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Complex tangential characterizations of Hardy-Sobolev Spaces of holomorphic functions

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Introduction and results.

Let Ω be a C^∞-domain in $\mathbb{C}^n$. It is well known that a holomorphic function on Ω behaves twice as well in complex tangential directions (see [GS] and [Kr] for instance). It follows from well known results (see [H], [RS]) that some converse is true for any kind of regular functions when Ω satisfies

(P) The tangent space is generated by the Lie brackets of real and imaginary parts of complex tangent vectors.

In this paper, we are interested in the behavior of holomorphic Hardy-Sobolev functions in complex tangential directions. Our aim is to give a characterization of these spaces, defined on a domain which satisfies the property (P), involving only complex tangential derivatives. Our method, which is elementary, is to prove pointwise estimates between gradients and tangential gradients of holomorphic functions and, next, to use them to obtain the characterization of Hardy-Sobolev spaces for $1 \le p < \infty$. To give precise statements, let us introduce some notations.

Write $\Omega = \{r < 0\}$, where r is a C^∞ function satisfying $dr \neq 0$ on $\partial\Omega = \{r = 0\}$.

Define the holomorphic complex normal vector field

$$N = \sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z_j}} \frac{\partial}{\partial z_j}$$

and the (holomorphic) complex tangential gradient of order k of u, $\{\nabla_T^k u\}$, as follows. It is the vector $\{L_{I,J}\, u \;:\; I, J \in \{1, \ldots, n\}^k\}$ where

$$L_{i,j} = \frac{\partial r}{\partial z_i} \frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j} \frac{\partial}{\partial z_i}, \qquad i, j \in \{1, \ldots, n\}$$

and $L_{I,J} = L_{i_1,j_1} \ldots L_{i_k,j_k}$ when $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k)$.

The family $\{L_{i,j} \;:\; i, j \in \{1, \ldots, n\}, i \neq j\}$ gives a total system of complex tangential vector fields on $\partial\Omega$ (respectively on $\partial\Omega_\varepsilon = \{r = -\varepsilon\}, \quad 0 < \varepsilon < \varepsilon_0$).

For z in Ω near $\partial\Omega$, we set

$$C_2(z) = \max\left\{\left|\partial\overline{\partial} r(L_{i,j}, \overline{L_{k,l}})\right|(z) :\; i, j, k, l \in \{1, \ldots, n\}, i \neq j, k \neq l\right\}.$$

It is known that C_2 is different from zero on $\partial\Omega$ if and only if Ω satisfies (P).

Denote by $\delta(\cdot)$ the distance to the boundary $\partial\Omega$. We use the following mean-value operator for z in Ω near $\partial\Omega$

$$\mathrm{Mean}^{Q(z)}(u) = \frac{1}{|Q(z)|} \int_{Q(z)} |u(\zeta)|\, dV(\zeta)$$

where $Q(z)$ is a polydisc centered at z whose size is $c\,\delta(z)$ in the complex normal direction and $\sqrt{c\,\delta(z)}$ in the complex tangential ones, c chosen so that, in particular, $Q(z) \subset \Omega$.

Now, let us state our pointwise estimates.

Pointwise estimates. *Let $k \in \mathbb{N}$, $0 < p < \infty$. For each $z_0 \in \partial\Omega$, there exist a neighborhood $V(z_0)$ and a constant C such that, for every holomorphic function g in Ω and every $z \in V(z_0) \cap \Omega$*

$$\delta(z)^{kp/2} \left|\nabla_T^k g(z)\right|^p \leq C\, \mathrm{Mean}^{Q(z)}(|g|^p), \tag{1}$$

$$
\begin{aligned}
(2) \qquad & C_2(z)^{kp}\delta(z)^{kp}\left|\nabla^k g(z)\right|^p \\
& \qquad \leq C\,\mathrm{Mean}^{Q(z)}\left(\delta^{kp/2}\left|\nabla_T^k g\right|^p + \mathrm{Rest}^k(g)^p\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Rest}^k(g) = \delta^{1/2}\Big(&\sum_{r=0}^{k-1}\sum_{\substack{1\leq j+r<(k+r)/2\\ j\geq 0}} \mathcal{O}(\delta^{(k-1)/2})\left|\nabla^j\nabla_T^r g\right| \\
&+\sum_{r=0}^{k-1}\sum_{\substack{(k+r)/2\leq j+r\leq k\\ j\geq 0}} \mathcal{O}(\delta^{j+r/2})\left|\nabla^j\nabla_T^r g\right|\Big).
\end{aligned}
$$

Inequality (1) is what we call the direct estimates. Such an estimate is implicit in some works but is not explicitely written (see [GS] and [Kr]).

Inequality (2) is the new estimate we prove; it says that, up to a rest, the complex tangential gradient controls all the gradient. It is what we call the converse estimates.

The main terms in these estimates are homogeneous in the following sense: each derivative of order r in the complex tangential directions appears with a factor $\delta^{r/2}$ and each one in the other directions with a factor δ^r. In the remaining terms, they appear with a smaller factor. Compared with the usual mean-value property of holomorphic functions, these pointwise estimates show that $\nabla_T^k g$ behaves as a complete gradient of order $k/2$. Obviously, by the mean-value property and inequality (1), we can majorize $\mathrm{Rest}^k(g)$ by the mean-value of $\delta^{p/2}|g|^p$. However, for technical reasons, we will need this complicated form of $\mathrm{Rest}^k(g)$ (in order to be able to apply Hardy inequalities for example).

Now, we give the precise definition of the Hardy-Sobolev spaces. We will identify a small neighborhood of $\partial\Omega$ in $\overline{\Omega}$, with $\partial\Omega\times[0,s_0[$. More precisely, we choose a map $\Phi:\partial\Omega\times[0,s_0[\to\overline{\Omega}$ such that

- Φ is a diffeomorphism of $\partial\Omega\times[0,s_0[$ onto a neighborhood $\overline{\Omega}\cap U$ of $\partial\Omega$ in $\overline{\Omega}$,
- $\Phi(\zeta,0)=\zeta$ for every $\zeta\in\partial\Omega$,
- $\delta(\Phi(\zeta,t))\simeq t$ for every $\zeta\in\partial\Omega$ and every $0<t<s_0$.

For $0<p<\infty$ and $k\in\mathbb{N}$, the holomorphic Hardy-Sobolev space $\mathcal{H}_k^p(\Omega)$ is defined to be the space of holomorphic functions g which satisfy

$$\sup_{0<t<s_0}\left|\nabla^k g\circ\Phi(\cdot,t)\right|\in L^p(\partial\Omega),$$

We will see that this definition does not depend on the choice of the function Φ.

To state our main theorem, we need some other definitions.

We know, by [NSW] for instance, that we can define a non-isotropic metric $d(\cdot,\cdot)$ on $\partial\Omega$ which satisfies $d(p,q) \simeq |p-q|^2 + |\langle \operatorname{Im} N_p, p-q\rangle|$ (see [S1]).

We define the following quantities for every smooth function u and every aperture $\alpha > 0$.

- The maximal admissible function:

$$\text{for every } \zeta \in \partial\Omega, \quad \mathcal{M}_\alpha u(\zeta) = \sup\{|u(z)| : \quad z \in \mathcal{A}_\alpha(\zeta)\}$$

where $\mathcal{A}_\alpha(\zeta)$ denotes the admissible approach region

$$\mathcal{A}_\alpha(\zeta) = \{\Phi(\eta,t) : \quad \eta \in \partial\Omega,\ 0 < t < s_0, \quad d(\eta,\zeta) < \alpha\, t\}.$$

- The admissible area function:

$$\text{for every } \zeta \in \partial\Omega, \quad S_\alpha^q u(\zeta) = \left(\int_{\mathcal{A}_\alpha(\zeta)} |u|^2\, \delta^q \frac{dV}{\delta^{n+1}}\right)^{1/2}.$$

- The Littlewood-Paley function:

$$\text{for every } \zeta \in \partial\Omega, \quad G^q(u)(\zeta) = \left(\int_0^{s_0} |u \circ \Phi(\zeta,t)|^2\, t^q \frac{dt}{t}\right)^{1/2}.$$

- The non-isotropic maximal operator on $\partial\Omega$:

$$\text{for every } \zeta \in \partial\Omega, \quad Mf(\zeta) = \sup_{t>0} \frac{1}{|B^d(\zeta,t)|} \int_{B^d(\zeta,t)} |f|\, d\sigma$$

where $B^d(\zeta,t)$ is a non-isotropic ball on $\partial\Omega$, defined with the aid of the metric d, centered at ζ, of radius t.

Since d is a metric and defines a space of homogeneous type, the non-isotropic maximal operator on $\partial\Omega$ is bounded from $L^p(\partial\Omega)$ into itself, for every $1 < p \leq \infty$ and is of weak type $(1,1)$ (see [S1]).

Before stating our results in terms of complex tangential derivatives, we recall some known results about Hardy-Sobolev spaces (where S_α and G stand respectively for S_α^0 and G^0).

Auxiliary Theorem. *Let α be a fixed aperture, $k \in \mathbb{N}$. For every $0 < p < \infty$ and every holomorphic function g, the following are equivalent:*

(1) $g \in \mathcal{H}_k^p(\Omega)$
(2) $S_\alpha(\delta \nabla^{k+1} g) \in L^p(\partial\Omega)$
(3) $G(\delta \nabla^{k+1} g) \in L^p(\partial\Omega)$
(4) $\mathcal{M}_\alpha(\nabla^k g) \in L^p(\partial\Omega)$

and the corresponding norms are equivalent.

Now, we can state our main result describing $\mathcal{H}_k^p(\Omega)$ only in terms of complex tangential derivatives.

Main Theorem. *Let α be a fixed aperture, $k \in \mathbb{N}$. For every $1 \le p < \infty$ and every holomorphic function g, the following are equivalent:*

(1) $g \in \mathcal{H}_k^p(\Omega)$
(2) $S_\alpha\left(\delta \nabla \nabla_T^{2k} g\right) \in L^p(\partial\Omega)$
(3) $\mathcal{M}_\alpha\left(\nabla_T^{2k} g\right) \in L^p(\partial\Omega)$
(4) $\sup_{0<t<s_0} \left|\nabla_T^{2k} g\right| \in L^p(\partial\Omega)$

and the corresponding norms are equivalent.

REMARK. The results of Main Theorem are true for a larger class of p. We will give later the details and precise statements. For instance, it follows from our results the following corollary.

Corollary. *Let α be a fixed aperture. For every $0 < p < \infty$ and every holomorphic function g, we have*

$$g \in \mathcal{H}^p(\Omega) \quad \text{if and only if} \quad \|S_\alpha(\delta \nabla_T^2 g)\|_{L^p(\partial\Omega)} < \infty .$$

For the unit ball in $\mathbb{C}^n$, a characterization of Hardy-Sobolev spaces in terms of complex tangential derivatives is given by Ahern and Bruna in [AB]. But, in this particular case, it is easier since the complex tangential derivatives of holomorphic functions are also harmonic. In the case of strictly pseudoconvex domains, Cohn gives a characterization of Hardy-Sobolev spaces $\mathcal{H}_k^p$ with $p > 1$ in terms of maximal function of complex tangential gradients of order $2k$. But his proof uses the representation of the Szegö kernel given by Kerzman and Stein and, so, needs pseudoconvexity (see [Co]).

Our method is to use the pointwise estimates essentially to show that one can define the Hardy-Sobolev spaces in terms of the admissible area function of ordinary gradients as well as in terms of the admissible area function of complex tangential gradients. Then, for the other characterizations, we adapt, when it is possible, the method of [FS]. The technical difficulties are due to the fact that, for a holomorphic function g, $\nabla_T^k g$ is no longer holomorphic nor harmonic, but we can show that, locally and up to a rest, it satisfies some mean-value properties analogous to the ones satisfied by holomorphic functions. When the technics of [FS] do not work, as far as we know, we use a trick which consists in writing $\nabla_T^k g$ as the sum of the solution of a Dirichlet's Problem with data $\Delta\nabla_T^k g$ and a harmonic function -the idea being that the harmonic part is the principal term and the other part a rest. To estimate this rest, we prove an estimate on the Dirichlet's problem in mixed L^p norms with weight.

In a previous paper (see [G1]), we gave analogous pointwise estimates in the more general context of domains of finite type and we applied them to characterize Lipschitz, Besov and Sobolev spaces of holomorphic functions. These estimates allow to generalize some of the results of Main Theorem to domains of finite type. But, as we are not able to deal completely with this case, we restrict ourselves to the case of the (P) property. For more details on finite type domains, see [G2].

1. Pointwise estimates.

1.1 Preliminaries. Change of coordinates and polydiscs.

Let $z_0 \in \partial\Omega$. As $dr(z_0) \neq 0$, we can assume that $\partial r/\partial z_1 \neq 0$ on a neighborhood $V(z_0)$ of z_0. We shall need the following lemma which is well known (see [C] for instance).

Lemma 1.1. *For each $z \in V(z_0) \cap \Omega$, there exists a polynomial biholomorphism Φ_z from $\mathbb{C}^n$ to itself such that*

1) *The coefficients of Φ_z are C^∞ with respect to z and the jacobian of Φ_z is uniformly bounded from both side for $z \in V(z_0)$.*

2) *The defining function $\varrho = r \circ \Phi_z$ of $\Omega_z = \Phi_z^{-1}(\Omega)$ satisfies*

$$\varrho(\zeta) = r(z) + \operatorname{Re}\zeta_1 + \sum_{j,k=2}^{n} a_{j,k}(z)\,\zeta_j\overline{\zeta_k} + \mathcal{O}\left(|\zeta_1|\,|\zeta| + |\zeta'|^3\right)$$

where $\zeta' = (\zeta_2, \dots, \zeta_n)$.

3) *There exists a constant* c *on* $V(z_0)$ *such that the polydisc* $R(z)$ *defined by*

$$R(z) = \left\{ \zeta \in \mathbb{C}^n : \quad |\zeta_1| < c\,\delta(z) , \quad \sum_{k=2}^{n} |\zeta_k|^2 < c\,\delta(z) \right\}$$

is included in Ω_z. *So*

$$Q(z) = \Phi_z\left(R(z)\right) \subset \Omega ,$$

and there exist $C_1, C_2 > 0$ *such that*

$$P_{C_1}(z) \subset Q(z) \subset P_{C_2}(z)$$

where

$$P_C(z) = \left\{ \zeta \in \mathbb{C}^n : \quad |(z-\zeta)N_z| \le C\,\delta(z) , \quad |z-\zeta|^2 \le C\,\delta(z) \right\} .$$

4) $C_2(z) \simeq \max\left\{ |a_{j,k}(z)| : \ j,k \in \{2, \dots, n\} \right\}$ *for every* $z \in V(z_0)$.

REMARKS. This lemma allows to estimate ϱ and its derivatives; it shows that, since ϱ is C^∞,

$$\frac{\partial \varrho}{\partial \zeta_j}(0) = 0 \quad \text{for } j = 2, \dots, n , \quad \frac{\partial \varrho}{\partial \zeta_1}(0) = \frac{1}{2} ,$$
$$\frac{\partial^2 \varrho}{\partial \zeta_j \partial \zeta_k}(0) = 0 , \quad \frac{\partial^2 \varrho}{\partial \zeta_j \partial \overline{\zeta_k}}(0) = a_{j,k}(z) \quad \text{for } j,k = 2, \dots, n .$$

Let us denote by $Q_t(z)$ the set

$$\Phi_z\left(\left\{ \zeta \in \mathbb{C}^n : \quad |\zeta_1| < t, \sum_{k=2}^{n} |\zeta_k|^2 < t \right\} \right).$$

It is shown in [NSW] that, for every $\eta \in \partial\Omega$, $Q_t(\eta) \cap \Omega$ is comparable with the "tent"

$$\hat{B}(\eta, t) = \left\{ z \in \overline{\Omega} \cap U : \quad d(\pi(z), \eta) \le t, \ \delta(z) \le t \right\} ,$$

where π denotes the projection on $\partial\Omega$. This allows to see that there exist $c_1, c_2, c_3 > 0$ such that

$$\Phi^{-1}\left(Q(\Phi(\eta,t))\right) \subset B^d(\eta, c_1 t)\times]c_2 t, c_3 t[\,.$$

(It suffices to remark the two following properties:

- $\eta \in Q(\Phi(\eta, \tilde{C}t))$ for some constant $\tilde{C}$; so, $Q(\Phi(\eta,t)) \subset Q_{c_1 t}(\eta)$ for some constant c_1,
- $\delta(\cdot) \simeq t$ on $Q(\Phi(\eta,t))$.)

