a fundamental solution (provided the solution u satisfies the local regularity assumptions quoted above). This follows from the proofs.

It is also clear that the assumption that u be nonnegative, can be relaxed to $u \geq -c$, $c > 0$. Indeed $v = u + c \geq 0$ fulfils

$$v_t - \Delta(v - c - 1)_+ = 0 \ .$$

The paper is organised as follows: in Section 1 we recall some known facts and establish some further regularity results. In Section 2 we prove the inequality (0.2). In Section 3 we prove some consequences of inequality (0.2); namely, existence and uniqueness of the initial trace, and, conversely, existence and uniqueness of a nonnegative solution to the Cauchy problem for (0.1) taking an initial datum $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \cap \mathcal{G}_C$.

1. Regularity results.

In this section we summarize some known facts about integrability and local boundedness of solutions, and we also prove some regularity results, which we will need in Section 2.

The following sequence of results is obtained in [13] by following the methods of [10]. For future reference we will state them as a theorem.

Theorem 1.1. *Let $0 \le u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ be a solution in the sense of $\mathcal{D}'(\mathbb{R}^n \times (0,T))$ of*

$$u_t = \Delta(u-1)_+ \,, \tag{1.1}$$

i.e.

$$\int_{\mathbb{R}^n} \int_0^T (u\varphi_t + (u-1)_+ \Delta\varphi)\, dx\, dt = 0 \,,$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n \times (0,T))$. Then

i) *For any smooth bounded domain $D \subset \mathbb{R}^n$ and $0 < a < b < T$, there exist nonnegative measures ν_a, ν_b on D and μ on $\partial D \times [a,b]$ such that*

$$\begin{aligned} \int_D \psi(x,b) d\nu_b = \int_D \psi(x,a)\, d\nu_a & \\ + \iint_{D\times(a,b)} \Big((u-1)_+\Delta\psi + u\frac{\partial\psi}{\partial t}\Big)\, dx\, dt & \\ + \iint_{\partial D\times[a,b]} \frac{\partial\psi}{\partial n} d\mu \,, & \end{aligned} \tag{1.2}$$

for any $\psi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ such that $\psi\big|_{\partial D\times[a,b]} = 0$. Here $\dfrac{\partial}{\partial n}$ denotes differentiation with respect to the inward unit normal to ∂D.

ii) *For a.e. t, $d\nu_t = u(x,t)\, dx$, $0 < t < T$. We remark that by considering a countable sequence of domains $\{D_m\}$ invading $\mathbb{R}^n$, ν_t may be taken independent of the domain D.*

iii) *$u \in L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$.*

iv) *If for $\bar{t} > 0$ and $E \subset \mathbb{R}^n$ measurable, $|E| > 0$, $u(y,\bar{t}) < 1$ a.e. in E, then $u(x,t) \le u(x,\bar{t})$ for a.e. $x \in E$ and $0 < t < \bar{t}$ (this was found independently also in* [1]*).*

v) (Comparison) *If $0 \le v \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ is a solution to* (1.1) *such that the traces of u and v on the parabolic boundary of $D \times (\bar{t}, t)$ (D a smooth bounded domain in $\mathbb{R}^n$, $\bar{t} > 0$) are ordered, the same order holds for u and v a.e. in $D \times (\bar{t}, t)$.*

vi) *$(u-1)_+$ satisfies*

$$\Delta(u-1)_+ - \frac{\partial}{\partial t}(u-1)_+ \ge 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0,T)) \,.$$

vii) $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$.

REMARK. In fact Theorem 1.1 applies to local solutions of (1.1) defined in $\Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, with the obvious modifications.

Lemma 1.2. *For any solution $0 \le u$ to* (1.1) *in the sense made precise in Theorem* 1.1, $(u-1)_+$ *belongs to* $W_2^{1,1}(K)$, *for any compact* $K \subset \mathbb{R}^n \times (0,T)$.

