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Multiplicative structure of de Branges's spaces

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1. Introduction

L. de Branges has originated a viewpoint one of whose repercussions has been the detailed analysis of certain Hilbert spaces of holomorphic functions contained within the Hardy space H^2 of the unit disk. The initial study of the spaces was made by de Branges and J. Rovnyak [4] about 25 years ago. Although neglected for a while, the spaces are now attracting considerable attention because of their beautiful internal structure and their relevance to function theory [21]. Our aim in this paper is to investigate their multipliers.

The starting point is a nonconstant function b in $B(H^\infty)$, the unit ball in the space H^∞ of all bounded holomorphic functions in the open unit disk, D, of the complex plane. The de Branges space $\mathsf{H}(b)$ consists by definition of the range of the operator $(1 - T_b T_{\bar{b}})^{1/2}$ (where, for ϕ in L^∞ of the unit circle, T_ϕ denotes the Toeplitz operator on H^2 with symbol ϕ). The space $\mathsf{H}(b)$ is given the Hilbert space structure that makes the operator $(1 - T_b T_{\bar{b}})^{1/2}$ a coisometry of H^2 onto $\mathsf{H}(b)$. By a multiplier of $\mathsf{H}(b)$ we mean a holomorphic function m in D such that mh is in $\mathsf{H}(b)$ whenever h is. Since the evaluation functionals on $\mathsf{H}(b)$ at the points of D are bounded, one sees from the closed graph theorem that the multiplication operator on $\mathsf{H}(b)$ induced by such an m is bounded, from which it follows that m must be in H^∞ [23].

There are two extreme cases. If $\|b\|_\infty < 1$, then $\mathsf{H}(b)$ is just a renormed version of H^2 and every function in H^∞ is a multiplier of it. At the other extreme, if b is an inner function, then $\mathsf{H}(b)$ is an ordinary subspace of H^2, namely, the orthogonal complement of the Beurling invariant subspace bH^2.

It is thus the typical invariant subspace of S^*, the adjoint of the unilateral shift operator, S, on H^2 $((Sf)(z) = zf(z))$. In this case, $H(b)$ has no nonconstant multipliers. (Proof: If b is an inner function and m is a multiplier of $H(b)$ then, because S^*b is in $H(b)$ [21], we have, for all f in H^2, the equality $0 = \langle mS^*b, bf \rangle$. One easily sees that the right side equals $\langle S^*m, (1 - \overline{b(0)} b)f \rangle$. Setting $f = S^*m/(1 - \overline{b(0)}b)$, we find that $S^*m = 0$).

The spaces $H(b)$ break naturally into two classes according to whether b is or is not an extreme point of $B(H^\infty)$, or, what is equivalent, according to whether the function $1 - |b|^2$ is not or is log-integrable on ∂D [14]. A few results in the latter case can be found in [18]. It is shown there, for example, that if b is not an extreme point of $B(H^\infty)$ then any function holomorphic in a neighborhood of $\bar{D}$ is a multiplier of $H(b)$, and those b for which every function in H^∞ is a multiplier of $H(b)$ are characterized. Further progress has recently been made by B. M. Davis and J. E. McCarthy [1] who, among other things, have characterized the functions that are multipliers of every space $H(b)$ with b nonextreme. For the case where b is an extreme point, on the other hand, next to nothing has been known up to now beyond the negative result for inner functions cited above. In particular, it has been an open question whether there is any extreme point b such that $H(b)$ has nonconstant multipliers.

In this paper we shall concentrate mainly on the case where b is an extreme point but not an inner function. The main thrust of our results is that $H(b)$ has an abundance of multipliers in that case.

A space closely related to $H(b)$, called $H(\bar{b})$, arises naturally in the search for multipliers. By definition, $H(\bar{b})$ is the range of the operator $(1 - T_{\bar{b}}T_b)^{1/2}$, with the Hilbert space structure that makes this operator a coisometry of H^2 onto $H(\bar{b})$. The space $H(\bar{b})$ is trivial if b is an inner function, but otherwise it is infinite dimensional. It turns out that every multiplier of $H(b)$ differs by at most a constant from a function in $H(\bar{b})$. The culmination of our efforts will be a proof, for the case where b is an extreme point of $B(H^\infty)$, that the multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.

In Section 2 the place of the spaces $H(b)$ and $H(\bar{b})$ in the general scheme of de Branges is described. A lemma concerning that scheme is established and used to obtain information about $H(b)$ and $H(\bar{b})$. (Some of the results here can be found in the literature, but the present proofs seem particularly apt.).

Section 3 explains the relation between $H(b)$ and $H(\bar{b})$ and certain spaces of Cauchy integrals. The multipliers of $H(b)$ coincide with the multipliers of its related space of Cauchy integrals. Cauchy integrals in the unit disk have been studied extensively beginning with V. P. Havin [12], but from a viewpoint rather different from ours. In Section 4 we show how our methods provide a simple proof of a theorem of S. A. Vinogradov.