PROOF. There exists a biholomorphism

$$\begin{aligned}\Phi_z(\zeta) = \Big(& z_1 + d_0(z)\,\zeta_1 + \sum_{j=2}^{n} d_1^j(z)\,\zeta_j \\ & + \sum_{j,k=2}^{n} d_2^{j,k}(z)\,\zeta_j\zeta_k,\; z_2+\zeta_2, \ldots, z_n+\zeta_n\Big)\end{aligned}$$

such that $\varrho(\zeta) = r\circ\Phi_z(\zeta)$ takes the given form; explicitely

$$d_0(z) = \frac{1}{2}\left(\frac{\partial r}{\partial z_1}(z)\right)^{-1}, \quad d_1^j(z) = \left(\frac{\partial r}{\partial z_1}(z)\right)^{-1}\frac{\partial r}{\partial z_j}(z),$$

$$\begin{aligned}d_2^{j,k}(z) = -d_0(z)\Big[&\frac{\partial^2 r}{\partial z_1^2}(z)\,d_1^j(z)\,d_1^k(z) + \frac{\partial^2 r}{\partial z_1\partial z_k}(z)\,d_1^j(z) \\ &+ \frac{\partial^2 r}{\partial z_1\partial z_j}(z)\,d_1^k(z) + \frac{\partial^2 r}{\partial z_j\partial z_k}(z)\Big]\,.\end{aligned}$$

Properties 3) and 4) follow from a direct computation.

Our aim, now, is to show that, after this change of coordinates, $\nabla_T^k g$, for g holomorphic, is, locally and up to a rest, a function which satisfies some mean-value properties.

Let $z \in V(z_0)\cap\Omega$. Let us consider the family

$$L_i' = \frac{\partial\varrho}{\partial\zeta_1}\frac{\partial}{\partial\zeta_i} - \frac{\partial\varrho}{\partial\zeta_i}\frac{\partial}{\partial\zeta_1}, \quad i\in\{2,\ldots,n\},$$

of complex tangential vector fields in Ω_z. Since by assumption $\partial\varrho/\partial\zeta_1 \neq 0$ on a neighborhood of $0\in\Omega_z$, the family L_i' for $i\in\{2,\ldots,n\}$ gives a total system of complex tangential vector fields in a neighborhood of $0 \in \Omega_z$. We need the following technical lemma which allows to write locally the field $L'^K = L_2'^{k_2} \dots L_n'^{k_n}$ as a sum of a field with coefficients which are almost harmonic and a rest. As before, we write $\zeta = (\zeta_1, \zeta')$ where $\zeta' = (\zeta_2, \dots, \zeta_n)$.

Lemma 1.2. *Let $K = (k_2, \dots, k_n) \in \mathbb{N}^{n-1}$. On the set*

$$R(z) = \left\{ \zeta \in \mathbb{C}^n : \quad |\zeta_1| \le c\,\delta(z), \sum_{k=2}^{n} |\zeta_k|^2 \le c\,\delta(z) \right\} \subset \Omega_z ,$$

we have

$$\begin{aligned} L'^K &= \prod_{j=2}^{n} \left[\frac{1}{2} \frac{\partial}{\partial \zeta_j} - \left(\sum_{l=2}^{n} a_{j,l}(z) \overline{\zeta_l} \right) \frac{\partial}{\partial \zeta_1} \right]^{k_j} + \mathrm{Rest}_{|K|}^K \\ &= F^K + \mathrm{Rest}_{|K|}^K \end{aligned}$$

where Rest_k^K *stands for*

$$\mathrm{Rest}_k^K = \sum_{\substack{1 \le j+|R| \le k \\ R \le K}} b_{j,R} \frac{\partial^{j+|R|}}{\partial \zeta_1^j \partial \zeta^R}$$

with $C^\infty(\Omega_z)$-functions $b_{j,R}$, $1 \le j + |R| \le k$, which satisfy the following properties: they are uniformly bounded and if $(2j + |R| - k + 1)/2 > 0$, $b_{j,R}(0) = 0$ and, for every $\zeta \in R(z)$, $|b_{j,R}(\zeta)| \le C\,\delta(z)^{(2j+|R|-k+1)/2}$ for some uniform constant C.

PROOF. We will give the proof in $\mathbb{C}^2$ to simplify.

By convention, we will denote by $\mathcal{O}_r$ any regular function defined on Ω_z, uniformly bounded, satisfying, if $r > 0$, $\mathcal{O}_r(0) = 0$ and, for every $\zeta \in R(z)$, $|\mathcal{O}_r(\zeta)| \le C\delta(z)^r$ for some constant C. Let

$$L' = \frac{\partial \varrho}{\partial \zeta_1} \frac{\partial}{\partial \zeta_2} - \frac{\partial \varrho}{\partial \zeta_2} \frac{\partial}{\partial \zeta_1}$$

be a complex tangential vector field in Ω_z. We can show by induction on k that there exist some constants $c_{j,r}$, $1 \le j + r \le k$, such that

$$L'^k = \sum_{1 \le j+r \le k} \sum_{E_{k,j,r}} c_{j,r} \left(\prod_{i=1}^{k} \frac{\partial^{m_i+n_i} \varrho}{\partial \zeta_1^{m_i} \partial \zeta_2^{n_i}} \right) \frac{\partial^{j+r}}{\partial \zeta_1^j \partial \zeta_2^r} ,$$

where $E_{k,j,r}$ denotes the set of couples (m_i, n_i), $i = 1, \dots, k$, which are in alphabetical order and satisfy $\sum_{i=1}^k m_i = k - j$, $\sum_{i=1}^k n_i = k - r$ with $m_i + n_i \geq 1$.

When $j + r = k$, necessarily, there are j couples which are equal to $(0,1)$ and r couples which are equal to $(1,0)$. So, the corresponding terms get the following form

$$C\left(\frac{\partial \varrho}{\partial \zeta_1}\right)^r \left(\frac{\partial \varrho}{\partial \zeta_2}\right)^{k-r} \frac{\partial^k}{\partial \zeta_1^{k-r} \partial \zeta_2^r} .$$

But, we know by the Taylor expansion of ϱ given in Lemma 1.1 that, for $\zeta \in R(z)$,

$$\frac{\partial \varrho}{\partial \zeta_1}(\zeta) = \frac{1}{2} + \mathcal{O}_{1/2}(\zeta)$$

and

$$\frac{\partial \varrho}{\partial \zeta_2}(\zeta) = a_{2,2}(z)\overline{\zeta_2} + \mathcal{O}_1(\zeta) = \mathcal{O}_{1/2}(\zeta) .$$

This allows to see that the terms of order k take the form given in the lemma. Let us look at the terms with $j + r < k$. Since ϱ is a C^∞ function, the coefficients are uniformly bounded on $V(z_0)$. So, it suffices to consider the case when $2j + r \geq k$ and to show that, in this case, the corresponding coefficients are equal to zero at the origin and are bounded by $C\,\delta(z)^{(2j+r-k+1)/2}$ on $R(z)$.

So, let j, r fixed with $j + r < k$ and $2j + r \geq k$. Let us denote by J the number of couples (m_i, n_i) with m_i equal to zero. As $\sum m_i = k - j$, necessarily $J \geq j$. Assume that $m_1 = \cdots = m_J = 0$, necessarily $n_i \geq 1$ for $i \leq J$. Let us denote by K the number of couples $(0, n_i)$, $i \leq J$ with $n_i = 1$. We have $n_1 = \cdots = n_K = 1$ and

$$k - r = \sum_{i=1}^k n_i = K + \sum_{j=K+1}^{J} n_i + \sum_{j=J+1}^{k} n_i \geq K + 2(J - K) .$$

So, $K \geq 2J - k + r \geq 2j - k + r$. So, if $K \geq 2j - k + r + 1$, the corresponding coefficient which is known to be a $\mathcal{O}_{K/2}$ (as there are at least K factors $\partial\varrho/\partial\zeta_2$), is bounded by $C\delta^{(2j-k+r+1)/2}$ and is equal to zero at the origin since, by assumption on the indices, $K \geq 1$. Otherwise, if $K < 2j - k + r + 1$ then, necessarily $J = j$, $K = 2j - k + r$ and there exists at least one couple $(0, n_i)$ with $n_i = 2$ for $K + 1 \leq i \leq j$ (since otherwise all the n_i, for $K+1 \le i \le j$, should be strictly bigger than 2 and we would have

$$\begin{aligned} k-r &= \sum_{i=1}^{k} n_i \\ &> K + 2(J-K) \\ &= 2j-k+r+2(k-j-r) = k-r\,, \end{aligned}$$

which is impossible).

So, we use the fact that $\partial^2\varrho/\partial\zeta_2^2\,(\zeta) = \mathcal{O}_{1/2}(\zeta)$ for every $\zeta \in R(z)$. This gives that the corresponding coefficient is a $\mathcal{O}_{(K+1)/2}$ and so, is equal to zero at the origin and is bounded by $C\delta^{(2j-k+r+1)/2}$ since there are $K = 2j-k+r$ factors $\partial\varrho/\partial\zeta_2$ and at least one $\partial^2\varrho/\partial\zeta_2^2$.

In the new system of coordinates, near the origin, $\partial/\partial\zeta_i \simeq L'_i$, this allows to show the following corollary.

Corollary 1.3. *For every $l \in \mathbb{N}$, every $K \in \mathbb{N}^{n-1}$ and every function $u \in C^\infty(\overline{\Omega_z})$, we have, on $R(z)$*

$$\left|\frac{\partial^{l+|K|}u}{\partial\zeta_1^l\partial\zeta'^K}\right| \le C \sum_{\substack{1\le j+|R|\le l+|K| \\ R\le K}} \left|L'^R\frac{\partial^j u}{\partial\zeta_1^j}\right| .$$

PROOF. Lemma 1.2 allows to write on $R(z)$

$$\begin{aligned}\left(\frac{\partial\varrho}{\partial\zeta_1}\right)^{|K|}\frac{\partial^{|K|}}{\partial\zeta'^K} = L'^K + \sum_{|J|=0}^{|K|-1}\mathcal{O}(\delta(z)^{(|K|-|J|)/2})\frac{\partial^{|K|}}{\partial\zeta_1^{|K|-|J|}\partial\zeta'^J} \\ + \mathrm{Rest}_{|K|-1}^{K} f\,,\end{aligned}$$

and we can assume that $\partial\varrho/\partial\zeta_1 \neq 0$ on $R(z)$. So, this allows us to estimate each derivative $|\partial^{l+|K|}u/\partial\zeta_1^l\partial\zeta'^K|$ in terms of $|L'^K\partial^l u/\partial\zeta_1^l|$ and of derivatives either of order strictly less than $l+|K|$, or of order with respect to ζ' strictly less than $|K|$.

Applying successively this estimate, we obtain the lemma.

Now, we are going to see that, for f holomorphic in Ω_z, $F^K f$ satisfies some mean-value property. Let us give the following definition (*cf.* [AB] and [G1] for instance).

Definition. *Let Ω be an open set in $\mathbb{C}^n$. Let $K = (k_1, \ldots, k_n)$ be a multi-index of integers. A function $F \in C^\infty(\Omega)$ is called* $(\mathrm{AB})_K$ *if*

$$\frac{\partial^{k_j} F}{\partial \overline{\zeta_j}^{k_j}} = 0, \qquad \text{for } j = 1, \ldots, n.$$

To simplify, we will assume that K is fixed in the following and we will write (AB) instead of $(\mathrm{AB})_K$. For every $\zeta \in \mathbb{C}$, $r > 0$, we denote by $D(\zeta, r)$ the disc $\{z \in \mathbb{C} : |z - \zeta| \leq r\}$. Then, we have the following lemma (see [AB]).

Lemma 1.4. *For every $L, M \in \mathbb{N}^n$, $0 < p < \infty$, there exists a constant C such that, for every* (AB) *function F in Ω, every $\zeta = (\zeta_1, \ldots, \zeta_n) \in \Omega$ and every $r = (r_1, \ldots, r_n) \in (]0, +\infty[)^n$ with $D(\zeta_1, r_1) \times \cdots \times D(\zeta_n, r_n) \subset \Omega$, we have*

$$\left| \frac{\partial^{|L|+|M|} F}{\partial \overline{\zeta}^L \partial \zeta^M} (\zeta) \right|^p \leq \frac{C}{\prod_{j=1}^n r_j^{p(L_j + M_j) + 2}} \int_{D(\zeta_1, r_1) \times \cdots \times D(\zeta_n, r_n)} |F|^p dV.$$

So, for f holomorphic, $F^K f$ is an (AB) function. This allows to prove the following result for example which gives a mean-value property with rest for $\nabla_T^k \nabla^l g$.

Corollary 1.5. *For every $l, k \in \mathbb{N}$ and every holomorphic function g in Ω, we have, for every $z \in V(z_0) \cap \Omega$,*

$$\delta(z)^{k/2+l} \left| \nabla_T^k \nabla^l g \right| (z) \leq C \operatorname{Mean}^{Q(z)} \left(\delta^{k/2+l} \left| \nabla_T^k \nabla^l g \right| + \delta^l \operatorname{Rest}^k (\nabla^l g) \right).$$

(Rest^k *has the same meaning as in the Pointwise Estimates*).

PROOF. It suffices to consider the case $l = 0$. The general case follows replacing g by $\nabla^l g$.

We set $f = g \circ \Phi_z$. Since, by assumption, the family

$$L_i' = \frac{\partial \varrho}{\partial \zeta_1} \frac{\partial}{\partial \zeta_i} - \frac{\partial \varrho}{\partial \zeta_i} \frac{\partial}{\partial \zeta_1}, \qquad i \in \{2, \ldots, n\}$$

gives a total system of complex tangential vector fields in a neighborhood of $0 \in \Omega_z$, each iterated complex tangential vector field of order

k at 0 is obtained as a linear combination with smooth coefficients of ${L'}^K$ at 0, where $K \in \mathbb{N}^{n-1}$, $1 \leq |K| \leq k$.

So, to estimate $\nabla_T^k g(z)$, we have to estimate ${L'}^K f(0)$ for every $K \in \mathbb{N}^{n-1}$ with $1 \leq |K| \leq k$.

To simplify, we will only estimate ${L'}_i^k f(0)$ and we will write ${L'}_i^k f = F_i^k + \mathrm{Rest}_i^k$.

By Lemma 1.2, we have

$$\begin{aligned}\left|{L'}_i^k f(0)\right| &\leq C\left(\left|\frac{\partial^k f}{\partial \zeta_i^k}(0)\right| + \sum_{1\leq 2j+r\leq k-1}\left|\frac{\partial^{j+r} f}{\partial \zeta_1^j \partial \zeta_i^r}(0)\right|\right)\\ &= C\left(\left|F_i^k f(0)\right| + \sum_{1\leq 2j+r\leq k-1}\left|\frac{\partial^{j+r} f}{\partial \zeta_1^j \partial \zeta_i^r}(0)\right|\right)\end{aligned}$$

Now $F_i^k f$ is an (AB) function since f is holomorphic and so, by Lemma 1.4,

$$\begin{aligned}\left|{L'}_i^k f(0)\right| &\leq C\,\mathrm{Mean}^{R(z)}\left(\left|F_i^k f\right| + \sum_{1\leq 2j+r\leq k-1}\left|\frac{\partial^{j+r} f}{\partial \zeta_1^j \partial \zeta_i^r}\right|\right)\\ &\leq C\,\mathrm{Mean}^{R(z)}\left(\left|{L'}_i^k f\right| + \mathrm{Rest}_i^k f\right).\end{aligned}$$

Now, we will show that, after a change of coordinates, $\delta(z)^{k/2}\sum_i \mathrm{Rest}_i^k f$ is bounded by $\mathrm{Rest}^k(g)$. By Corollary 1.3, we have on $R(z)$

$$\begin{aligned}\mathrm{Rest}_i^k f \leq C\Big(&\sum_{r=0}^{k-1}\sum_{1\leq j+r<(k+r)/2}\left|{L'}_i^r \frac{\partial^j f}{\partial \zeta_1^j}\right|\\ &+\sum_{r=0}^{k-1}\sum_{(k+r)/2\leq j+r\leq k}\delta(z)^{(2j+r-k+1)/2}\left|{L'}_i^r\frac{\partial^j f}{\partial \zeta_1^j}\right|\Big).\end{aligned}$$

This inequality allows to conclude.

1.2. Direct estimates.

We are going to prove the following theorem.