PROOF. For the sake of notational simplicity, we assume that u is a solution to (1.1) in $\mathbb{R}^n \times (0, T+\delta)$, $\delta > 0$. Defining u and $(u-1)_+$ as zero for $t < 0$, we let

$$u_m(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} u(y,s)\rho_m(x-y)\tau_m(t-s)\,dy\,ds\,,$$

and analogously we set $w_m = (u-1)_+ * \rho_m\tau_m$, where ρ_m and τ_m are the usual (compactly supported) mollifiers. Therefore we have by (1.1)

$$\frac{\partial}{\partial t}u_m(x,t) - \Delta w_m(x,t) = 0\,, \qquad \text{in } B_R \times (t_0, T)\,, \tag{1.3}$$

where $R > 0$ and $T > t_0 > 0$ are arbitrarily fixed, and m is large enough.

Let $\alpha(s) = (s-1)_+$, and $\alpha_m(s)$ a smooth regularization of α such that $\alpha_m(s) = \alpha(s)$ for $s \ge 1 + 1/m$ and $\alpha'_m(s) > 0$ for $s \ge 0$. Define v_m as the solution to

$$\begin{cases} v_t = \Delta\alpha_m(v) & \text{in } B_R \times (t_0,T)\,, \\ v(x,t_0) = u_m(x,t_0) & \text{for } x \in B_R\,, \\ \alpha_m(v) = w_m & \text{on } \partial B_R \times (t_0,T)\,. \end{cases} \tag{1.4}$$

By standard calculations (see *e.g.* [6]) it follows that for any $K \subset B_R \times (t_0,T)$, K compact,

$$\|v_m\|_{L^\infty(B_R\times(t_0,T))} \le C(M)\,,$$
$$\|\nabla\alpha_m(v_m)\|_{L^2(K)} + \|\alpha_m(v_m)_t\|_{L^2(K)} \le C(M,K)\,,$$

where $M = \|u\|_{L^\infty(B_{2R}\times(t_0/2,T))} < \infty$.

Therefore, choosing a subsequence of $\{v_m\}$, which we denote by $\{v_m\}$ again, we may prove that a $v \in L^\infty(B_R \times (t_0,T))$ exists such that

$$
\begin{aligned}
v_m &\to v && \text{weakly in } L^2(B_R \times (t_0,T))\,,\\
\alpha_m(v_m) &\to (v-1)_+ && \text{a.e. in } B_R \times (t_0,T)\,,\\
\nabla\alpha_m(v_m) &\to \nabla(v-1)_+ && \text{weakly in } L^2_{\text{loc}}(B_R \times (t_0,T))\,,\\
\frac{\partial}{\partial t}\alpha_m(v_m) &\to \frac{\partial}{\partial t}(v-1)_+ && \text{weakly in } L^2_{\text{loc}}(B_R \times (t_0,T))\,.
\end{aligned}
\tag{1.5}
$$

Next we show that $(u-1)_+ \equiv (v-1)_+$ in $B_R \times (t_0,T)$. We follow [16], and introduce the smooth function

$$
\psi_m(x,t) = \int_t^T (\alpha_m(v_m) - w_m)(x,\tau)\,d\tau\,.
$$

We subtract the first equation of (1.4) from (1.3), multiply by ψ_m and integrate by parts over $B_R \times (t_0,T)$, finding

$$
\begin{aligned}
\int_{t_0}^T\int_{B_R} &(v_m - u_m)(\alpha_m(v_m) - w_m)\,dx\,dt\\
&= -\int_{t_0}^T\int_{B_R} \nabla(\alpha_m(v_m) - w_m)\\
&\qquad\cdot\int_t^T \nabla(\alpha_m(v_m) - w_m)(x,\tau)\,d\tau\,dx\,dt\\
&= \frac{1}{2}\int_{t_0}^T\int_{B_R}\frac{\partial}{\partial t}\Big|\int_t^T \nabla(\alpha_m(v_m) - w_m)d\tau\Big|^2 dx\,dt\\
&= -\frac{1}{2}\int_{B_R}\Big|\int_{t_0}^T \nabla(\alpha_m(v_m) - w_m)d\tau\Big|^2 dx \le 0\,.
\end{aligned}
\tag{1.6}
$$

Then we let $m \to \infty$ in (1.6) to get, taking into account (1.5),

$$
\int_{t_0}^T\int_{B_R}(v-u)((v-1)_+ - (u-1)_+)\,dx\,d\tau \le 0\,.
\tag{1.7}
$$

Since

$$
(v-u)((v-1)_+ - (u-1)_+) \ge [(v-1)_+ - (u-1)_+]^2 \ge 0\,,
$$

(1.7) implies $(u-1)_+ \equiv (v-1)_+$, and due to (1.5), the proof is completed.