The remainder of our paper addresses mainly the case where b is an extreme point of $B(H^\infty)$. Section 5 contains two negative results for that case that say, very roughly speaking, that nonconstant multipliers cannot behave too nicely. Section 6 pertains to decompositions of the space $H(b)$ and Section 7 to the case where b is invertible. In the latter case b is shown to be a multiplier of $H(b)$, and the converse is shown to hold when b is an extreme point. In Section 8 the multiplication operator on $H(b)$ induced by a multiplier is discussed, for the extreme point case.

In Section 9, again for the extreme point case, we introduce two conjugations, one on $H(b)$ and another on the one-dimension extension of $H(\bar{b})$ by the constant functions. It is shown that to each multiplier m of $H(b)$ there corresponds a conjugate multiplier, m_*. The multipliers of $H(b)$ thus form a $*$-algebra, although not a C^*-algebra. A certain algebra of Cauchy integrals is introduced which contains all the multipliers of $H(b)$.

Section 10 contains two needed lemmas on Cauchy integrals. They are used in Section 11 to obtain more information on the space $H(\bar{b})$ and in Section 12 to establish a criterion that, among other things, enables us to construct a set of multipliers of $H(b)$ that is dense in $H(\bar{b})$ (again, for the extreme point case).

If u is an inner function, then, as is explained in Section 6, every multiplier of $H(ub)$ is a multiplier of $H(b)$. In Section 13, in the extreme point case, a criterion is obtained for a multiplier of $H(b)$ to be a multiplier of $H(ub)$. The functions that are multipliers of $H(ub)$ for every inner function u are characterized. A sufficient condition on u is found for $H(b)$ and $H(ub)$ to have the same multipliers.

In Section 14 we give a complete description of the multipliers of $H(b)$ for a certain class of extreme points b. This result is related to a well-known theorem of H. Helson and G. Szegö. We also give an example to show that, even when b is an extreme point, an inner function u can exist such that not every multiplier of $H(b)$ is one of $H(ub)$. (Such an example with b not an extreme point can be extracted from [18].)

The concluding Section 15 contains a short list of open questions.

Besides the notations already introduced, the following additional ones are needed.

L^2 denotes the L^2 space of normalized Lebesgue measure on $\partial \mathrm{D}$ and P_+ denotes the orthogonal projection in L^2 with range H^2. The norm and inner product in L^2 are denoted by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$.

The norm and inner product in $H(b)$ are denoted by $\|\cdot\|_b$ and $\langle\cdot,\cdot\rangle_b$, and those in $H(\bar{b})$ by $\|\cdot\|_{\bar{b}}$ and $\langle\cdot,\cdot\rangle_{\bar{b}}$.

The kernel function in H^2 for the point w of D is denoted by

$$k_w \; k_w(z) = (1 - \bar{w}z)^{-1}.$$

The kernel functions in $H(b)$ and $H(\bar{b})$ for w are denoted by k_w^b, as a simple argument shows. For k_w^b one has the expression $k_w^b = (1 - \overline{b(w)}b)k_w$ [4], [19].

The term «operator» will always mean «bounded operator».

The following two simple properties of $H(b)$, the first of which was mentioned earlier, can be found in [19].

1. S^*b belongs to $H(b)$.
2. b belongs to $H(b)$ if and only if b is not an extreme point of $B(H^\infty)$.

Alternative proofs of some of the results below have recently been found by A. V. Lipin (private communication).

2. Relations between $H(b)$ and $H(\bar{b})$

It is helpful to fit the spaces $H(b)$ and $H(\bar{b})$ into the general scheme promulgated by de Branges (for example, in [2]). If H and H_1 are Hilbert spaces and A is an operator in $L(H_1, H)$, then de Branges's space $M(A)$ consists of the range of A, with the Hilbert space structure that makes A into a coisometry of H_1 onto $M(A)$. Thus, for example, if y is in H_1 and is orthogonal to the kernel of A, then $\|Ay\|_{M(A)} = \|y\|_{H_1}$. If $\|A\| \leq 1$, then the space $M((1 - AA^*)^{1/2})$ is called by de Branges the complementary space of $M(A)$ and denoted by $H(A)$. Our spaces $H(b)$ and $H(\bar{b})$, therefore, coincide with $H(T_b)$ and $H(T_{\bar{b}})$, respectively. We shall denote $M(T_b)$ by $M(b)$.

A factorization criterion of R. G. Douglas [5] is often useful in establishing containment relations between de Branges's spaces, and in showing a given operator maps one of these spaces into another one.

Douglas's criterion. *Let H, H_1, and H_2 be Hilbert spaces and A and B operators in $L(H_1, H)$ and $L(H_2, H)$, respectively. Then the operator inequality $BB^* \leq AA^*$ is necessary and sufficient for the existence of a factorization $B = AR$ with R in $L(H_2, H_1)$ and $\|R\| \leq 1$.*

This tells us, for example, that the two spaces $M(A)$ and $M(B)$ coincide as Hilbert spaces if and only if $AA^* = BB^*$. In virtue of the operator inequality $1 - T_{\bar{b}}T_b \leq 1 - T_bT_{\bar{b}}$, it tells us also that $H(\bar{b})$ is contained in $H(b)$, with the inclusion map a contraction.