Theorem A. *Let Ω be a C^∞ domain in $\mathbb{C}^n$. Let $k \in \mathbb{N}$, $l \in \mathbb{N}^*$, $0 < p < \infty$. For every $z_0 \in \partial\Omega$, there exist a neighborhood $V(z_0)$ of z_0 and a constant C such that, for every holomorphic functions g in Ω and every $z \in V(z_0) \cap \Omega$, we have*

$$(1) \quad \delta(z)^{kp/2} \left|\nabla_T^k g(z)\right|^p \leq C \operatorname{Mean}^{Q(z)}(|g|^p).$$

$$(2) \quad \delta(z)^{kp/2+lp} \left|\nabla^l \nabla_T^k g(z)\right|^p \leq C \operatorname{Mean}^{Q(z)} \left(\delta^{lp} \sum_{j=1}^{l} \left|\nabla^j g\right|^p\right).$$

PROOF. Let $z \in V(z_0)$. We set $f(\zeta) = g(\Phi_z(\zeta))$, then f is holomorphic in $\Omega_z = \Phi_z^{-1}(\Omega)$.

In order to show the first part of Theorem A, we will apply Cauchy Formula to f. Since f is holomorphic, by subharmonicity property of f^p, we have, for every $j \in \mathbb{N}$, every $R \in \mathbb{N}^{n-1}$ and every $p > 0$

$$\left|\frac{\partial^{j+|R|} f}{\partial\zeta_1^j \partial\zeta'^R}(0)\right|^p \leq \frac{1}{(c\,\delta(z))^{n+1+p(j+|R|/2)}} \int_{|\zeta_1|, |\zeta'|^2 \leq c\delta(z)} |f(\zeta)|^p \, dV(\zeta).$$

The domain of integration is $R(z)$. In order to conclude, we recall that each iterated complex tangential vector field of order k at 0 is obtained as a linear combination with smooth coefficients of L'^K at 0, where $K \in \mathbb{N}^{n-1}$, $1 \leq |K| \leq k$. Furthermore, by Lemma 1.2, $L'^K f(0)$ is almost equal to $(1/2)^{|K|} \partial^{|K|} f / \partial\zeta'^K(0)$. More precisely, for each $K \in \mathbb{N}^{n-1}$, we can write, with the help of Lemma 1.2, that

$$\begin{aligned}
\left|L'^K f(0)\right| &\leq C \left(\left|\frac{\partial^{|K|} f}{\partial\zeta'^K}(0)\right| + \sum_{1 \leq 2j+|R| \leq |K|-1} \left|\frac{\partial^{j+|R|} f}{\partial\zeta_1^j \partial\zeta'^R}(0)\right| \right) \\
&\leq C \left(\delta(z)^{-|K|p/2} + \sum_{1 \leq 2j+|R| \leq |K|-1} \delta(z)^{-p(j+|R|/2)} \right) \\
&\qquad + \left(\frac{1}{|R(z)|} \int_{R(z)} |f|^p dV \right) \\
&\leq C\, \delta(z)^{-|K|p/2} \left(\frac{1}{|R(z)|} \int_{R(z)} |f|^p dV \right).
\end{aligned}$$

This allows to conclude for the first part of Theorem A.

To show the second part, it suffices to apply the preceding result to the derivatives of g and then, to use the first part of the following elementary lemma.

Lemma 1.6. *For every $l, k \in \mathbb{N}$ and every $u \in C^\infty(\Omega)$, we have*

$$|\nabla^l \nabla_T^k u| \le |\nabla_T^k \nabla^l u| + \mathcal{O}\Big(\sum_{\substack{1\le j\le l,\\ 0\le r\le k-1}} |\nabla_T^r \nabla^j u| \Big)$$

$$|\nabla_T^k \nabla^l u| \le |\nabla^l \nabla_T^k u| + \mathcal{O}\Big(\sum_{\substack{1\le j\le l,\\ 0\le r\le k-1}} |\nabla_T^r \nabla^j u| \Big)$$

where the $\mathcal{O}$ are uniform on $V(z_0)$.

1.3. Converse estimates.

We are going to show the following theorem.

Theorem B. *Let Ω be a C^∞ domain in $\mathbb{C}^n$. Let $k \in \mathbb{N}$, $0 < p < \infty$. For every $z_0 \in \partial\Omega$, there exist a neighborhood $V(z_0)$ of z_0 and a constant C such that, for every holomorphic function g in Ω and every $z \in V(z_0) \cap \Omega$, we have*

$$C_2(z)^{kp} \delta(z)^{kp} \left|\nabla^k g(z)\right|^p \le C \operatorname{Mean}^{Q(z)} \left(\delta^{kp/2} \left|\nabla_T^k g\right|^p + \operatorname{Rest}^k(g)^p \right),$$

where

$$\operatorname{Rest}^k(g) = \delta^{1/2} \Big(\sum_{r=0}^{k-1} \sum_{\substack{1\le j+r<(k+r)/2\\ j\ge 0}} \mathcal{O}(\delta^{(k-1)/2}) \left|\nabla^j \nabla_T^r g\right| + \sum_{r=0}^{k-1} \sum_{\substack{(k+r)/2\le j+r\le k\\ j\ge 0}} \mathcal{O}(\delta^{j+r/2}) \left|\nabla^j \nabla_T^r g\right| \Big).$$

PROOF. As before, for every $z \in V(z_0)$ fixed, we set $f(\zeta) = g(\Phi_z(\zeta))$. We begin with the following lemma.

Lemma 1.7. *Let Ω be a C^∞ domain in $\mathbb{C}^n$, let $k \in \mathbb{N}$ and $0 < p < \infty$. For each $z_0 \in \partial\Omega$, there exist a neighborhood $V(z_0)$ and a constant C*

such that, for every $z \in V(z_0) \cap \Omega$, there exists a transverse vector field M_z, with $M_z r(z) = 1$, such that, for every holomorphic function g in Ω, we have

$$C_2(z)^{kp}\delta(z)^{kp}\left|M_z^k g(z)\right|^p \leq C\,\mathrm{Mean}^{Q(z)}\left(\delta^{kp/2}|\nabla_T^k g|^p + \mathrm{Rest}^k(g)^p\right)$$

where $\mathrm{Rest}^k(g)$ *take the form given in Theorem* B.

PROOF. We write $L'^k_i f(\zeta) = F_i^k f(\zeta) + \mathrm{Rest}_i^k f(\zeta)$ for $\zeta \in R(z)$, where

$$F_i^k f(\zeta) = \left(\frac{1}{2}\frac{\partial}{\partial \zeta_i} - \sum_{l=2}^{n} a_{i,l}(z)\overline{\zeta_l}\frac{\partial}{\partial \zeta_1}\right)^k f(\zeta)\,.$$

Since f is holomorphic, $F_i^k f$ is an (AB) function. Furthermore

$$\frac{\partial^k F_i^k f}{\partial \overline{\zeta_l}^k}(0) = (-a_{i,l}(z))^k\,\frac{\partial^k f}{\partial \zeta_1^k}(0)\,, \qquad l = 2, \dots, n\,.$$

So, Lemma 1.4 allows us to deduce that, for $l = 2, \dots, n$,

$$\begin{aligned}\delta(z)^{kp/2}\,|a_{i,l}(z)|^{kp}\left|\frac{\partial^k f}{\partial \zeta_1^k}(0)\right|^p &\leq C\,\mathrm{Mean}^{R(z)}(|F_i^k f|^p)\\ &\leq C\,\mathrm{Mean}^{R(z)}\left(|L'^k_i f|^p + (\mathrm{Rest}_i^k f)^p\right).\end{aligned}$$

Then, summing on i and l, we obtain

$$\begin{aligned}&C_2(z)^{kp}\delta(z)^{kp/2}\left|\frac{\partial^k f}{\partial \zeta_1^k}(0)\right|^p\\ &\qquad\leq C\,\mathrm{Mean}^{R(z)}\left(\sum_{i=2}^{n}\left(|L'^k_i f|^p + (\mathrm{Rest}_i^k f)^p\right)\right).\end{aligned}$$

And we estimate $\mathrm{Rest}_i^k f$ as in the proof of Corollary 1.5. In the ordinary system of coordinates, this gives

$$\begin{aligned}&C_2(z)^{kp}\delta(z)^{kp}\left|\left(\frac{\partial r}{\partial z_1}(z)\right)^{-k}\frac{\partial^k g}{\partial z_1^k}(z)\right|^p\\ &\qquad\leq C\,\mathrm{Mean}^{Q(z)}\left(\delta(z)^{kp/2}|\nabla_T^k g|^p + \mathrm{Rest}^k(g)^p\right).\end{aligned}$$

Lemma 1.7 follows with

$$M_z = \left(\frac{\partial r}{\partial z_1}(z)\right)^{-1} \frac{\partial}{\partial z_1}.$$

In order to conclude for Theorem B, we remark that there exists a constant C such that, for every $z \in V(z_0)$, we have

$$|\nabla^k g(z)| \leq C\Big\{|M_z^k g(z)| + \sum_{\substack{j+r=k \\ r\geq 1}} |\nabla_T^r \nabla^j g(z)| + \sum_{1\leq j\leq k-1} |\nabla^j g(z)|\Big\}.$$

But, by Corollary 1.5, we can estimate each $|\nabla_T^r \nabla^j g(z)|$ by its mean-value on $Q(z)$, disregarding some remaining terms. This allows to see that the terms

$$\sum_{\substack{j+r=k \\ r\geq 1}} \delta(z)^k |\nabla_T^r \nabla^j g(z)| + \sum_{1\leq j\leq k-1} \delta(z)^k |\nabla^j g(z)|$$

can be majorized by $C\,\mathrm{Mean}^{Q(z)}(\mathrm{Rest}^k(g))$.

REMARK. The estimate of Theorem B is intrinsic: explicitely, if $\hat{\nabla}_T^k$ is a tangential gradient defined with the help of another defining function $\hat{r}$, then the right member of the estimate defined with the help of these $\hat{\nabla}_T$ is equivalent to the one defined with the help of ∇_T.

We will assume now that Ω is bounded in $\mathbb{C}^n$; so, the estimates of Theorem A and B are uniformly true on $\Omega \cap U$, where U is a neighborhood of $\partial\Omega$ sufficiently small such that the projection on $\partial\Omega$ is well defined. Furthermore, if Ω satisfies (P), we assume U sufficiently small so that C_2 is uniformly bounded from below on $\overline{\Omega} \cap U$. In this case, we obtain the following corollary.

Corollary. *Let Ω be a bounded C^∞ domain satisfying* (P). *For every $0 < p < \infty$, $k \in \mathbb{N}$, there exists a constant C such that, for every holomorphic function g in Ω and every $z \in \Omega \cap U$, we have*

$$\delta(z)^{kp} |\nabla^k g(z)|^p \leq C\,\mathrm{Mean}^{Q(z)}\left(\delta^{kp/2} |\nabla_T^k g|^p + \mathrm{Rest}^k(g)^p\right).$$

2. Hardy-Sobolev Spaces.

In the following K will denote a compact set contained in the complement of $\Omega \cap U$.

2.1. Proof of the Auxiliary Theorem.

We give the main ideas and references for the proof of this theorem, which follows from standard methods but which is nowhere explicitely written (as far as we know).

Assume that $k = 0$ to simplify. The equivalence between (1) and (3) is well known from Fefferman-Stein work and is valid for harmonic functions (see [FS]). We have to prove that (1) implies (4).

For every $0 < p < \infty$, every $\Phi(\eta, t) \in \mathcal{A}_\alpha(\zeta)$, we have, by subharmonicity of $|g|^{p/2}$

$$\begin{aligned} |g \circ \Phi(\eta, t)|^{p/2} &\leq C \operatorname{Mean}^{Q(\Phi(\eta,t))}(|g|^{p/2}) \\ &\leq \frac{C}{|Q(\Phi(\eta, t))|} \int_{\Phi^{-1}(Q(\Phi(\eta,t)))} |g \circ \Phi(\eta', s)|^{p/2} d\sigma(\eta') ds \\ &\leq \frac{C}{|B^d(\zeta, ct)|} \int_{B^d(\zeta,ct)} \sup_{0<s<s_0} |g \circ \Phi(\eta', s)|^{p/2} d\sigma(\eta'), \end{aligned}$$

since if $\Phi(\eta, t) \in \mathcal{A}_\alpha(\zeta)$, the projection of $\Phi^{-1}(Q(\Phi(\eta, t)))$ on $\partial\Omega$ is contained in $B^d(\zeta, ct)$, for some constant c.

So,

$$\mathcal{M}_\alpha(|g|)^{p/2}(\zeta) \leq C\, M \left(\sup_{0<s<s_0} |g \circ \Phi(\cdot, s)|^{p/2} \right) (\zeta),$$

where M is the non-isotropic maximal operator. We conclude by the L^2-continuity of the operator M.

The fact that (2) implies (3) follows from a similar argument. For $p > 2$, (3) implies (2) is true for any kind of regular functions (see Lemma 2.5 further on) and follows from an argument of duality and from the L^q-continuity of the non-isotropic maximal operator for $q > 1$. It remains to prove that (4) implies (2) when $p \leq 2$. The proof of Fefferman-Stein can be adapted in this context. We postpone this proof as we shall adapt the method of Fefferman-Stein in a more general context for Theorem D. We can also see [B1] and [B2].

REMARK. As an immediate consequence of the equivalence between

$$\left\| \sup_{0<t<s_0} \left|\nabla^k g \circ \Phi(\cdot, t)\right| \right\|_{L^p(\partial\Omega)} \quad \text{and} \quad \left\|S_\alpha(\delta\nabla^{k+1} g)\right\|_{L^p(\partial\Omega)},$$

we obtain that the spaces $\mathcal{H}^p_k(\Omega)$ are independent of the choice of the map Φ, and that different choices of Φ yield equivalent norms.

2.2. Admissible area functions.

First, we give an auxiliary result which can be proved by the same method as in the case of classical area functions (we do not give the proof because the method will be largely used in the following, we can also see [CMS] for instance).

Lemma 2.1 *Let Ω be a C^2-domain in $\mathbb{C}^n$. For every apertures $\alpha, \beta > 0$, every $0 < p < \infty$, every $q \in \mathbb{R}$,*

$$\|S^q_\alpha(u)\|_{L^p(\partial\Omega)} \ \textit{and} \ \left\|S^q_\beta(u)\right\|_{L^p(\partial\Omega)}$$

a. equivalent for every regular function u defined on Ω.

Now, we need a particular Hardy Inequality.

Lemma 2.2. (A Hardy Inequality on a region over a graph). *Let $\mathcal{R}$ be a region over a graph in $\Omega \cap U$, $\mathcal{R}$ given by*

$$\Phi^{-1}(\mathcal{R}) = \{(\eta, t) \in \partial\Omega \times]0, s_0[: \quad t \geq \phi(\eta)\}$$

for some function ϕ.

Let $q > 0$. There exists a constant C such that, for every mesurable function u on Ω, we have

$$\iint_{\mathcal{R}} \delta^q |u|^2 \frac{dV}{\delta} \leq C\Big(\iint_{\mathcal{R}} \delta^{q+2k} |\nabla^k u|^2 \frac{dV}{\delta} + \sum_{j=0}^{k-1} \|\nabla^j u\|^2_{L^2(K)}\Big).$$

PROOF OF LEMMA 2.2. First, we recall the usual Hardy-inequality:

Let $p \geq 1$. For every $q > 0$, there exists a constant C such that, for any positive, mesurable function v defined on $\mathbb{R}^+$, we have

$$\int_0^\infty t^q V(t)^p \frac{dt}{t} \leq C \int_0^\infty t^q \, (tv(t))^p \, \frac{dt}{t} \, ,$$

where $V(t) = \int_t^\infty v(s)ds$ for $t \geq 0$.

In order to obtain the lemma, for each $\eta \in \partial\Omega$, we apply this inequality with $p = 2$ successively to the function

$$V_\eta(t_1) = \int_{t_1}^\infty \cdots \int_{t_k}^\infty |v_\eta(t)| \, dt dt_k \dots dt_2$$

$$\text{where } v_\eta(t) = \begin{cases} \dfrac{d^k u}{dt^k} \, (\Phi(\eta, t)) & \text{if } \phi(\eta) \leq t \leq s_0 \, , \\ 0 \quad \text{otherwise.} \end{cases}$$

Integrating over $\partial\Omega$, this gives the result.

We are going to prove, now, that the admissible area functions of different orders are equivalent. This equivalence follows from standard arguments and from an appropriate Hardy Inequality.

Theorem 2.3. *Let Ω be a bounded C^∞ domain in $\mathbb{C}^n$. For every holomorphic function g, every $k \in \mathbb{N}$, every $0 < p < \infty$, every $q > 0$ and every aperture $\alpha > 0$*

$$\|S_\alpha^q(g)\|_{L^p(\partial\Omega)} \quad \text{and} \quad \left\|S_\alpha^q(\delta^k \nabla^k g)\right\|_{L^p(\partial\Omega)}$$

are equivalent, modulo an error of $\|g\|_{L^2(K)}$.