REMARK. It follows from the proof above that Lemma 1.2 applies to any nonnegative distributional solution of the Stefan problem (1.1), defined in a cylinder $B_R \times (0,T)$, since such solutions are locally bounded, according to Theorem 1.1.

Corollary 1.3. *Let $0 \le u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ be a (distributional) solution to (1.1). Then $(u-1)_+$ is continuous.*

Corollary 1.3 is a consequence of the results of [8] (which apply to solutions with first derivatives in L^2) and of Lemma 1.2, and it holds in fact for local solutions as pointed out in the remark above.

A useful consequence of the regularity results given above is the following

Lemma 1.4. *Let u be as in Corollary 1.3. For all $R > 0$, $0 < \varepsilon < T/2$,*

$$\operatorname*{ess\,sup}_{|t-t_0|<\delta;\ 0<\varepsilon<t_0,\, t<T-\varepsilon} \int_{B_R} |u(x,t) - u(x,t_0)| dx \to 0 \qquad \text{as} \quad \delta \downarrow 0 .$$

PROOF. We define $h = u - (u-1)_+$; because of Theorem 1.1. iv) we have for a.e. $t_0, t \in (0,T)$, $t > t_0$,

$$\begin{aligned} \int_{B_R} |u(x,t) - u(x,t_0)|\, dx &\le \int_{B_R} |(u-1)_+(x,t) - (u-1)_+(x,t_0)|\, dx \\ &\quad + \int_{B_R} (h(x,t) - h(x,t_0))\, dx \\ &\equiv A(t_0,t) + B(t_0,t) . \end{aligned}$$

Note that $A(\cdot,\cdot)$ is continuous, owing to Corollary 1.3; then $A(t_0,t) \to 0$ as $t \downarrow t_0$. Also

$$B(t_0,t) \le \int_{B_{2R}} \psi(x)(h(x,t) - h(x,t_0))\, dx ,$$

for a nonnegative $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(x) \equiv 1$ for $|x| \le R$ and $\psi(x) \equiv 0$ for $|x| > 2R$. Employing (1.2) and the local integrability of u, the claim follows.

REMARK. In the following we say that a family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of Radon measures belongs *uniformly* to $\mathcal{G}_c$ if

$$\int_{\mathbb{R}^n} e^{-c|x|^2} d\mu_\varepsilon \leq M < \infty, \qquad \text{for all } \varepsilon > 0,$$

for a constant M independent of ε.

2. The main estimate.

This section will be devoted to the proof of our main result (see (2.1) below), enabling us to identify the growth at infinity of any (distributional, $L^1_{\rm loc}(\mathbb{R}^n \times (0,T))$) nonnegative solution to (1.1). Therefore we can identify the natural class to which such solutions belong, without any a priori requirement on initial values. We want to single out that, as a consequence of this result, even though compactly supported solutions to (1.1) propagate with finite speed, solutions of (1.1) cannot grow at infinity faster than solutions to the heat equation.

Theorem 2.1. *Let $0 \leq u \in L^1_{\rm loc}(\mathbb{R}^n \times (0,T))$, be a solution to* (1.1) *in the sense of $\mathcal{D}'(\mathbb{R}^n \times (0,T))$. Then for any $0 < \beta < 1$, there exists a constant $M = M(u,n,T,\beta)$ such that*

$$\int u(x,t) e^{-|x|^2/(4\beta(T-t))}\, dx \leq M\,, \tag{2.1}$$

for any $0 < t < T/2$.

Moreover, this inequality is optimal, i.e. there exist solutions actually exhibiting the maximal growth allowed by (2.1).

REMARK. The constant M in (2.1) actually may be assumed to depend on u only through a bound for $|u|$ over $B_1(x_1) \times (T/2, 3T/4)$, through $|x_1|$, and $u(x_1, 3T/4)$. Here $x_1 \in \mathbb{R}^n$ is chosen such that $u(x_1, 3T/4) > 1$. This follows from the proof below, and from the results in [8] on the modulus of continuity of $(u-1)_+$.

PROOF. For the sake of notational simplicity, we may assume that u is defined in $\mathbb{R}^n \times (0, T+\lambda)$, for some $\lambda > 0$. Choose $0 < t_0 < T$. For any $\rho \geq 1$, define v_ρ as the (compactly supported) solution to

$$\begin{cases} \dfrac{\partial}{\partial t} v_\rho - \Delta(v_\rho - 1)_+ = 0\,, & \text{in } \mathcal{D}'(\mathbb{R}^n \times (t_0, T+\lambda))\,, \\ \|v_\rho(\cdot,t) - u(\cdot,t_0)\chi_{B_\rho}(\cdot)\|_{L^1(\mathbb{R}^n)} \to 0\,, & t \downarrow t_0\,. \end{cases} \tag{2.2}$$

The existence of v_ρ is provided *e.g.* by semigroup arguments (see [5] and references given therein).

We remark that as a consequence of Theorem 1.1.v),

$$\text{(2.3)} \quad \rho_1 \geq \rho_2 \ \text{implies} \ v_{\rho_1} \geq v_{\rho_2}\,, \qquad \text{a.e. in } \mathbb{R}^n \times (t_0, T+\lambda)\,,$$

$$\text{(2.4)} \quad v_\rho \leq u\,, \qquad \text{a.e. in } \mathbb{R}^n \times (t_0, T+\lambda)\,.$$

Hence, due to monotonicity (implied by (2.3)) and local boundedness (implied by (2.4) and vii) of Theorem 1.1) there exists

$$u[t_0] = \lim_{\rho \to \infty} v_\rho \qquad \text{in } \mathbb{R}^n \times (t_0, T+\lambda)\,,$$

and $u[t_0]$ solves

$$\text{(2.5.a)} \quad w_t - \Delta(w-1)_+ = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (t_0, T+\lambda))\,,$$

$$\text{(2.5.b)} \quad w(x,t) \to u(x,t_0)\,, \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^n)\,, \qquad \text{as } t \downarrow t_0\,,$$

$$\text{(2.6)} \quad u[t_0] \leq u\,, \qquad \text{a.e. in } \mathbb{R}^n \times (t_0, T+\lambda)\,.$$

Equation (2.5.a) is obviously implied by the definition of $u[t_0]$. The convergence of $u[t_0](\cdot,t) \to u(\cdot,t_0)$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ follows by

$$v_\rho(x,t) - u(x,t_0) \leq u[t_0](x,t) - u(x,t_0) \leq u(x,t) - u(x,t_0)\,,$$

for all $\rho \geq 1$, when we take into account Lemma 1.3 and (2.2). Alternatively, the second of (2.5) may be derived subtracting the weak formulations of (2.2) and (2.5.a), and using again $u[t_0] \geq v_\rho$, with a suitable choice of the test functions.
Assume first that a point $P_0 \equiv (x_0, T)$ exists such that $u[t_0](P_0) > 1$: then we may find an $\varepsilon > 0$ such that

$$\text{(2.7)} \quad u[t_0](x,t) \geq 1 + 2\varepsilon\,, \quad \text{for all } (x,t) \in B_\varepsilon(x_0) \times (T-\varepsilon\,,\ T)\,.$$

Here we are using the continuity of $(u[t_0](x,t) - 1)_+$, provided by [8] and Lemma 1.2. We may assume that for $\rho \geq \rho_0$ large enough,

$$\text{(2.8)} \quad v_\rho(x,t) \geq 1 + \varepsilon\,, \quad \text{for all } (x,t) \in B_\varepsilon(x_0) \times (T-\varepsilon\,,\ T)\,.$$