In [S2], Stein gives this result for $p = 2$ and harmonic functions in $\mathbb{R}^n$. For general p, the corresponding result for harmonic functions in $\mathbb{R}^n$ follows from equivalent definitions of $\mathcal{H}^p$ (see [CT] for instance).

PROOF. The proof will be given in two steps. The first step is devoted to show that

$$\left\|S_\alpha^q(\delta^k \nabla^k g)\right\|_{L^p(\partial\Omega)} \leq C \left(\|S_\alpha^q(g)\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \right)$$

and the second to the converse inequality.

First inequality. In this part, we are going to show that, for every $\zeta \in \partial\Omega$, every $\alpha > 0$, there exists $\beta > \alpha$ such that

$$S_\alpha^q(\delta^k \nabla^k g)(\zeta) \leq C \left(S_\beta^q(g)(\zeta) + \|g\|_{L^2(K)} \right).$$

Lemma 2.1 will allow to conclude that, for every $0 < p < \infty$,

$$\left\| S_\alpha^q(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)} \leq C \left(\|S_\alpha^q(g)\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \right).$$

This inequality follows easily from the fact that, since g is holomorphic in Ω, for every $z \in \Omega \cap U$,

$$\delta(z)^{2k} |\nabla^k g|^2(z) \leq C \operatorname{Mean}^{Q(z)}(|g|^2).$$

So, it suffices to choose β sufficiently large such that, for every $z \in \mathcal{A}_\alpha(\zeta)$, $Q(z) \subset \mathcal{A}_\beta(\zeta)$.

Converse inequality. In order to show the converse inequality, we are going to distinguish three cases: $p = 2$, $0 < p < 2$ and $p > 2$.

1. *Case* $p = 2$. This case is the simplest one since

$$\begin{aligned} \|S_\alpha^q(g)\|^2_{L^2(\partial\Omega)} &\simeq \iint_{\mathcal{R}_\alpha} \delta^q(z) \, |g(z)|^2 \, \sigma\left(\{\zeta \in \partial\Omega : \, z \in \mathcal{A}_\alpha(\zeta)\}\right) \frac{dV(z)}{\delta^{n+1}(z)} \\ &\simeq \iint_{\mathcal{R}_\alpha} |g|^2 \delta^q \frac{dV}{\delta} \end{aligned}$$

where $\mathcal{R}_\alpha$ is the set $\cup_{\zeta\in\partial\Omega} \mathcal{A}_\alpha(\zeta) = \Omega \cap U$. So

$$\|S_\alpha^q(g)\|^2_{L^2(\partial\Omega)} \simeq \int_{\Omega\cap U} |g|^2 \delta^q \frac{dV}{\delta}$$

and it suffices to apply Hardy-inequality to conclude.

2. *Case* $0 < p < 2$. We use again ideas of [FS] and [CT]. The proof will be given in two parts.

2.1. *First part:* For every $\lambda > 0$ and every $\beta > \alpha > 0$, let $E \left(= E^\lambda\right)$ be the set of points of $\partial\Omega$ where $S_\beta^q(\delta^k \nabla^k g) \leq \lambda$. Now, let $E_0 \left(= E_0^\lambda\right)$ be the points of E of relative density $1/2$; more precisely E_0 is the set

$$\left\{ \zeta \in \partial\Omega : \text{ for every ball } B^d \text{ containing } \zeta, \, \sigma(E \cap B^d) \geq \frac{1}{2}\sigma(B^d) \right\}.$$

If χ is the characteristic function of $D\,(= D^\lambda) = E^c$ (complementary of E), then

$$D_0\,(= D_0^\lambda) = E_0^c = \left\{\zeta \in \partial\Omega :\ M(\chi) > \frac{1}{2}\right\},$$

where M is the non-isotropic maximal operator. Thus, there exists a constant c such that $\sigma(D_0) \leq c\,\sigma(D)$.

We are going to prove the following lemma.

Lemma 2.4. *Under the assumptions of Theorem* 2.3, *there exists a constant* C *such that*

$$\int_{E_0} S_\alpha^q(g)^2 d\sigma \leq C\left(\int_E S_\beta^q(\delta^k \nabla^k g)^2 d\sigma + \int_K |g|^2 dV\right).$$

PROOF. We have

$$\begin{aligned}
(*) &= \int_{E_0} S_\alpha^q(g)^2 d\sigma \\
&= \iint_{\mathcal{R}_\alpha} \delta^q(z)\,|g(z)|^2\,\sigma\left(\{\zeta \in E_0 :\ z \in \mathcal{A}_\alpha(\zeta)\}\right)\frac{dV(z)}{\delta^{n+1}(z)} \\
&\leq C \iint_{\mathcal{R}_\alpha} \delta^q\,|g|^2\,\frac{dV}{\delta}\,,
\end{aligned}$$

where we denote by $\mathcal{R}_\alpha$ the set $\cup_{\zeta\in E_0}\mathcal{A}_\alpha(\zeta)$. Observe that

$$\Phi^{-1}\left(\mathcal{R}_\alpha\right) = \left\{(\eta, t) \in \partial\Omega\times]0, s_0[:\quad t \geq \frac{1}{\alpha} d(\eta, E_0)\right\}$$

(where $d(\eta, E_0) = \inf_{\zeta\in E_0} d(\eta,\zeta)$) and apply Lemma 2.2 in order to obtain

$$\iint_{\mathcal{R}_\alpha} \delta^q |g|^2 \frac{dV}{\delta} \leq C\left(\iint_{\mathcal{R}_\alpha} \delta^{q+2k} |\nabla^k g|^2 \frac{dV}{\delta} + \int_K |g|^2 dV\right),$$

since, by assumption $q > 0$. So, we have

$$\int_{E_0} S_\alpha^q(g)^2 d\sigma \leq C\left(\iint_{\mathcal{R}_\alpha} \delta^{q+2k} |\nabla^k g|^2 \frac{dV}{\delta} + \int_K |g|^2 dV\right).$$

Now, it is sufficient to observe that $z = \Phi(\eta, t) \in \mathcal{R}_\alpha$ if and only if $d(\eta, \zeta) \leq \alpha t$ for some $\zeta \in E_0$. But then $z = \Phi(\eta, t) \in \mathcal{A}_\beta(w)$ whenever $d(\zeta, w) \leq (\beta - \alpha)t$. Thus

$$\begin{aligned}\sigma\left(\{w \in E : \ z \in \mathcal{A}_\beta(w)\}\right) &\geq \sigma(E \cap B^d(\zeta, (\beta - \alpha)t)) \\ &\geq \frac{1}{2}\, \sigma(B^d(\zeta, (\beta - \alpha)t))\end{aligned}$$

in view of the definition of E_0. So, the later quantity exceeds $Ct^n \simeq C\delta(z)^n$. So,

$$\begin{aligned}(*) &\leq C\Big(\iint_{\mathcal{R}_\alpha} \delta^{q+2k} |\nabla^k g(z)|^2 \sigma\left(\{w \in E : \ z \in \mathcal{A}_\beta(w)\}\right) \frac{dV}{\delta^{n+1}} \\ &\qquad + \int_K |g|^2 dV\Big) \\ &\leq C \left(\int_E S^q_\beta(\delta^k \nabla^k g)^2 d\sigma + \int_K |g|^2 dV \right).\end{aligned}$$

2.2. *Second part:* We conclude from Lemma 2.4 as in [FS] that, when $0 < p < 2$,

$$\|S^q_\alpha(g)\|_{L^p(\partial\Omega)} \leq C \left(\left\|S^q_\alpha(\delta^k \nabla^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \right).$$

Let us give the proof once for all for completeness.

Observe that

$$\begin{aligned}\sigma\left(\{S^q_\alpha(g) \geq \lambda\}\right) &\leq \sigma(D^\lambda_0) + \sigma\left(\{\zeta \in E^\lambda_0 : \quad S^q_\alpha(g)(\zeta) \geq \lambda\}\right) \\ &\leq \sigma(D^\lambda_0) + \frac{1}{\lambda^2} \int_{E^\lambda_0} S^q_\alpha(g)^2 d\sigma\,.\end{aligned}$$

Then, we write

$$\begin{aligned}\|S^q_\alpha(g)\|^p_{L^p(\partial\Omega)} &= p \int_0^\infty \lambda^{p-1} \sigma\left(\{S^q_\alpha(g) \geq \lambda\}\right) d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} \sigma(D^\lambda_0)\, d\lambda \\ &\qquad + p \int_M^\infty \lambda^{p-3} \int_{E^\lambda_0} S^q_\alpha(g)^2\, d\sigma d\lambda \\ &\qquad + \sigma(\partial\Omega) M^p\end{aligned}$$

$$\leq C\Big(p\int_0^\infty \lambda^{p-1}\sigma(D^\lambda)\,d\lambda + p\int_M^\infty \lambda^{p-3}\int_{E^\lambda} S^q_\beta(\delta^k\nabla^k g)^2\,d\sigma d\lambda + \|g\|^2_{L^2(K)}M^{p-2} + \sigma(\partial\Omega)M^p\Big)$$

$$\leq C\left(\left\|S^q_\beta(\delta^k\nabla^k g)\right\|^p_{L^p(\partial\Omega)} + \|g\|^p_{L^2(K)}\right)$$

(by choosing $M = \|g\|_{L^2(K)}$). Lemma 2.1 allows to conclude.

3. *Case* $2 < p < \infty$. First, it follows easily from the usual Hardy-inequality in $L^2(0, s_0)$ that, for every $q > 0$ and every $\zeta \in \partial\Omega$,

$$G^q(g)(\zeta) \leq C\left(G^q(\delta^k\nabla^k g)(\zeta) + \|g\|_{L^2(K)}\right).$$

So, for every $0 < p < \infty$,

$$\|G^q(g)\|_{L^p(\partial\Omega)} \leq C\left(\left\|G^q(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

Now, since for every function $u \in C^\infty(\Omega)$ and every $\zeta \in \partial\Omega$,

$$G^q(\mathrm{Mean}^Q(u))(\zeta) \leq C\left(S^q_\alpha(u)(\zeta) + \|u\|_{L^2(K)}\right)$$

for some $\alpha > 0$, we have

$$\|G^q(g)\|_{L^p(\partial\Omega)} \leq C\left(\|S^q_\alpha(g)\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right)$$

for every holomorphic function g and every $0 < p < \infty$. So, it remains to show the following lemma.

Lemma 2.5. *For every $\alpha > 0$, every $2 < p < \infty$, every $u \in C^\infty(\Omega)$ and every $q \in \mathbb{R}$, there exists a constant C such that*

$$\|S^q_\alpha(u)\|_{L^p(\partial\Omega)} \leq C\,\|G^q(u)\|_{L^p(\partial\Omega)}.$$

PROOF. As in [S2], we use the fact that

$$\|S^q_\alpha(u)\|^2_{L^p(\partial\Omega)} = \left\|S^q_\alpha(u)^2\right\|_{L^{p/2}(\partial\Omega)}$$

$$= \sup \int_{\partial\Omega} S_\alpha^q(u)(\zeta)^2 v(\zeta) d\sigma(\zeta)$$

where the supremum is taken over all the functions $v \in L^{p'}(\partial\Omega)$, with $2/p + 1/p' = 1$ and $\|v\|_{L^{p'}(\partial\Omega)} \leq 1$.

$$\begin{aligned}
&\int_{\partial\Omega} S_\alpha^q(u)^2(\zeta) v(\zeta) d\sigma(\zeta) \\
&\quad = C \int_{\partial\Omega} \int_{\Phi^{-1}(\mathcal{A}_\alpha(\zeta))} t^q |u|^2 \circ \Phi(\eta, t) \frac{dt}{t^{n+1}} \, d\sigma(\eta) v(\zeta) \, d\sigma(\zeta) \\
&\quad = C \int_{\partial\Omega} \int_0^{s_0} t^q |u|^2 \circ \Phi(\eta, t) \left(\frac{1}{t^n} \int_{B^d(\eta, \alpha t)} v(\zeta) d\sigma(\zeta) \right) \frac{dt}{t} \, d\sigma(\eta) \\
&\quad \leq C \int_{\partial\Omega} Mv(\eta) G^q(u)(\eta)^2 d\sigma(\eta) \, ,
\end{aligned}$$

where M is the non-isotropic maximal operator,

$$\begin{aligned}
&\leq C \, \|Mv\|_{L^{p'}(\partial\Omega)} \, \|G^q(u)\|^2_{L^p(\partial\Omega)} \\
&\leq C \|v\|_{L^{p'}(\partial\Omega)} \, \|G^q(u)\|^2_{L^p(\partial\Omega)} \\
&\leq C \, \|G^q(u)\|^2_{L^p(\partial\Omega)}
\end{aligned}$$

by the $L^{p'}$-continuity of the non-isotropic maximal operator.

2.3. Proof of the first part of Main Theorem.

We are going to prove the following result.

Theorem C. *Let Ω be a bounded C^∞-domain in $\mathbb{C}^n$ satisfying* (P), *$\alpha > 0$ be a fixed aperture and $0 < p < \infty$. For every holomorphic function g in Ω, every $q \in \mathbb{R}$, $k, l, r \in \mathbb{N}$ with $q + k + 2l > 0$ and $q + 2r > 0$,*

$$\left\| S_\alpha^q(\delta^{k/2+l} \nabla_T^k \nabla^l g) \right\|_{L^p(\partial\Omega)}, \quad \left\| S_\alpha^q(\delta^{k/2+l} \nabla^l \nabla_T^k g) \right\|_{L^p(\partial\Omega)},$$
$$\text{and} \qquad \|S_\alpha^q(\delta^r \nabla^r g)\|_{L^p(\partial\Omega)}$$

are equivalent, modulo an error of $\|g\|_{L^2(K)}$.

These results are also true for any permutation of ∇ and ∇_T in a product $\nabla^l \nabla_T^k$.

As an immediate application, we obtain the following corollary.

Corollary. *Let Ω be a bounded C^∞-domain in $\mathbb{C}^n$ satisfying* (P). *For every $0 < p < \infty$, every $k \in \mathbb{N}$ and every holomorphic function g, we have*

$$g \in \mathcal{H}_k^p(\Omega) \quad \text{if and only if} \quad \left\| S_\alpha^{-2k}(\delta^{j+l/2} \nabla^j \nabla_T^l g) \right\|_{L^p(\partial\Omega)} < \infty$$

for every $l, j \in \mathbb{N}$ with $j + l/2 > k$.

In particular for $l = 2k$ and $j = 1$, this corollary gives the equivalence between (1) and (2) of Main Theorem and for $k = 0$, $l = 2$ and $j = 0$, this gives the corollary stated in the introduction.

PROOF OF THEOREM C. We are going to show that

$$\left\| S_\alpha^q(\delta^{k/2+l} \nabla_T^k \nabla^l g) \right\|_{L^p(\partial\Omega)} \quad \text{and} \quad \| S_\alpha^q(\delta^r \nabla^r g) \|_{L^p(\partial\Omega)}$$

are equivalent, under the assumption that $q + 2r > 0$ and $q + 2l + k > 0$.

In order to obtain the last equivalence of Theorem C, we just have to use Lemma 2.2 which allows us to write $\nabla^l \nabla_T^k g$ as the sum of $\nabla_T^k \nabla^l g$ and of terms involving smaller derivatives like $\nabla_T^r \nabla^j g$, with $0 \le r \le k - 1$ and $1 \le j \le l$ (the terms involving smaller derivatives are smaller than $\| S_\alpha^q(\delta^r \nabla^r g) \|_{L^p(\partial\Omega)}$).

So, let us show this equivalence.

First inequality. We are going to show that

$$\left\| S_\alpha^q(\delta^{k/2+l} \nabla_T^k \nabla^l g) \right\|_{L^p(\partial\Omega)} \le C \left(\| S_\alpha^q(\delta^r \nabla^r g) \|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \right),$$

under the assumption that $q + k + 2l > 0$ and $q + 2r > 0$. As before, we will distinguish the cases $p = 2$, $0 < p < 2$ and $p > 2$.

The case $p = 2$ is the simplest one as before. Since it follows easily from the results of [G1], we will not repeat the proof here.

1. *Case* $0 < p < 2$. In view of the proof of Theorem 2.3, it is sufficient to prove the following inequality, the second step being the same as in the proof of Theorem 2.3.

There exists $\gamma > \alpha$ such that

$$\int_{E_0} S^q_\alpha(\delta^{k/2+l}\nabla^k_T\nabla^l g)^2 d\sigma \leq C\left(\int_E S^q_\gamma(\delta^r\nabla^r g)^2 d\sigma + \|g\|^2_{L^2(K)}\right)$$

with E, E_0 corresponding to $S^q_\gamma(\delta^r\nabla^r g)$. This inequality will follow from the following lemma.