Then, since $v_\rho(\cdot,t)$ is compactly supported in $\mathbb{R}^n$, we have for all $t_0 \leq t < T$, $0 < \delta < T - t$,

$$\text{(2.9)} \quad \begin{aligned} 0 &= \int_{\mathbb{R}^n} v_\rho\,\eta\,dx\Big|_t^{T-\delta} - \int_{\mathbb{R}^n}\int_t^{T-\delta} (v_\rho\eta_\tau + (v_\rho - 1)_+\Delta\eta)\,dx\,d\tau \\ &= \int_{\mathbb{R}^n} v_\rho\,\eta\,dx\Big|_t^{T-\delta} - \int_{\mathbb{R}^n}\int_t^{T-\delta} (v_\rho - (v_\rho - 1)_+)\,\eta_\tau\,dx\,d\tau\,, \end{aligned}$$

where we have defined

$$\eta(x,\tau) = \frac{1}{(4\pi)^{n/2}} \frac{1}{(T-\tau)^{n/2}} e^{-|x-x_0|^2/(4(T-\tau))} ,$$

so that $\eta_\tau + \Delta\eta = 0$, $\tau < T$.

Note that since

$$v_\rho - (v_\rho - 1)_+ = (v_\rho - (v_\rho - 1)_+ - 1) + 1 = -(1 - v_\rho)_+ + 1 ,$$

it holds

$$\begin{aligned} - \int_{\mathbb{R}^n} \int_t^{T-\delta} & (v_\rho - (v_\rho - 1)_+) \eta_\tau \, dx \, d\tau \\ &= \int_{\mathbb{R}^n} \int_t^{T-\delta} (1 - v_\rho)_+ \eta_\tau - \int_{\mathbb{R}^n} \eta(x,\tau)\, dx \Big|_t^{T-\delta} \\ &= \int_{\mathbb{R}^n} \int_t^{T-\delta} (1 - v_\rho)_+ \eta_\tau \\ &\leq \iint_{A_\varepsilon} |\eta_\tau| \, dx \, d\tau , \end{aligned}$$

where

$$A_\varepsilon = \Big(\mathbb{R}^n \times (0,T)\Big) \setminus \Big(B_\varepsilon(x_0) \times (T-\varepsilon, T)\Big) .$$

The last integral is majorized by a constant depending on T, n, and ε only; since ε depends in turn on t_0 (through $u[t_0]$), we denote this constant $c_1(\varepsilon(t_0))$. Thus, letting $\delta \to 0$ in (2.9), we have for $t_0 < t < T$

$$\begin{aligned} v_\rho(x_0,T) &+ c_1(\varepsilon(t_0)) \\ &\geq (4\pi)^{-n/2}(T-t)^{-n/2} \int_{\mathbb{R}^n} v_\rho(x,t)\, e^{-|x-x_0|^2/(4(T-t)}\, dx , \end{aligned} \tag{2.10}$$

implying, owing to (2.6)

$$\begin{aligned} (4\pi)^{n/2}&[u(x_0,T) + c_1(\varepsilon(t_0))] \\ &\geq (T-t)^{-n/2} \int_{\mathbb{R}^n} u[t_0](x,t) e^{-|x-x_0|^2/(4(T-t))}\, dx , \end{aligned} \tag{2.11}$$

for $t_0 < t < T$.

Due to Theorem 1.1. iv), estimate (2.11) is trivial if $u[t_0](x,T) \leq 1$ for all $x \in \mathbb{R}^n$. Of course, the same conclusions hold for any time level

$t_1 \in (t_0, T)$ and the relative $u[t_1]$, but the constant c_1 in (2.11), as well as the point x_0, are a priori different from the ones found above.

Next we show that actually the same x_0 and ε as those employed in estimating $u[t_0](x,t)$ may be used in estimating $u[t_1](x,t)$, $t_1 < t < T$. Note that, because of (2.6),

$$u[t_0](x,t_1) \leq u(x,t_1) = u[t_1](x,t_1) \ ,$$

for a.e. $x \in \mathbb{R}^n$.