Lemma 2.6. *Let $k, l, r \in \mathbb{N}$, $q \in \mathbb{R}$ with $q + 2l + k > 0$ and $q + 2r > 0$. Then, for every $\alpha > 0$, there exist $\beta > \alpha$ and a constant C such that, for every $E_0 \subset \partial\Omega$ and every holomorphic function g in Ω, we have*

$$\iint_{\mathcal{R}_\alpha} \delta^{q+2l+k}\left|\nabla^k_T\nabla^l g\right|^2 \frac{dV}{\delta} \leq C\left(\iint_{\mathcal{R}_\beta} \delta^{q+2r}\left|\nabla^r g\right|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right),$$

$$\iint_{\mathcal{R}_\alpha} \delta^{q+2l+k}\left|\nabla^l\nabla^k_T g\right|^2 \frac{dV}{\delta} \leq C\left(\iint_{\mathcal{R}_\beta} \delta^{q+2r}\left|\nabla^r g\right|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right),$$

where $\mathcal{R}_\alpha = \cup_{\zeta\in E_0}\mathcal{A}_\alpha(\zeta)$.

PROOF. We only give the proof for $l = 0$. The general result for the first inequality will follow applying the result with $l = 0$ to the components of $\nabla^l g$, changing q into $q + 2l$ and using the same method as in the proof of Theorem 2.3. For the second inequality, we use Lemma 1.6 as before.

Let us apply the Hardy Inequality of Lemma 2.2, this gives

$$\iint_{\mathcal{R}_\alpha} \delta^{q+k}\left|\nabla^k_T g\right|^2 \frac{dV}{\delta} \leq C\left(\iint_{\mathcal{R}_\alpha} \delta^{q+2r+k}\left|\nabla^r\nabla^k_T g\right|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right),$$

under the assumption that $q + k > 0$.

Now, we apply successively the results of Theorem A and the Hardy Inequality of Lemma 2.2 to the terms involving derivatives less than r in order to obtain

$$\iint_{\mathcal{R}_\alpha} \delta^{q+k} \left|\nabla_T^k g\right|^2 \frac{dV}{\delta} \le C\left(\sum_{j=1}^{r} \iint_{\mathcal{R}_\beta} \delta^{q+2r} \left|\nabla^j g\right|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right)$$
$$\le C\left(\iint_{\mathcal{R}_\beta} \delta^{q+2r} \left|\nabla^r g\right|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right)$$

since, by assumption, $q + 2r > 0$.

2. *Case* $2 < p < \infty$. As before, we will only consider the case $l = 0$. We have, by Lemma 2.5,

$$\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)} \le C \left\|G^q(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)}.$$

But, by Hardy inequality in $L^2(0, s_0)$, we have, for every $\zeta \in \partial\Omega$,

$$G^q(\delta^{k/2}\nabla^k_T g)(\zeta) \le C\left(G^q(\delta^{k/2+r}\nabla^r\nabla^k_T g)(\zeta) + C\|g\|_{L^2(K)}\right)$$
$$\le C\left(S^q_\alpha(\delta^r \sum_{j=1}^{r} |\nabla^j g|)(\zeta) + \|g\|_{L^2(K)}\right)$$

by Theorem A, since for every function $u \in C^\infty(\Omega)$ and every $\zeta \in \partial\Omega$, there exists $\alpha > 0$ such that

$$G^q(\mathrm{Mean}^Q(u))(\zeta) \le C\left(S^q_\alpha(u)(\zeta) + \|u\|_{L^2(K)}\right).$$

So, by Theorem 2.3, we obtain the result.

$$\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)} \le C\left(\|S^q_\alpha(\delta^r\nabla^r g)\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

Converse inequality. We want to show that,

$$\|S^q_\alpha(\delta^r\nabla^r g)\|_{L^p(\partial\Omega)} \le C\left(\left\|S^q_\alpha(\delta^{k/2+l}\nabla^k_T\nabla^l g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right),$$

under the assumptions that $q + k + 2l > 0$ and $q + 2r > 0$. We only consider the case $l = 0$. For general l, we use the same method as before.

By Theorem 2.3, it suffices to show that

$$\left\|S^q_\alpha(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)} \le C\left(\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

We begin with the following lemma.

Lemma 2.7. *Under the assumptions of Theorem* C, *we have*

$$\begin{aligned}\left\|S^q_\alpha(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)} \le C\Big(&\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)}\\ &+ \left\|S^{q+1}_\alpha(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)}\|g\|_{L^2(K)}\Big).\end{aligned}$$

PROOF. 1. *Case* $0 < p < 2$. Lemma 2.7 will follow from the following estimate.

There exists $\gamma > \alpha$ and a constant C such that,

$$\begin{aligned}\int_{E_0} S^q_\alpha(\delta^k\nabla^k g)^2 d\sigma \le C\Big(&\int_E S^q_\gamma(\delta^{k/2}\nabla^k_T g)^2\, d\sigma\\ &+ \int_E S^{q+1}_\gamma(\delta^k\nabla^k g)^2 d\sigma + \|g\|^2_{L^2(K)}\Big)\end{aligned}$$

where

$$E = \left\{S^{q+1}_\gamma(\delta^k\nabla^k g) \le \lambda \text{ and } S^q_\gamma(\delta^{k/2}\nabla^k_T g) \le \lambda\right\}$$

and E_0 is the set of points of E of relative density $1/2$. As in the proof of Theorem 2.3, we will denote by D_0 and by D the complements of E_0 and E respectively; then $\sigma(D_0) \le c\,\sigma(D)$ by the non-isotropic maximal Theorem.

The preceding inequality will follow from the following estimate.

There exists $\beta' > \beta > \alpha$ such that

$$\begin{aligned}(*) &= \iint_{\mathcal{R}_\alpha} \delta^{q+2k}|\nabla^k g|^2 \frac{dV}{\delta}\\ &\le C\Big(\iint_{\mathcal{R}_\beta} \delta^{q+k}|\nabla^k_T g|^2 \frac{dV}{\delta}\\ &\qquad + \iint_{\mathcal{R}_{\beta'}} \delta^{q+2k+1}|\nabla^k g|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\Big).\end{aligned}$$

Let us prove this inequality. By Theorem B, we have

$$(*) \le C\Big(\iint_{\mathcal{R}_\beta} \delta^{q+k}|\nabla_T^k g|^2 \frac{dV}{\delta} + \iint_{\mathcal{R}_\beta} \delta^q |\mathrm{Rest}^k(g)|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\Big).$$

Let us estimate the remaining term. We are going to show that it is bounded by

$$C\left(\iint_{\mathcal{R}_{\beta'}} \delta^{q+2k+1}|\nabla^k g|^2 \frac{dV}{\delta} + \|g\|^2_{L^2(K)}\right).$$

We have

$$\begin{aligned}(**) &= \iint_{\mathcal{R}_\beta} \delta^q |\mathrm{Rest}^k(g)|^2 \frac{dV}{\delta} \\ &\le C\Big(\iint_{\mathcal{R}_\beta} \sum_{1\le j+r<(k+r)/2} \delta^{k+q} \left|\nabla^j \nabla_T^r g\right|^2 \frac{dV}{\delta} \\ &\qquad + \iint_{\mathcal{R}_\beta} \sum_{(k+r)/2\le j+r\le k} \delta^{2j+r+1+q} \left|\nabla^j \nabla_T^r g\right|^2 \frac{dV}{\delta}\Big).\end{aligned}$$

By Hardy inequality of Lemma 2.2, we have

$$\begin{aligned}(**) \le C &\left(\iint_{\mathcal{R}_\beta} \sum_{1\le j+r<(k+r)/2} \delta^{k+q+2(k-j)} \left|\nabla^k \nabla_T^r g\right|^2 \frac{dV}{\delta}\right. \\ &\left.\quad + \iint_{\mathcal{R}_\beta} \sum_{(k+r)/2\le j+r\le k} \delta^{2k+r+1+q} \left|\nabla^k \nabla_T^r g\right|^2 \frac{dV}{\delta}\right).\end{aligned}$$

Then, by Theorem A, we obtain

$$\begin{aligned}(**) \le C\Big(&\iint_{\mathcal{R}_\beta} \sum_{1\le j+r<(k+r)/2} \sum_{l=1}^{k} \delta^{k+q+2(k-j)-r} \left|\nabla^l g\right|^2 \frac{dV}{\delta} \\ &+ \iint_{\mathcal{R}_\beta} \sum_{l=1}^{k} \delta^{2k+1+q} \left|\nabla^l g\right|^2 \frac{dV}{\delta}\Big)\end{aligned}$$

$$\leq C \iint_{\mathcal{R}_{\beta'}} \delta^{q+2k+1} |\nabla^k g|^2 \frac{dV}{\delta}.$$

So, we obtain the good estimate.

Using this inequality, we obtain Lemma 2.7 by the same method as in the proof of Theorem 2.3 when $0 < p < 2$ since

$$E \subset \left\{ S^q_\gamma(\delta^{k/2} \nabla^k_T g) \leq \lambda \right\}, \quad E \subset \{ S^{q+1}_\gamma(\delta^k \nabla^k g) \leq \lambda \}$$

and

$$\sigma(D) \leq \sigma\left(\{ S^{q+1}_\gamma(\delta^k \nabla^k g) \geq \lambda \} \right) + \sigma\left(\left\{ S^q_\gamma(\delta^{k/2} \nabla^k_T g) \geq \lambda \right\} \right).$$

2. *Case* $2 < p < \infty$. We have

$$\left\| S^q_\alpha(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)} \leq C \left\| G^q(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)}.$$

But, by Theorem B, we have, for every $\zeta \in \partial\Omega$,

$$G^q(\delta^k \nabla^k g)(\zeta) \leq C \left(S^q_\alpha(\delta^{k/2} \nabla^k_T g)(\zeta) + S^q_\alpha(\mathrm{Rest}^k(g))(\zeta) + \|g\|_{L^2(K)} \right).$$

We estimate the remaining terms $\left\| S^q_\alpha(\mathrm{Rest}^k(g)) \right\|_{L^p(\partial\Omega)}$ by Theorem 2.3 and the first inequality we have just proved, in order to obtain

$$\left\| G^q(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)} \leq C \Big(\left\| S^q_\alpha(\delta^{k/2} \nabla^k_T g) \right\|_{L^p(\partial\Omega)} + \left\| S^{q+1}_\alpha(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \Big).$$

Lemma 2.7 follows.

END OF THE PROOF OF THEOREM C. It remains to show that Theorem C follows from Lemma 2.7. We distinguish two cases.

1. *Case* $q + k \geq 1$. Observe that, since $\delta(z) \leq s_0$ on $\mathcal{A}_\alpha(\zeta)$ for every $\zeta \in \partial\Omega$,

$$\left\| S^{q+1}_\alpha(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)} \leq C s_0^{1/2} \left\| S^q_\alpha(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega)}.$$

So, we apply Lemma 2.7 in $\Omega_\varepsilon = \{\Phi(\zeta, t) \in \Omega,\, t > \varepsilon\}$ to g (which belongs to $C^\infty(\overline{\Omega_\varepsilon})$). Reducing s_0 if necessary, we get

$$\left\| S^q_{\alpha,\varepsilon}(\delta^k \nabla^k g) \right\|_{L^p(\partial\Omega_\varepsilon)} \leq C \left(\left\| S^q_{\alpha,\varepsilon}(\delta^{k/2} \nabla^k_T g) \right\|_{L^p(\partial\Omega_\varepsilon)} + \|g\|_{L^2(K)} \right)$$

where $S^q_{\alpha,\varepsilon}$ denotes the admissible area function corresponding to Ω_ε. We want to let $\varepsilon \to 0$.

Using Fatou's Lemma, it is sufficient to show that, for ε small enough,

$$\left\|S^q_{\alpha,\varepsilon}(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega_\varepsilon)} \leq C\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)}.$$

We have

$$\left\|S^q_{\alpha,\varepsilon}(\delta^{k/2}\nabla^k_T g)\right\|^p_{L^p(\partial\Omega_\varepsilon)} = \int_{\partial\Omega_\varepsilon}\left(\int_{\mathcal{A}_\alpha(\zeta_\varepsilon)} \delta_\varepsilon^{k+q}\left|\nabla^k_T g\right|^2 \frac{dV}{\delta_\varepsilon}\right)^{p/2} d\sigma_\varepsilon.$$

Now, for $\zeta_\varepsilon = \Phi(\zeta,\varepsilon) \in \partial\Omega_\varepsilon$, $\mathcal{A}_\alpha(\zeta_\epsilon) \subset \mathcal{A}_\beta(\zeta)$ for some $\beta > \alpha$ and obviously, $\delta_\varepsilon \leq \delta$. This allows to conclude.

2. *Case* $q + k < 1$. We are going to use the method called "bootstrapping". Without loss of generality, we can assume that

$$\left\|S^q_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)} < \infty.$$

Then, in particular, for any ϱ sufficiently large such that $k + \varrho \geq 1$,

$$\left\|S^\varrho_\alpha(\delta^{k/2}\nabla^k_T g)\right\|_{L^p(\partial\Omega)} < \infty.$$

and by the preceding result

$$\left\|S^\varrho_\alpha(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)} < \infty.$$

Now, choose $\varrho = q + m$ with $m \in \mathbb{N}$. Then, we apply Lemma 2.7 in order to obtain

$$\left\|S^{\varrho-1}_\alpha(\delta^k\nabla^k g)\right\|_{L^p(\partial\Omega)} < \infty.$$

Now, we repeat the same argument as long as the index of the weight in S_α is different from q.

As a corollary of Theorem C, we obtain the following.

Corollary 2.8. *Let Ω be a bounded C^∞-domain in $\mathbb{C}^n$, satisfying* (P). *For every $0 < p < \infty$, for every $\alpha > 0$, every $k, l \in \mathbb{N}$ and $q \in \mathbb{R}$ with*

$q+k+2l > 0$, *there exists a constant* C *such that, for every holomorphic function* g *in* Ω, *we have*

$$\left\|G^q(\delta^{l+k/2}\nabla^l\nabla_T^k g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|S_\alpha^q(\delta^{l+k/2}\nabla^l\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

PROOF. As in Corollary 1.5, we write

$$\delta^{2l+k}(z)|\nabla^l\nabla_T^k g|^2(z) \leq C\,\mathrm{Mean}^{Q(z)}\Big(\delta^{2l+k}|\nabla^l\nabla_T^k g|^2 + \delta^l\left[\mathrm{Rest}^k(\nabla^l g)\right]^2\Big).$$

So,

$$\left\|G^q(\delta^{l+k/2}\nabla^l\nabla_T^k g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|S_\alpha^q(\delta^{l+k/2}\nabla^l\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \left\|S_\alpha^q(\delta^l\mathrm{Rest}^k(\nabla^l g))\right\|_{L^p(\partial\Omega)}\right).$$

So, it suffices to estimate the remaining term by Theorem C. We obtain

$$\left\|S_\alpha^q(\delta^l\mathrm{Rest}^k(\nabla^l g))\right\|_{L^p(\partial\Omega)} \leq C\,s_0^{1/2}\left(\left\|S_\alpha^q(\delta^{l+k/2}\nabla^l\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

2.4. Admissible area and maximal functions.

In this paragraph, we continue the proof of Main Theorem.

Theorem D. *Let* Ω *be a bounded* C^∞*-domain in* $\mathbb{C}^n$ *satisfying* (P), α *be an aperture* > 0. *For every* $0 < p < \infty$, *there exists a constant* C *such that, for every holomorphic function* g, *we have*

$$\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

As an immediate application of this result, we obtain that (3) implies (2) in Main Theorem.

PROOF OF THEOREM D. The proof of Theorem D will follow the same lines as the corresponding one of Fefferman-Stein (see [FS]). The differences are due to the fact that $\nabla_T^k g$ is no longer holomorphic or even harmonic in Ω although g is holomorphic.

First, we assume that $g \in C^\infty(\overline{\Omega})$ and we are going to show the following a priori inequality:

$$\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

Let us assume this inequality proved. Then, it remains to show that this inequality is still valid for general g. We apply this inequality in Ω_ε to g holomorphic in Ω. One can verify that the constant involved is independent of $\varepsilon > 0$. We want to let $\varepsilon \to 0$ in the inequality. Let us observe that, for $\zeta_\varepsilon = \Phi(\zeta, \varepsilon) \in \partial\Omega_\varepsilon$, $\mathcal{A}_\alpha(\zeta_\epsilon) \subset \mathcal{A}_\beta(\zeta)$, for some $\beta > \alpha$. This allows to show that

$$\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega_\varepsilon)} \leq C\left\|\mathcal{M}_\beta(\nabla_T^k g)\right\|_{L^p(\partial\Omega)}.$$

Now, we just have to apply Fatou's Lemma and the following usual result (see [FS] for instance).

Let $0 < p < \infty$ and α, $\beta > 0$. There exists a constant C such that, for every function u on Ω

$$\|\mathcal{M}_\alpha u\|_{L^p(\partial\Omega)} \leq C\,\|\mathcal{M}_\beta u\|_{L^p(\partial\Omega)}.$$

So, let us show the a priori inequality. As in the preceding, we are going to distinguish three cases: $p = 2$, $0 < p < 2$, $p > 2$. As the case when $p = 2$ is the simplest one and follows the same line as the case when $0 < p < 2$, we will only consider the cases $0 < p < 2$ and $p > 2$.

1. *Case* $0 < p < 2$. In the following, it will be convenient to have a defining function for Ω which is harmonic near $\partial\Omega$. We choose a point $x_0 \in K$ and denote by δ the Green's function for Ω with singularity x_0. Thus, δ is harmonic in $\Omega \setminus \{x_0\}$ and $\delta(z)$ is comparable with the distance to the boundary, for $z \in \Omega \cap U$.

Let λ and ε be any real positive numbers and E be the set

$$\{\zeta \in \partial\Omega : \quad \mathcal{M}_\alpha(\nabla_T^k g)(\zeta) \leq \lambda\,, \quad S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1} g)(\zeta) \leq C(\varepsilon, s_0)\,\lambda\}\ ,$$

for some $\gamma > \alpha$ and $C(\varepsilon, s_0) = (\varepsilon^2 + s_0)^{-1/2}$.

Let E_0 be those points of E of relative density $1/2$, D_0, D their complements. We are going to prove the following lemma which gives a "good λ" inequality of a new type.

Lemma 2.9. *Under the assumptions of Theorem* D, *there exists a constant C such that, for every $\varepsilon > 0$*

$$\begin{aligned}\int_{E_0} S_\alpha(\delta\nabla\nabla_T^k g)^2(z)d\sigma(z) \leq C\Big(\Big(\frac{1}{\varepsilon^2}+1\Big)\lambda^2\sigma(D_0) \\ + \int_0^\lambda t\sigma\left(\{\mathcal{M}_\alpha(\nabla_T^k g) \geq t\}\right) dt \\ + (\varepsilon^2+s_0)\int_E S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)^2(z)\,d\sigma(z) \\ + \|g\|^2_{L^2(K)}\Big)\,.\end{aligned}$$

PROOF. We denote by $\mathcal{R}_\alpha$ the set $\cup_{z\in E_0}\mathcal{A}_\alpha(z)$ and

$$I_{E_0} = \int_{E_0} S_\alpha(\delta\nabla\nabla_T^k g)^2(\zeta)d\sigma(\zeta)\,.$$

Then

$$\begin{aligned}I_{E_0} =& \iint_{\mathcal{R}_\alpha} \delta^2|\nabla\nabla_T^k g|^2(z)\,\sigma\left(\{\zeta\in E_0 :\ z\in\mathcal{A}_\alpha(\zeta)\}\right)\frac{dV(z)}{\delta^{n+1}} \\ \leq& C\iint_{\mathcal{R}_\alpha} \delta\,|\nabla\nabla_T^k g|^2(z)dV(z).\end{aligned}$$

We write that

$$2\,|\nabla\nabla_T^k g|^2 \leq 2\,|\Delta(\nabla_T^k g)\cdot\nabla_T^k g| + \Delta|\nabla_T^k g|^2$$

and, following the method of Fefferman and Stein, we will estimate

$$\iint_{\mathcal{R}_\alpha} \delta\,\Delta|\nabla_T^k g|^2 dV$$

by applying Green's Theorem (we recall that δ is a Green's function for Ω). Let us denote by $d\hat{\sigma}$ the surface measure on $\partial\mathcal{R}_\alpha$. So, we obtain

$$\begin{aligned} I_{E_0} \le C \Big(& \iint_{\mathcal{R}_\alpha} \delta\, |\nabla_T^k g \cdot \Delta(\nabla_T^k g)| dV \\ & + \Big(\int_{\partial\mathcal{R}_\alpha} \delta \frac{\partial |\nabla_T^k g|^2}{\partial \nu} d\hat{\sigma} - \int_{\partial\mathcal{R}_\alpha} \frac{\partial \delta}{\partial \nu} |\nabla_T^k g|^2 d\hat{\sigma} \Big)\Big) \\ = (1) + (2) + (3)\,, \end{aligned}$$

where $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\mathcal{R}_\alpha$.

1.1. *Estimate of the first term* (1). As g is holomorphic in Ω, we have $\Delta\nabla_T^k g = [\Delta, \nabla_T^k]g$ and so,

$$|\Delta\nabla_T^k g| \le C \sum_{\substack{0\le j\le 2\\ 0\le r\le k-1}} |\nabla^j \nabla_T^r g|\,.$$

Now

$$\begin{aligned} (1) &\le \iint_{\mathcal{R}_\alpha} \delta |\nabla_T^k g| \sum_{\substack{0\le j\le 2\\ 0\le r\le k-1}} |\nabla^j \nabla_T^r g|\, dV \\ &\le \Big(\iint_{\mathcal{R}_\alpha} \delta^{-1+\mu} |\nabla_T^k g|^2 dV \Big)^{1/2} \\ &\quad \cdot \Big(\iint_{\mathcal{R}_\alpha} \delta^{3-\mu} \sum_{\substack{0\le j\le 2\\ 0\le r\le k-1}} |\nabla^j \nabla_T^r g|^2 dV \Big)^{1/2} \end{aligned}$$

for every $\mu > 0$. Let us apply Lemma 2.6 with $0 < \mu < 1$, in order to obtain

$$(1) \le C\Big(s_0 \iint_{\mathcal{R}_\beta} \delta^{k+2} |\nabla^{k+1} g|^2 \frac{dV}{\delta} + \int_K |g|^2 dV \Big)$$

for some $\beta > \alpha$.

1.2. *Estimate of the second term* (2).

$$(2) \le C \int_{\partial\mathcal{R}_\alpha} \delta\, |\nabla\nabla_T^k g|\, |\nabla_T^k g|\, d\hat{\sigma}\,,$$

We split $\partial\mathcal{R}_\alpha$ into three pieces $\partial\mathcal{R}_\alpha = F \cup F^{E_0} \cup F^{D_0}$ where

$$\Phi^{-1}(F) \subset \partial\Omega \times \{s_0\}, \quad \Phi^{-1}\left(F^{E_0}\right) \subset E_0$$

and

$$\Phi^{-1}\left(F^{D_0}\right) \subset D_0 \times (0, s_0).$$

So, we write

$$(2) \leq C\left(\int_F + \int_{F^{E_0}} + \int_{F^{D_0}}\right).$$

First, we have trivially

$$\int_F \delta\, |\nabla\nabla_T^k g|\, |\nabla_T^k g|\, d\hat{\sigma} \leq C\, \|g\|^2_{L^2(K)}.$$

Then, as $d\hat{\sigma} \leq C\, d\sigma$ along $F^{E_0} \cup F^{D_0}$, we have

$$\int_{F^{E_0}} \delta\, |\nabla\nabla_T^k g|\, |\nabla_T^k g|\, d\hat{\sigma} = 0 \qquad textsince\ F^{E_0} \subset \partial\Omega.$$

For every $\varepsilon > 0$, the last part is majorized by

$$\leq C\left(\frac{1}{\varepsilon^2}\int_{F^{D_0}} |\nabla_T^k g|^2 d\hat{\sigma} + \varepsilon^2 \int_{F^{D_0}} \delta^2 |\nabla\nabla_T^k g|^2 d\hat{\sigma}\right).$$

As $\mathcal{M}_\alpha(\nabla_T^k g) \leq \lambda$ on E, we deduce that

$$\frac{1}{\varepsilon^2}\int_{F^{D_0}} |\nabla_T^k g|^2 d\hat{\sigma} \leq \frac{1}{\varepsilon^2}\lambda^2 \int_{F^{D_0}} d\hat{\sigma} \leq \frac{C}{\varepsilon^2}\lambda^2 \sigma(D_0).$$

We are going to prove now that, under the assumptions of Theorem D

$$\varepsilon^2 \int_{F^{D_0}} \delta^2 |\nabla\nabla_T^k g|^2 d\hat{\sigma} \leq C\left(\varepsilon^2 \iint_{\mathcal{R}_\beta} \delta^{k+2} |\nabla^{k+1} g|^2 \frac{dV}{\delta} + \int_K |g|^2 dV\right).$$

This will follow from the fact that, by Corollary 1.5,

$$|\nabla\nabla_T^k g|^2(\zeta) \leq C\, \mathrm{Mean}^{Q(\zeta)}\left(|\nabla\nabla_T^k g|^2 + \left[\delta^{-k/2}\mathrm{Rest}^k(\nabla g)\right]^2\right),$$

and the fact that

$$\int_{\partial\mathcal{R}_\alpha} \delta^{l+1}\mathrm{Mean}^Q(|f|^2)\, d\hat{\sigma} \leq C \iint_{\mathcal{R}_\beta} \delta^l |f|^2 dV,$$

for some β sufficiently large. Then, we apply Lemma 2.6 in order to obtain the result. So

$$(2) \le \frac{C}{\varepsilon^2}\lambda^2\sigma(D_0) + C\left(\varepsilon^2 \iint_{\mathcal{R}_\beta} \delta^{k+2}|\nabla^{k+1}g|^2\frac{dV}{\delta} + \int_K |g|^2 dV\right).$$

1.3. *Estimate of the third term.* The third term is majorized by

$$\begin{aligned}(3) &\le C\int_{\partial\mathcal{R}_\alpha} |\nabla_T^k g|^2 d\hat{\sigma} \le C\left(\int_F + \int_{F^{E_0}} + \int_{F^{D_0}}\right)\\ &\le C\left(\int_K |g|^2 dV + \int_0^\lambda t\,\sigma\left(\{\mathcal{M}_\alpha(\nabla_T^k g) \ge t\}\right)dt + \lambda^2\,\sigma(D_0)\right),\end{aligned}$$

since $\mathcal{M}_\alpha(\nabla_T^k g) \le \lambda$ on E.

To conclude for Lemma 2.9, it suffices to remark that, as in the proof of Theorem 2.3,

$$\iint_{\mathcal{R}_\beta} \delta^{k+2}\left|\nabla^{k+1}g\right|^2\frac{dV}{\delta} \le \int_E S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)^2 d\sigma,$$

for some $\gamma > \beta$.

1.4. Let us prove now the a priori inequality in case $0 < p < 2$.

$$\begin{aligned}&\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|^p_{L^p(\partial\Omega)}\\ &\quad = p\int_0^\infty \lambda^{p-1}\sigma\left(\{S_\alpha(\delta\nabla\nabla_T^k g) \ge \lambda\}\right)d\lambda\\ &\quad \le p\int_0^\infty \lambda^{p-1}\sigma(D_0)d\lambda + M^p\sigma(\partial\Omega)\\ &\qquad + p\int_M^\infty \lambda^{p-3}\int_{E_0} S_\alpha(\delta\nabla\nabla_T^k g)^2(\zeta)d\sigma(\zeta)\,d\lambda\\ &\quad \le C\Big(\Big(\frac{1}{\varepsilon^2}+1\Big)p\int_0^\infty \lambda^{p-1}\sigma(D)\,d\lambda + M^p\sigma(\partial\Omega)\\ &\qquad + (\varepsilon^2 + s_0)p\int_M^\infty \lambda^{p-3}\int_E S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)^2(\zeta)\,d\sigma(\zeta)d\lambda\\ &\qquad + p\int_M^\infty \lambda^{p-3}\int_0^\lambda t\,\sigma\left(\{\mathcal{M}_\alpha(\nabla_T^k g) \ge t\}\right)dt\,d\lambda\end{aligned}$$

$$+ \|g\|^2_{L^2(K)} M^{p-2}\Big)$$

$$\leq C\Big(\Big(\frac{1}{\varepsilon^2}+1\Big)\Big(\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|^p_{L^p(\partial\Omega)}$$

$$+ \frac{1}{C(\varepsilon,s_0)^p}\left\|S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)\right\|^p_{L^p(\partial\Omega)}\Big)$$

$$+ M^p\sigma(\partial\Omega)$$

$$+ \|g\|^2_{L^2(K)} M^{p-2}$$

$$+ \frac{(\varepsilon^2+s_0)}{C(\varepsilon,s_0)^{p-2}}\left\|S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)\right\|_{L^p(\partial\Omega)}\Big),$$

since

$$\sigma(D) \leq \sigma\left(\{\mathcal{M}_\alpha(\nabla_T^k g) \geq \lambda\}\right) + \sigma\left(\{S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g) \geq C(\varepsilon,s_0)\lambda\}\right).$$

By Theorem C and under the assumptions of Theorem D, we have

$$\left\|S_\gamma^{-k}(\delta^{k+1}\nabla^{k+1}g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|S_\gamma(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

So, it is sufficient to apply Lemma 2.1 and to choose ε and s_0 sufficiently small and $M = \|g\|_{L^2(K)}$ in order to conclude that, for $g \in C^\infty(\overline{\Omega})$ holomorphic in Ω, we have

$$\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

2. *Case* $2 < p < \infty$. In this part, we will use an auxiliary result on Dirichlet's problem. This will be proved in the appendix since we did not find any reference. We need the following definitions.

- We denote by $L_\alpha^{(p,2)}(\Omega)$ the space of all functions u on Ω such that

$$u \circ \Phi \in L^p\left(\partial\Omega; L^2(]0,s_0[,t^\alpha dt)\right)$$

with the induced Banach norm on $L_\alpha^{(p,2)}(\Omega)$

$$\|u\|^p_{L_\alpha^{(p,2)}(\Omega)} = \int_{\partial\Omega}\left(\int_0^{s_0}|u\circ\Phi(\zeta,t)|^2\, t^\alpha dt\right)^{p/2} d\sigma(\zeta).$$

- We denote by $W_\alpha^{l;(p,2)}(\Omega)$, $l \in \mathbb{N}$, the space of all functions u such that

$$D^j u \in L_\alpha^{(p,2)}(\Omega) \qquad \text{for } |j| \le l\,,$$

where $D^j u$ denotes the distribution derivative. We define a Banach norm on $W_\alpha^{l;(p,2)}(\Omega)$ by

$$\|u\|^p_{W_\alpha^{l;(p,2)}} = \Big(\sum_{|j|\le l} \|D^j u\|^p_{L_\alpha^{(p,2)}} \Big)^{1/p}.$$

- For $l \in \mathbb{N}$, we denote by $\overset{\circ}{W}{}_\alpha^{l;(p,2)}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W_\alpha^{l;(p,2)}(\Omega)$.
- For $l \in \mathbb{N}$, we denote by $W_\alpha^{-l;(p,2)}(\Omega)$ the dual space of $\overset{\circ}{W}{}_{-\alpha}^{l;(p',2)}(\Omega)$, $1/p + 1/p' = 1$.

Now, we can state our result.

Theorem on Dirichlet's Problem. *Let $1 < p < \infty$, Ω be a bounded C^∞-domain and A be a differential operator of order 2, strongly elliptic, with smooth coefficients. For every smooth function v defined on Ω, let u be the solution of the problem*

$$\begin{cases} Au = v & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then, for every $-1 < \theta < 1$, there exists a constant C independent of v such that

$$\int_{\partial\Omega} \left(\int_0^{s_0} |\nabla u \circ \Phi(z,t)|^2 t^\theta \, dt \right)^{p/2} d\sigma(z) \le C \left(\|v\|_{W_\theta^{-1;(p,2)}(\Omega)} + \sup_K |u|^p \right).$$

To prove our estimate, we have to majorize, for $g \in C^\infty(\overline{\Omega})$,

$$\left\| S_\alpha \left(\delta \nabla \nabla_T^k g \right) \right\|_{L^p(\partial\Omega)} \quad \text{by} \quad \left\| \sup_{0<t<s_0} \left| \nabla_T^k g \right| \right\|_{L^p(\partial\Omega)},$$

then the a priori estimate will follow.

First, we are going to show that

$$\begin{aligned}&\left\|G\left(\delta\nabla\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)}\\&\quad\leq C\left(\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)}+s_0^{1/2}\left\|S_\alpha\left(\delta\nabla\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)}\right).\end{aligned}$$

The desired estimate will follow since

$$\left\|S_\alpha\left(\delta\nabla\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)}\leq C\left\|G\left(\delta\nabla\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)}.$$

We write $\nabla_T^k g=\left(\nabla_T^k g\right)_0+\left(\nabla_T^k g\right)_h$ where $\left(\nabla_T^k g\right)_0$ is the solution of the problem

$$\begin{cases}\Delta w=\Delta(\nabla_T^k g) & \text{in } \Omega\,,\\ w=0 & \text{on } \partial\Omega\,.\end{cases}$$

We know that such a solution exists in $C^\infty(\overline{\Omega})$ (since $g\in C^\infty(\overline{\Omega})$ by assumption).