Taking into account (2.11) and the analogous inequality valid for $u[t_1]$, $t_1 \leq t < T$, for all $T - t_1 > \sigma > 0$ we may find constants $\gamma = \gamma(\sigma)$, $M = M(\sigma, t_0, t_1)$ such that

$$\sup_{t_1 \leq t \leq T-\sigma} \int_{\mathbb{R}^n} \left(u[t_0](x,t) + u[t_1](x,t)\right) e^{-\gamma|x|^2} dx \leq M \ . \tag{2.12}$$

Therefore a comparison principle may be applied ([7] and Remark 2.2), giving

$$u[t_0] \leq u[t_1] \qquad \text{a.e. in } \mathbb{R}^n \times (t_1, T)$$

(indeed σ in (2.12) may be chosen arbitrarily small). Hence

$$u[t_1](x,t) \geq 1 + 2\varepsilon \ , \quad \text{for all } (x,t) \in B_\varepsilon(x_0) \times (T - \varepsilon, T) \ , \tag{2.13}$$

where x_0 and ε are the same as in (2.7). We may now repeat the estimation of $u(\cdot, t_1) = u[t_1](\cdot, t_1)$ with this choice of x_0 and ε and, taking into account the arbitrariness of $t_1 \in (t_0, T)$, we get

$$\begin{aligned} &(4\pi)^{n/2}\Big(u(x_0,T) + c_1(\varepsilon(t_0))\Big) \\ &\qquad \geq (T-t)^{-n/2} \int_{\mathbb{R}^n} u(x,t) e^{-|x-x_0|^2/(4(T-t))} \, dx \ , \end{aligned} \tag{2.14}$$

for almost all $t_0 < t < T$. We still have to get rid of the dependence on t_0 of both x_0 and ε in the left hand side of (2.14). We reason as follows.

Estimate (2.14) implies that, for all $0 < \sigma < T - t_0$,

$$u(\cdot, t) \in \mathcal{G}_{1/2\sigma} \quad \text{uniformly for} \quad t_0 < t < T - \sigma \ .$$

Then, since

$$\|u[t_0](x, t + t_0) - u(x, t + t_0)\|_{L^1_{\text{loc}}(\mathbb{R}^n)} \to 0 \ , \qquad \text{as } t \downarrow 0 \ ,$$

the uniqueness result in [7] can be applied (see Remark 2.2), to find

$$u[t_0] = u \qquad \text{a.e. in } \mathbb{R}^n \times (t_0, T) \,.$$

Next we choose (x_1, T) and $\varepsilon_1 > 0$ such that for $t_0 < T$,

$$u[t_0](x,t) \equiv u(x,t) \geq 1 + 2\varepsilon_1 \,, \tag{2.15}$$

for all $(x,t) \in B_{\varepsilon_1}(x_1) \times (T - \varepsilon_1, T)$. We remark that the choice of x_1 and ε_1 does not depend on t_0. We may now repeat the arguments leading to (2.10) and (2.11), to find

$$\begin{aligned} &(4\pi)^{n/2}\big(u(x_1,T) + c_1(\varepsilon_1)\big) \\ &\qquad \geq (T - t_0)^{-n/2} \int_{\mathbb{R}^n} u(x,t_0) e^{-|x-x_1|^2/(4(T-t_0))}\, dx \,, \end{aligned} \tag{2.16}$$

for a.e. $0 < t_0 < T$. We note again that in (2.16) x_1, ε_1 may be chosen without any further constraint than (2.15). Inequality (2.1) follows easily from (2.16) (see [11, Lemma 4, p. 25]).

In order to show that (2.1) is optimal, just consider the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0,T)\,, \\ u(x,0) = e^{|x|^2} & \text{for } x \in \mathbb{R}^n \,. \end{cases}$$

It is well known that a solution u exists in $\mathbb{R}^n \times (0,T)$, $T = 1/4$. Since $u \geq 1$ in $\mathbb{R}^n \times (0,T)$, u may be seen as a solution to (1.1) in $\mathbb{R}^n \times (0,T)$, $u(\cdot,t) \sim e^{c(t)|x|^2}$ as $|x| \to \infty$.

REMARK 2.2. (Comparison in $\mathcal{G}_c$, personal communication of J.E. Bouillet). In order to adapt the proof of [7] to our situation it has to be shown that for suitable $0 < \tau < t < T$

$$\int_{R-1}^{R+1} \int_{\tau}^{t} \oint_{\partial B_{\tilde{R}}} f^r\, \nabla\phi \cdot \nu \, dS\, d\theta\, d\tilde{R} \to 0\,, \qquad \text{as } R \to \infty \,,$$