So,

$$\begin{aligned}\left\|G\left(\delta\nabla\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)}&\leq\left\|G\left(\delta\nabla(\nabla_T^k g)_0\right)\right\|_{L^p(\partial\Omega)}\\&\quad+\left\|G\left(\delta\nabla(\nabla_T^k g)_h\right)\right\|_{L^p(\partial\Omega)}.\end{aligned}$$

2.1. *Estimate of* $\left\|G(\delta\nabla(\nabla_T^k g)_h)\right\|_{L^p(\partial\Omega)}$. It is well known that the Littlewood-Paley function of a harmonic function is majorized, in L^p-norm, by the L^p-norm of its trace on the boundary. So, we have

$$\begin{aligned}\left\|G(\delta\nabla(\nabla_T^k g)_h)\right\|_{L^p(\partial\Omega)}&\leq\left\|\left|(\nabla_T^k g)_h\right|\right\|_{L^p(\partial\Omega)}\\&\leq\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)}\end{aligned}$$

since, by assumption, $\nabla_T^k g=(\nabla_T^k g)_h$ on $\partial\Omega$ and $g\in C^\infty(\overline{\Omega})$.

2.2. *Estimate of* $\left\|G(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)}$. As g is holomorphic in Ω,

$$\Delta(\nabla_T^k g)=[\Delta,\nabla_T^k]g\,,\quad\left|\Delta(\nabla_T^k g)\right|\leq C\left|\nabla v\right|$$

with

$$|v|\leq C\sum_{\substack{0\leq j\leq 1\\0\leq r\leq k-1}}\left|\nabla^j\nabla_T^r g\right|$$

and we can apply the Theorem on the Dirichlet's problem. For every $0<\theta<1$

$$
\begin{aligned}
\|G\left(\delta\nabla(\nabla_T^k g)_0\right)\|_{L^p(\partial\Omega)} &\le s_0^{(1-\theta)/2}\,\|G^{\theta-1}\left(\delta\nabla(\nabla_T^k g)_0\right)\|_{L^p(\partial\Omega)}\\
&= s_0^{(1-\theta)/2}\Big(\int_{\partial\Omega}\Big(\int_0^{s_0}\left|\nabla(\nabla_T^k g)_0\right|^2 t^\theta dt\Big)^{p/2}d\sigma\Big)^{1/p}\\
&\le C\,s_0^{(1-\theta)/2}\Big(\int_{\partial\Omega}\Big(\int_0^{s_0}\sum_{\substack{0\le j\le 1\\ 0\le r\le k-1}}|\nabla^j\nabla_T^r g|^2 t^\theta dt\Big)^{p/2}d\sigma\Big)^{1/p}\\
&\quad + C\,\|g\|_{L^2(K)}\\
&\le C\,s_0^{(1-\theta)/2}\Big(\int_{\partial\Omega}\Big(\int_0^{s_0}\sum_{0\le r\le k-1}|\nabla^k\nabla_T^r g|^2 t^{\theta+2k-2}dt\Big)^{p/2}d\sigma\Big)^{1/p}\\
&\quad + C\,\|g\|_{L^2(K)},
\end{aligned}
$$

(by Hardy inequality in $L^2(]0,s_0[)$),

$$
\begin{aligned}
&\le C\,s_0^{(1-\theta)/2}\Big(\int_{\partial\Omega}\Big(\int_0^{s_0}\mathrm{Mean}^Q\big(\sum_{j=1}^k|\nabla^j g|^2\big)t^{\theta+k-1}dt\Big)^{p/2}d\sigma\Big)^{1/p}\\
&\quad + C\,\|g\|_{L^2(K)}\ ,
\end{aligned}
$$

(by Theorem A),

$$
\begin{aligned}
&= C\,s_0^{(1-\theta)/2}\Big(\Big\|S_\alpha^{\theta+k}\big(\sum_{j=1}^k|\nabla^j g|\big)\Big\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\Big)\\
&\le C\,s_0^{1/2}\left(\|S_\alpha(\delta\nabla\nabla_T^k g)\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right)\ ,
\end{aligned}
$$

(by Theorem 2.3).

2.5. Admissible and radial maximal functions.

We are going to show the following theorem.

Theorem E. *Let Ω be a bounded C^∞-domain in $\mathbb{C}^n$, satisfying* (P). *For every $0<p<\infty$, every $k\in\mathbb{N}$, every $\alpha>0$, there exists a constant C such that, for every holomorphic function g, the following holds*

$$
\|\mathcal{M}_\alpha(\nabla_T^k g)\|_{L^p(\partial\Omega)} \le C\left(\Big\|\sup_{0<t<s_0}|\nabla_T^k g|\Big\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).
$$

As an application, we obtain that (4) implies (3) in Main Theorem.

Lemma 2.10. *Let $l > 0$, α be a fixed aperture. There exists a constant C such that, for every function $u \in C^\infty(\Omega)$, we have, for every $\mu > 0$, every $\zeta \in \partial\Omega$*

$$\begin{aligned}\mathcal{M}_\alpha\left(\delta^\mu|u|\right)(\zeta) \leq & C\mathcal{M}_\alpha\left(\delta\left|\frac{\partial u}{\partial\nu}\right|\right)(\zeta) + C\sup_K |u|,\\ \mathcal{M}_\alpha\left(\delta^l|u|\right)(\zeta) \leq & C\mathcal{M}_\alpha\left(\delta^{l+1}\left|\frac{\partial u}{\partial\nu}\right|\right)(\zeta) + C\sup_K |u|,\end{aligned}$$

where $\partial/\partial\nu$ denotes the normal derivative on $\partial\Omega$.

PROOF. Let $\zeta \in \partial\Omega$. We assume that

$$\mathcal{M}_\alpha\left(\delta^{l+1}\left|\frac{\partial u}{\partial\nu}\right|\right)(\zeta) \leq C\,.$$

For every $\Phi(\eta, t) \in \mathcal{A}_\alpha(\zeta)$, we have

$$\begin{aligned}|u\circ\Phi(\eta,t)| = & \left|\int_0^{s_0}\frac{d}{ds}u\circ\Phi(\eta, s+t)ds - u\circ\Phi(\eta, s_0+t)\right|\\ \leq & \mathcal{M}_\alpha\left(\delta^{l+1}\left|\frac{\partial u}{\partial\nu}\right|\right)(\zeta)\cdot\left|\int_0^{s_0}\frac{ds}{(t+s)^{l+1}}\right| + C\sup_K|u|\end{aligned}$$

(since $\Phi(\eta, s+t) \in \mathcal{A}_\alpha(\zeta) \cup K$)

$$\leq C\left(t^{-l}\mathcal{M}_\alpha\left(\delta^{l+1}\left|\frac{\partial u}{\partial\nu}\right|\right)(\zeta) + \sup_K|u|\right).$$

PROOF OF THEOREM E. As in the proof of Theorem D, we are going to prove an a priori estimate. Explicitely, for $g \in C^\infty(\overline{\Omega})$, we are going to show that

$$\left\|\mathcal{M}_\alpha(|\nabla_T^k g|)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)} + \|g\|_{L^p(K)}\right).$$

For general g, we apply the preceding inequality in Ω_ε and we let $\varepsilon \to 0$ as before.

So, let us assume that $g \in C^\infty(\overline{\Omega})$. By Lemma 1.2, after a change of coordinates, we can write, around any point of $\mathcal{A}_\beta(\zeta)$, each component of $\nabla_T^k g$ as a sum of an (AB) function and of a rest $\delta^{-k/2}\mathrm{Rest}^k(g)$. This allows us to show that, for every $\Phi(\eta,t) \in \mathcal{A}_\alpha(\zeta)$

$$\begin{aligned}
&\left|\nabla_T^k g \circ \Phi(\eta,t)\right|^{p/2} \\
&\leq \frac{C}{|Q(\Phi(\eta,t))|}\int_{Q(\Phi(\eta,t))}\left|\nabla_T^k g\right|^{p/2} dV \\
&\quad + \frac{C}{|Q(\Phi(\eta,t))|}\int_{Q(\Phi(\eta,t))}\left|\delta^{-k/2}\mathrm{Rest}^k(g)\right|^{p/2} dV \\
&\quad + \left|\delta^{-k/2}\mathrm{Rest}^k(g)\circ\Phi(\eta,t)\right|^{p/2} \\
&\leq \frac{C}{|B^d(\eta,t)|}\int_{B^d(\eta,t)}\sup_{0<s<s_0}\left|\nabla_T^k g\circ\Phi(\eta',s)\right|^{p/2} d\sigma(\eta') \\
&\quad + \frac{C}{|B^d(\eta,t)|}\int_{B^d(\eta,t)}\sup_{0<s<s_0}\left|\delta^{-k/2}\mathrm{Rest}^k(g)\circ\Phi(\eta',s)\right|^{p/2} d\sigma(\eta') \\
&\quad + \left|\delta^{-k/2}\mathrm{Rest}^k(g)\circ\Phi(\eta,t)\right|^{p/2}.
\end{aligned}$$

So, we obtain, by the L^2-continuity of the non-isotropic maximal operator

$$\begin{aligned}
&\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)}^p = \left\|\mathcal{M}_\alpha(\nabla_T^k g)^{p/2}\right\|_{L^2(\partial\Omega)}^p \\
&\leq C\Big(\Big\|\sup_{0<s<s_0}\left|\nabla_T^k g\right|\Big\|_{L^p(\partial\Omega)}^p + \Big\|\sup_{0<s<s_0}\left|\delta^{-k/2}\mathrm{Rest}^k(g)\right|\Big\|_{L^p(\partial\Omega)}^p \\
&\qquad + \left\|\mathcal{M}_\alpha(\delta^{-k/2}\mathrm{Rest}^k(g))\right\|_{L^p(\partial\Omega)}^p\Big) \\
&\leq C\left(\Big\|\sup_{0<s<s_0}\left|\nabla_T^k g\right|\Big\|_{L^p(\partial\Omega)}^p + \left\|\mathcal{M}_\alpha(\delta^{-k/2}\mathrm{Rest}^k(g))\right\|_{L^p(\partial\Omega)}^p\right).
\end{aligned}$$

So, it suffices to estimate $\left\|\mathcal{M}_\alpha(\delta^{-k/2}\mathrm{Rest}^k(g))\right\|_{L^p(\partial\Omega)}^p$ in terms of $\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)}$ with a small constant.

$$\begin{aligned}
\mathcal{M}_\alpha\left(\delta^{-k/2}\mathrm{Rest}^k(g)\right) \leq C\Big(&\mathcal{M}_\alpha\Big(\sum_{1\leq j+r<(k+r)/2}\left|\nabla^j\nabla_T^r u\right|\Big) \\
&+ \mathcal{M}_\alpha\Big(\sum_{(k+r)/2\leq j+r\leq k}\delta^{j+(r+1-k)/2}\left|\nabla^j\nabla_T^r u\right|\Big)\Big).
\end{aligned}$$

Let β be chosen so that if $z \in \mathcal{A}_\alpha(\zeta)$, $Q(z) \subset \mathcal{A}_\beta(\zeta)$. We apply successively Lemma 2.10 and Theorem A in order to obtain, for every $\mu > 0$,

$$\begin{aligned}\mathcal{M}_\alpha\left(\delta^{-k/2}\mathrm{Rest}^k(g)\right) \leq & C\Big(\mathcal{M}_\beta\Big(\sum_{1\leq j+r<(k+r)/2} \delta^{k-\mu-j-r/2}|\nabla^k g|\Big) \\ & + \mathcal{M}_\beta\Big(\sum_{(k+r)/2\leq j+r\leq k} \delta^{(k+1)/2}|\nabla^k g|\Big)\Big) \\ \leq & C\,\mathcal{M}_\beta\left(\delta^{(k+1)/2-\mu}|\nabla^k g|\right).\end{aligned}$$

So, we have to estimate

$$\|\mathcal{M}_\beta\left(\delta^{(k+1)/2-\mu}|\nabla^k g|\right)\|_{L^p(\partial\Omega)}$$

in terms of

$$\left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)}$$

with a small constant.

By the converse estimates of Theorem B, we have, by choosing $\mu < 1/2$,

$$\begin{aligned}(*) &= \mathcal{M}_\beta\left(\delta^{(k+1)/2-\mu}|\nabla^k g|\right) \\ &\leq C\left(\mathcal{M}_\gamma\left(\delta^{1/2-\mu}|\nabla_T^k g|\right) + \mathcal{M}_\gamma\left(\delta^{-\mu+(1-k)/2}|\mathrm{Rest}^k(g)|\right) + \sup_K |g|\right) \\ &\leq C\left(s_0^{1/2-\mu}\mathcal{M}_\gamma\left(|\nabla_T^k g|\right) + s_0^{1/2}\mathcal{M}_\gamma\left(\delta^{(k+1)/2-\mu}|\nabla^k g|\right) + \sup_K |g|\right).\end{aligned}$$

Then, we choose s_0 sufficiently small in order to obtain

$$\begin{aligned}&\left\|\mathcal{M}_\beta\left(\delta^{(k+1)/2-\mu}|\nabla^k g|\right)\right\|_{L^p(\partial\Omega)} \\ &\qquad\leq C\left(s_0^{1/2-\mu}\left\|\mathcal{M}_\alpha(|\nabla_T^k g|)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^p(K)}\right).\end{aligned}$$

Inserting this inequality in the estimate of

$$\left\|\mathcal{M}_\alpha(\delta^{-k/2}\mathrm{Rest}^k(g))\right\|_{L^p(\partial\Omega)},$$

we are able to conclude, reducing again s_0 if necessary, that, for $g \in C^\infty(\overline{\Omega})$

$$\left\|\mathcal{M}_\alpha(|\nabla_T^k g|)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\sup_{0<t<s_0}|\nabla_T^k g|\right\|_{L^p(\partial\Omega)} + \|g\|_{L^p(K)}\right).$$

2.6. End of the proof of Main Theorem.

In this paragraph, we are going to show the following theorem.

Theorem F. *Let Ω be a bounded C^∞-domain in $\mathbb{C}^n$, satisfying* (P). *For every $1-1/(2n+1)<p<\infty$, every $k\in\mathbb{N}$, there exists a constant C such that, for every holomorphic function g in Ω, we have*

$$\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)} \le C\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\,.$$

As an immediate corollary of this theorem, we obtain that (2) implies (4) in Main Theorem.

PROOF OF THEOREM F. In this part, we will use theorem on Dirichlet's problem stated in paragraph 2.4.

We want to majorize

$$\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)}$$

As before, we are going to prove Theorem F only for $g\in C^\infty(\overline{\Omega})$. We write $\nabla_T^k g = \left(\nabla_T^k g\right)_0 + \left(\nabla_T^k g\right)_h$ where $\left(\nabla_T^k g\right)_0$ is the solution of the problem

$$\begin{cases} \Delta w = \Delta(\nabla_T^k g) & \text{in } \Omega\,,\\ w=0 & \text{on } \partial\Omega\,.\end{cases}$$

So,

$$\begin{aligned}\left\|\sup_{0<t<s_0}\left|\nabla_T^k g\right|\right\|_{L^p(\partial\Omega)} &\le \left\|\sup_{0<t<s_0}\left|(\nabla_T^k g)_0\right|\right\|_{L^p(\partial\Omega)}\\ &\quad+\left\|\sup_{0<t<s_0}\left|(\nabla_T^k g)_h\right|\right\|_{L^p(\partial\Omega)}\\ &=(1)+(2)\,.\end{aligned}$$

First, for every $0<\theta<1$, we have

$$\left|(\nabla_T^k g)_0\circ\Phi(\eta,t)\right| \le \int_0^{s_0}\left|\nabla(\nabla_T^k g)_0\circ\Phi(\eta,s)\right|ds + \sup_K|g|$$

$$\begin{aligned}&\leq C\, s_0^{(1-\theta)/2} \left(\int_0^{s_0} \left|\nabla(\nabla_T^k g)_0 \circ \Phi(\eta, s)\right|^2 s^{\theta+1} \frac{ds}{s} \right)^{1/2} \\ &\quad + \sup_K |g| \\ &\leq C\, s_0^{(1-\theta)/2}\, G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)(\eta) + \sup_K |g|\,.\end{aligned}$$

So,

$$\begin{aligned}(1) &= \left\| \sup_{0<t<s_0} \left|(\nabla_T^k g)_0\right| \right\|_{L^p(\partial\Omega)} \\ &\leq C \left(s_0^{(1-\theta)/2} \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^p(K)} \right).\end{aligned}$$

It is well known that, for harmonic functions, the L^p-norm of the radial maximal functions are majorized by the L^p-norm of the Littlewood-Paley functions. So,

$$(2) = \left\| \sup_{0<t<s_0} \left|(\nabla_T^k g)_h\right| \right\|_{L^p(\partial\Omega)} \leq C \left\|G(\delta\nabla(\nabla_T^k g)_h)\right\|_{L^p(\partial\Omega)}$$

and we obtain

$$\begin{aligned}(2) &\leq C \left(\left\|G(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + s_0^{(1-\theta)/2} \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)} \right) \\ &\leq C \left(\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + s_0^{(1-\theta)/2} \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)} \right),\end{aligned}$$

by Corollary 2.8.