uniformly in $r > 0$, where

$$|\nabla\phi(x,t)| \leq k\, e^{-(R-R_1-1)^2/(8(t-\tau))}\,,$$

$|x| = R > R_1 > 0$, $k > 0$ given,

$$\sup_{0<t<T} \int_{\mathbb{R}^n} |f(x,t)|\, e^{-c|x|^2}\, dx < \infty \,, \tag{2.17}$$

and

$$f^r(x,t) = \int_{|x-z|<r} f(z,t)\rho_r(x-z)\,dz\,,$$

with ρ_r the standard (compactly supported) mollifier. But for R large enough, $0 < r < 1$, we have

$$\begin{aligned}
&\Big|\int_{R-1}^{R+1}\int_\tau^t \oint_{\partial B_{\tilde R}} f^r\,\nabla\phi\cdot\nu\,dS\,d\theta\,d\tilde R\Big| \\
&\quad\le k\int_{R-1}^{R+1}\int_\tau^t \oint_{\partial B_{\tilde R}}\int |f(z,\theta)|\rho_r(x-z) \\
&\qquad\qquad\cdot e^{-(R-R_1-1)^2/(8(t-\tau))}\,dz\,dS_x\,d\theta\,d\tilde R \\
&\quad\le k\int_\tau^t\int_{B_{R+1}\setminus B_{R-1}}\int |f(z,\theta)|\rho_r(x-z)\,e^{-|z|^2/(16(t-\tau))}\,dz\,dx d\theta \\
&\quad\le k\int_\tau^t\int_{B_{R+2}\setminus B_{R-2}} |f(z,\theta)|\,e^{-|z|^2/(16(t-\tau))}\,dz\,d\theta \to 0\,,
\end{aligned}$$

as $R\to\infty$ if $1/(16(t-\tau)) > c$ due to (2.17).

3. Applications.

In this section we will derive some consequences of Theorem 2.1. Corollary 3.1 (existence of a unique initial trace in the class $\mathcal{G}_c$) is an extension of the result of [12], valid under the a priori assumption $u(\cdot,t)\in\mathcal{G}_c$, $T>t>0$, to any nonnegative distributional solution $u\in L^1_{\rm loc}(\mathbb{R}^n\times(0,T))$ to (1.1). Corollary 3.2 extends the comparison result of [7] (uniqueness in $\mathcal{G}_c\cap L^\infty_{\rm loc}(\mathbb{R}^n\times(0,T))$) to distributional solutions belonging to $L^1_{\rm loc}(\mathbb{R}^n\times(0,T))$ (see also Remark 2.2). The proof of Theorem 3.4 (existence for the Cauchy problem when $u_0(x)\in\mathcal{G}_c\cap L^1_{\rm loc}(\mathbb{R}^n)$) yields as a by-product the fact that the growth at infinity "at most as $e^{c|x|^2}$" is actually pointwise.

Corollary 3.1. *For any nonnegative distributional solution $u\in L^1_{\rm loc}$ $(\mathbb{R}^n\times(0,T))$, there exists a unique nonnegative locally finite measure $\mu\in\mathcal{G}_c$, $c=1/(2T)$, such that*

$$\lim_{t\downarrow 0}\int u(x,t)\varphi(x)\,dx = \int\varphi(x)\,d\mu\,,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

PROOF. The proof of [12] for solutions in the class

$$\Big\{u(x,t) : \sup_{R>1} \frac{1}{|B_R|} \int_{B_R} u(x,t)\, e^{-c'|x|^2}\, dx < M\,, \quad 0 < t < T\Big\}$$

applies with only minor changes to the present situation. In fact the class $\{u(x,t) : u(\cdot,t) \in \mathcal{G}_c\,,$ uniformly on $t \in (0,T)\}$ is contained in a functional class of the type above, for a suitable c'.

Corollary 3.2. *Let $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ be two nonnegative solutions (in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$) to (1.1) such that*

i) $\|(u-v)_+(\cdot,t)\|_{L^1_{\mathrm{loc}}} \to 0$ *as* $t \downarrow 0$, *or*
ii) $n = 1$, *and for every* $\varphi \in C_0(\mathbb{R})$,

$$\int_{\mathbb{R}} (u-v)_+(x,t)\, \varphi(x)\, dx \to 0 \qquad as \quad t \downarrow 0\,.$$

Then $u(x,t) \le v(x,t)$ *a.e. in* $\mathbb{R}^n \times (0,T)$.