So, it remains to estimate $\left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)}$. Since in the proof of Theorem D, we have shown that, when $2 < p < \infty$,

$$\begin{aligned}&s_0^{(1-\theta)/2} \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)} \\ &\qquad\qquad \leq C \left(s_0^{1/2} \left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)} \right),\end{aligned}$$

it is sufficient to consider the case $0 < p \leq 2$. We are going to show that

$$\left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)}$$

is bounded by

$$\left\| \sup_{0<t<s_0} \left|\nabla_T^k g\right| \right\|_{L^p(\partial\Omega)}$$

when $1 - 1/(2n+1) < p \leq 2$.

First, let us give the following lemma.

Lemma 2.11. *Let u be any continuous function defined on Ω. Then, for every $\alpha > 0$, every $q \geq p > 0$ and every $\theta > 0$ satisfying $\theta \geq n/p - n/q$, we have*

$$\left\|\mathcal{M}_\alpha(\delta^\theta u)\right\|_{L^q(\partial\Omega)} \leq C \left\|\mathcal{M}_\alpha(u)\right\|_{L^p(\partial\Omega)}.$$

PROOF. The proof is based on the atomic decomposition of spaces of homogeneous type (see [AN]). We will denote by T^p the space of all continuous functions u on Ω such that $\mathcal{M}_\alpha(u) \in L^p(\partial\Omega)$ with norm $\|u\|_{T^p} = \|\mathcal{M}_\alpha(u)\|_{L^p(\partial\Omega)}$. For every $E \subset \partial\Omega$, we define the tent over E to be the subset $\hat{E}$ of $\partial\Omega \times (0, s_0)$ by $\partial\Omega \times (0, s_0) \setminus \hat{E} = \cup\{\mathcal{A}(\zeta),\ \zeta \in \partial\Omega \setminus E\}$. A non-negative function a on $\partial\Omega \times (0, s_0)$ is an atom over $\hat{B^d}$ if a vanishes outside $\hat{B^d}$ and if $a \leq \sigma(B^d)^{-1}$. The atomic decomposition Theorem can be formulated as follows.

Theorem. ([AN]). *There is a constant C such that, for every $u \in T^1$, there are a sequence of balls $\{B_j^d = B^d(x_j, \delta_j)\}$, a sequence of atoms a_j over B_j^d and a sequence $\{\lambda_j\}$ of positive numbers such that $|u \circ \Phi| \leq \sum \lambda_j a_j$ on $\partial\Omega \times (0, s_0)$ and $\sum \lambda_j \leq \|u\|_{T^1}$.*

Let $u \in T^p$ with $\|u\|_{T^p} \leq 1$, then $|u|^p \in T^1$, so by the preceding theorem, there are a sequence $\{a_j\}$ of atoms and a sequence $\{\lambda_j\}$ of positive numbers such that $|u \circ \Phi|^p \leq \sum \lambda_j a_j$ on $\partial\Omega \times (0, s_0)$ and $\sum \lambda_j \leq \|u\|_{T^p}^p \leq 1$. Thus, we have

$$\left[\mathcal{M}_\alpha(\delta^\theta |u|)\right]^p \leq \mathcal{M}_\alpha(\delta^{\theta p}|u|^p) \leq \left(\sum \lambda_j \mathcal{M}_\alpha(\delta^{\theta p}|a_j|)\right).$$

So

$$\begin{aligned}\left\|\mathcal{M}_\alpha(\delta^\theta |u|)\right\|_{L^q(\partial\Omega)} &= \left\|\mathcal{M}_\alpha(\delta^\theta |u|)^p\right\|_{L^{q/p}(\partial\Omega)} \\ &\leq \sum \lambda_j \left\|\mathcal{M}_\alpha(\delta^{\theta p}|a_j|)\right\|_{L^{q/p}(\partial\Omega)}\end{aligned}$$

and it suffices to see that

$$\left\|\mathcal{M}_\alpha(\delta^{\theta p}|a_j|)\right\|_{L^{q/p}(\partial\Omega)} \leq C.$$

But

$$\left\|\mathcal{M}_\alpha(\delta^{\theta p}\,|a_j|)\right\|_{L^{q/p}(\partial\Omega)} \le \delta_j^{\theta p}\,\sigma(B_j^d)^{-1}\,\sigma(B_j^d)^{p/q}$$

since $\delta(z) \le c\,\delta_j$ on $\hat{B_j^d}$. This allows to conclude, since $\sigma(B_j^d) \simeq \delta_j^n$.

1. *Estimate of* $\left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)}$ *when* $1-1/(2n+1) < p \le 2$. As g is holomorphic in Ω, $\Delta(\nabla_T^k g) = [\Delta, \nabla_T^k]g$, $\left|\Delta(\nabla_T^k g)\right| \le C\,|\nabla v|$ with

$$|v| \le C \sum_{\substack{0\le j\le 1\\ 0\le r\le k-1}} |\nabla^j \nabla_T^r g|$$

and we can apply the result of the theorem on Dirichlet's problem.

For every $0 < \theta < 1$

$$(*) = \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^p(\partial\Omega)} \le \left\|G^{\theta-1}(\delta\nabla(\nabla_T^k g)_0)\right\|_{L^q(\partial\Omega)}$$

where $q > 1$ will be chosen later. Then, we can apply our result on Dirichlet's problem to obtain

$$\begin{aligned}(*) &\le C\Big(\int_{\partial\Omega}\Big(\int_0^{s_0} \sum_{\substack{0\le j\le 1\\ 0\le r\le k-1}} |\nabla^j\nabla_T^r g|^2\, t^\theta\, dt\Big)^{q/2} d\sigma\Big)^{1/q}\\ &\quad + C\,\|g\|_{L^2(K)}\\ &\le C\Big(\int_{\partial\Omega}\Big(\int_0^{s_0} \sum_{0\le r\le k-1} |\nabla^k\nabla_T^r g|^2\, t^{\theta+2k-2}\, dt\Big)^{q/2} d\sigma\Big)^{1/q}\\ &\quad + C\,\|g\|_{L^2(K)}\,,\end{aligned}$$

(by Hardy inequality in $L^2(]0, s_0[)$),

$$\begin{aligned}&\le C\Big(\int_{\partial\Omega}\Big(\int_0^{s_0} \mathrm{Mean}^Q\big(\sum_{j=1}^{k} |\nabla^j g|^2\big) t^{\theta+k-1} dt\Big)^{q/2} d\sigma\Big)^{1/q}\\ &\quad + C\,\|g\|_{L^2(K)}\,,\end{aligned}$$

(by Theorem A),

$$\le C\left(\left\|S_\alpha^{\theta-k}(\delta^k\nabla^k g)\right\|_{L^q(\partial\Omega)} + \|g\|_{L^2(K)}\right),$$

(by Theorem 2.3).

So, we obtain by Theorem C

$$\left\|S_\alpha^{\theta-k}(\delta^k \nabla^k g)\right\|_{L^q(\partial\Omega)} \leq C\left(\left\|S_\alpha^\theta(\nabla_T^k g)\right\|_{L^q(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

So we have, for every $0 < \varepsilon < 1$,

$$(*) \leq C\left(\int_{\partial\Omega} \left(\mathcal{M}_\alpha(\delta^\beta |\nabla_T^k g|)^q \left(\int_{\mathcal{A}_\alpha(\zeta)} \delta^{2\varepsilon} \frac{dV}{\delta^{n+1}}\right)^{q/2} d\sigma\right)^{1/q} + C\,\|g\|_{L^2(K)}$$

(with $\beta = \theta/2 - \varepsilon$),

$$\leq C\left(s_0^\varepsilon \left\|\mathcal{M}_\alpha(\delta^\beta |\nabla_T^k g|)\right\|_{L^q(\partial\Omega)} + \|g\|_{L^2(K)}\right)$$
$$\leq C\left(s_0^\varepsilon \left\|\mathcal{M}_\alpha(\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right),$$

by choosing q such that Lemma 2.11 holds with $\theta/2 - \varepsilon \geq n/p - n/q$ which is possible if $1 - 1/(2n+1) < p < \infty$ (since we can choose $\theta - 1$ and ε arbitrarily close to 0). But, by Theorem E, we have

$$\left\|\mathcal{M}_\alpha\left(\nabla_T^k g\right)\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|\sup_{0<t<s_0} |\nabla_T^k g|\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

So, we obtain an a priori estimate for every $1 - 1/(2n+1) < p < \infty$ and every holomorphic function $g \in C^\infty(\overline{\Omega})$

$$\left\|\sup_{0<t<s_0} |\nabla_T^k g|\right\|_{L^p(\partial\Omega)} \leq C\left(\left\|S_\alpha(\delta\nabla\nabla_T^k g)\right\|_{L^p(\partial\Omega)} + \|g\|_{L^2(K)}\right).$$

3. Appendix: Dirichlet problem in mixed Sobolev Spaces with weights.

Let us recall the result we are going to prove.

Theorem on Dirichlet's Problem. *Let $1 < p < \infty$, Ω be a bounded C^∞-domain and A be a differential operator of order 2, strongly elliptic,* *with smooth coefficients. For every smooth function v defined on Ω, let u be the solution of the problem*

$$\begin{cases} Au = v & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then, for every $-1 < \theta < 1$, there exists a constant C independent of v such that

$$\int_{\partial\Omega} \left(\int_0^{s_0} |\nabla u \circ \Phi(z,t)|^2 t^\theta dt \right)^{p/2} d\sigma(z) \\ \leq C \left(\|v\|_{W_\theta^{-1;(p,2)}(\Omega)} + \sup_K |u|^p \right).$$

The proof follows the same line as the one given by Grisvard in his book for the usual problem (see [Gr]). By routine arguments (partition of unity, change of coordinates, freezing of coefficients) it suffices to solve the problem for the Laplacian and for smooth functions with compact support in

$$F = \left\{ (x', x_n) : \quad x' \in \mathbb{R}^{n-1},\, 0 \leq x_n \leq s_0 \right\} \subset \mathbb{R}^n_+, \quad s_0 \text{ fixed.}$$

Explicitely, we just have to show the following lemma.

Lemma 3.1. *Let $1 < p < \infty$. Then, for every $-1 < \theta < 1$, there exists a constant C such that, for any smooth function u with compact support in F, we have*

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{s_0} |\nabla u|^2 x_n^\theta dx_n \right)^{p/2} dx' \leq C \left(\|\Delta u\|_{W_\theta^{-1;(p,2)}(F)} + \sup_K |u|^p \right).$$

PROOF. Let u be any smooth function with compact support in F. We denote by v the function Δu. Assume that v belongs to $W_\theta^{-1,(p,2)}(\mathbb{R}^n_+)$; then there exists v_1, $v_2^{(J)} \in L_\theta^{(p,2)}(\mathbb{R}^n_+)$, with compact support in F, such that

$$v = v_1 + \sum_{|J|=1} D^J v_2^{(J)}$$

$$\text{with } \|v\|_{W_\theta^{-1,(p,2)}(\mathbb{R}^n_+)} \simeq \|v_1\|_{L_\theta^{(p,2)}(\mathbb{R}^n_+)} + \sum_{|J|=1} \|v_2^{(J)}\|_{L_\theta^{(p,2)}(\mathbb{R}^n_+)}\,.$$

Then, u can be written as the sum of two functions u_1, u_2 satisfying

$$\begin{cases} \Delta u_1 = v_1 & \text{in } \mathbb{R}^n_+ \,, \\ u_1 = 0 & \text{on } \{x_n = 0\} \,, \end{cases} \qquad \begin{cases} \Delta u_2 = \sum_{|J|=1} D^J v_2^{(J)} & \text{in } \mathbb{R}^n_+ \,, \\ u_2 = 0 & \text{on } \{x_n = 0\} \,. \end{cases}$$

We will only estimate the term corresponding to u_2 since the other term is better.

An argument of symetry allows us to write

$$u_2(x', x_n) = \int_{\mathbb{R}^n_+} [E(x'-y', x_n - y_n) - E(x'-y', x_n + y_n)] \cdot \sum_{|J|=1} D^J v_2^{(J)}(y', y_n) dy' dy_n \,,$$

where E is the normalized fundamental solution of Laplace's Equation. We can assume that u_2 is smooth with compact support in F.

By Green's Theorem, we have

$$u_2(x', x_n) = \int_{\mathbb{R}^n_+} \sum_{|J|=1} D^J [E(x'-y', x_n - y_n) - E(x'-y', x_n + y_n)] \cdot v_2^{(J)}(y', y_n) dy' dy_n \,,$$

so, for $|K| = 1$

$$\begin{aligned} D^K u_2(x', x_n) &= \int_{\mathbb{R}^n_+} \sum_{|J|=1} D^{K+J} [E(x'-y', x_n - y_n) \\ &\qquad - E(x'-y', x_n + y_n)] v_2^{(J)}(y', y_n) dy' dy_n \\ &= \sum_{|\mu|=2} (D^\mu E * V_2^{(\mu)} + D^\mu E * V'^{(\mu)}_2) \,, \end{aligned}$$

where $V_2^{(\mu)}$ and $V'^{(\mu)}_2$ are zero outside $\mathbb{R}^n_+$. It is usual to see that $K = \sum_{|\mu|=2} D^\mu E$ is a Calderón-Zygmund kernel. We are going to show that the corresponding operator is bounded from $L^p(dx', L^2(x_n^\theta dx_n))$ into itself for every $-1 < \theta < 1$. Assume it is done, then we obtain

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{s_0} |\nabla u_2|^2 x_n^\theta dx_n \right)^{p/2} dx'$$

$$\leq C \int_{\mathbb{R}^{n-1}} \left(\int_0^{s_0} \Big| \sum_{|J|=1} v_2^{(J)} \Big|^2 x_n^\theta dx_n \right)^{p/2} dx'$$
$$\leq C \, \|v\|_{W_\theta^{-1,(p,2)}}$$

and this gives the result.

So, it remains to show that a Calderón-Zygmund kernel K defines a bounded operator from $L^p(dx', L^2(x_n^\theta dx_n))$ into itself, for every $-1 < \theta < 1$ and every $1 < p \leq 2$: the result for general p follows from duality. Let us give the proof for completeness.

First, it is easy to see that the weight x_n^θ belongs to the class of Muckenhoupt (A_2), for every $-1 < \theta < 1$. This implies the continuity of the operator from $L^2(x_n^\theta dx_n dx')$ into itself. It remains to show (see [S2]) that

$$\|\nabla_{x'} K\| \leq \frac{C}{|x'|^n}$$

where $K(x')$ is the operator defined by

$$K(x')h(t) = \int K(x', t-s)h(s)ds = \int \frac{\omega(x', t-s)}{\left((t-s)^2 + |x'|^2\right)^{n/2}} h(s)ds$$

for $h \in L^2(x_n^\theta dx_n)$, where ω is homogeneous of order 0 and $\|\nabla_{x'} K\|$ denotes the norm, from $L^2(x_n^\theta dx_n)$ into itself, of the corresponding operator which satisfies, for every $h \in L^2(x_n^\theta dx_n)$ and every $t \in \mathbb{R}$,

$$|\nabla_{x'} K(x')h(t)| \leq C \int \frac{\omega(x', t-s)}{\left((t-s)^2 + |x'|^2\right)^{(n+1)/2}} h(s)ds \, .$$

So, by homogeneity, it suffices to show that $\|k\| \leq C$, where k is the convolution operator with kernel $1/(t^2+1)^{(n+1)/2}$, acting on $L^2(x_n^\theta dx_n)$.

It is well known that, for every $t \in \mathbb{R}$, every $h \in L^2(x_n^\theta dx_n)$, $k(h)(t) \leq C\, Mh(t)$; where M denotes the maximal Hardy-Littlewood operator. This finishes the proof since M is bounded in $L^2(x_n^\theta dx_n)$ as x_n^θ belongs to (A_2).

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