Corollary 3.2 follows as in [7], once we use the estimates provided by Theorem 2.1 and Remark 2.2.

Corollary 3.3. *Any nonnegative distributional solution $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ belongs in fact to $L^\infty((0,T-\varepsilon) : L^1_{\mathrm{loc}}(\mathbb{R}^n))$, for all $\varepsilon > 0$.*

Theorem 3.4. *Let $0 \le u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $u_0 \in \mathcal{G}_c$, $c > 0$. Then there exists a (unique) nonnegative solution to*

$$\text{(3.1.a)} \qquad u_t = \Delta(u-1)_+\,, \qquad in \quad \mathcal{D}'(\mathbb{R}^n \times (0,T))$$

$$\text{(3.1.b)} \qquad \|u(x,t) - u_0(x)\|_{L^1_{\mathrm{loc}}(\mathbb{R}^n)} \to 0, \qquad t \downarrow 0\,,$$

with $T = 1/(4\,c)$.

PROOF. Let

$$u_0^{(n)}(x) = \begin{cases} 0 & \text{if } |x| \ge n\,, \\ u_0(x) & \text{if } |x| < n \quad \text{and } u_0(x) < n\,, \\ n & \text{if } |x| < n \quad \text{and } u_0(x) \ge n\,, \end{cases}$$

and $u^{(n)}(x,t)$ be the (semigroup) solution to (3.1.a) with initial datum $u_0^{(n)}$ (see [5] and references given therein).

It should be pointed out that $(u^{(n)}-1)_+ \le v$, where v is the solution to the heat equation with initial datum $(u_0-1)_+$. This can be shown by local comparison between $(u^{(n)}-1)_+$ (which is compactly supported) and the solution $v^{(n)}$ to the heat equation with initial datum $(u_0^{(n)}-1)_+$:

$$(u^{(n)}(x,t)-1)_+ \le v^{(n)}(x,t) \uparrow v(x,t) ,$$

employing Theorem 1.1. vi). Since $\{u^{(n)}(x,t)\}$ is increasing in n and bounded (by $v+1$), there exists

$$u(x,t) = \lim_{n\to\infty} u^{(n)}(x,t) .$$

By Lebesgue's bounded convergence theorem, u is a solution in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$ to (0.1).

The convergence $u(\cdot,t) \to u_0(\cdot)$ in the sense of measures can be proved subtracting the weak formulations of (2.1) for $u(x,t)$ and $u^{(n)}(x,t)$ with a suitable choice of the test function, and using the fact that

$$\|u^{(n)}(x,t) - u_0^{(n)}(x)\|_{L^1_{\rm loc}(\mathbb{R}^n)} \to 0 , \qquad \text{as } t \downarrow 0 .$$

Then relation (3.1.b) follows using $(u-1)_+ \le v$ and reasoning as in the proof of Lemma 1.4 and of (2.5.b). Finally uniqueness follows from Corollary 3.2.

Corollary 3.5. (of the proof of Theorem 3.4). *For a.e.* $(x,t),(y,s) \in \mathbb{R}^n \times (0,T)$, $0 < s_0 < s < t < T$,

$$u(y,s) \le (v[s_0](x,t)+1)\, e^{c\left(|x-y|^2/(t-s)+\log((t-s_0)/(s-s_0))+1\right)} ,$$

where $v[s_0]$ *solves*

$$\begin{cases} v_t = \Delta v , \\ v(x,s_0) = (u(x,s_0)-1)_+ . \end{cases}$$

Here $c = c(n)$.

PROOF. Apply Theorem 3.4 with $u_0(x) = u(x,s_0)$ in $\mathbb{R}^n \times (s_0,T)$ and combine it with the known inequality for solutions to the heat equation (see [15]).

REMARK 3.6. The existence result in Theorem 3.4 can be extended to the case of initial datum a Radon measure $\mu \in \mathcal{G}_C$; in this case relation (3.1.b) is replaced by

$$u(\cdot, t) \to \mu \qquad \text{as } t \downarrow 0 \text{ in the sense of measures.}$$

The proof follows the lines of the one given above for $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, employing truncation and regularization of μ.

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