If A is a contraction in $L(H_1, H)$, then $M(A)$ is an ordinary subspace of H if and only if A is a partial isometry, in which case $H(A)$ is the ordinary orthogonal complement of $M(A)$. In the contrary case the intersection $M(A) \cap H(A)$, which we call an overlapping space, is nontrivial (de Branges and Rovnyak [3] use the term «overlapping space» in a slightly different way). A simple lemma establishes the relation between $H(A)$, $H(A^*)$, and their overlapping spaces.

Lemma 2.1. *Let H and H_1 be Hilbert spaces and A a contraction in $L(H_1, H)$. The vector x in H belongs to $H(A)$ if and only if A^*x belongs to $H(A^*)$, in which case.*

$$\|x\|^2_{H(A)} = \|x\|^2_H + \|A^*x\|^2_{H(A^*)}.$$

The overlapping space $M(A^*) \cap H(A^*)$ coincides with $A^*H(A)$.

The inclusion $A^*H(A) \subset H(A^*)$ follows from the operation identity

$$A^* (1 - AA^*)^{1/2} = (1 - A^*A)^{1/2} A^*,$$

which goes back at least to a paper of P. R. Halmos [10]. Suppose x is a vector in H such that A^*x is in $H(A^*)$, say $A^*x = (1 - A^*A)^{1/2}y$ with y in H_1 and orthogonal to the kernel of $(1 - A^*A)^{1/2}$. Then $AA^*x = (1 - AA^*)^{1/2} Ay$ (by the same identity used above), from which one concludes that

$$x = (1 - AA^*)^{1/2}[(1 - AA^*)^{1/2}x + Ay],$$

showing that x is in $H(A)$. As Ay is easily seen to be orthogonal to the kernel of $(1 - AA^*)^{1/2}$, we have

$$\|x\|_{H(A)} = \|(1 - AA^*)^{1/2}x + Ay\|_H.$$

The square of the right side equals

$$\begin{aligned}
&\langle (1 - AA^*)\, x,x\rangle_H + \|Ay\|^2_H + 2Re\langle (1 - A^*A)^{1/2}\, A^*\, x,\, y\rangle_{H_1} \\
&= \|x\|^2_H - \|A^*x\|^2_{H_1} + \|Ay\|^2_H + 2\langle (1 - A^*A)y,y\rangle_{H_1} \\
&= \|x\|^2_H - \|A^*x\|^2_{H_1} + \|y\|^2_{H_1} + \|(1 - A^*A)^{1/2}y\|^2_{H_1} \\
&= \|x\|^2_H + \|y\|^2_{H_1} \\
&= \|x\|^2_H + \|A^*x\|^2_{H(A^*)},
\end{aligned}$$

which gives the desired expression for $\|x\|_{H(A)}$. This completes the proof of the lemma.

For the situation of interest in this paper, we obtain the following immediate consequences of Lemma 2.1.

Lemma 2.2. *The H^2 function h belongs to $H(b)$ if and only if $T_{\bar{b}}h$ is in $H(\bar{b})$. If h is in $H(b)$, then*

$$\|h\|^2_b = \|h\|^2_2 + \|T_{\bar{b}}h\|^2_{\bar{b}}.$$

Lemma 2.3. *The overlapping space $M(b) \cap H(b)$ equals $T_{\bar{b}}H(\bar{b})$. The operator T_b acts as a contraction from $H(\bar{b})$ to $H(b)$.*

Two corollaries of the last lemma are worth recording.

Corollary 2.4. *Every multiplier of $H(b)$ is a multiplier of $H(\bar{b})$.*

Corollary 2.5. *The overlapping space $M(b) \cap H(b)$ is dense in $H(b)$ if and only if b is an outer function.*

Corollary 2.4 is immediate. To establish Corollary 2.5 it suffices to note that $T_b H(\bar{b})$ is the range of the operator $(1 - T_b T_{\bar{b}})^{1/2} T_b$, which is dense in $H(b)$ if and only if the range of T_b is dense in H^2, in other words, if and only if b is an outer function.

Lemma 2.6. *If ϕ is a function in H^∞ then the spaces $H(b)$ and $H(\bar{b})$ are invariant under the Toeplitz operator $T_{\bar{\phi}}$, whose norm as an operator in each of them does not exceed $\|\phi\|_\infty$.*

For the proof, we can assume with no loss of generality that $\|\phi\|_\infty = 1$. To settle the case of $H(\bar{b})$ it will be enough, by Douglas's criterion, to verify the operator inequality

$$T_{\bar{\phi}}(1 - T_{\bar{b}} T_b) T_\phi \leq 1 - T_{\bar{b}} T_b.$$

One easily sees that the difference between the right and left sides equals

$$1 - T_{|b|^2} - T_{|\phi|^2} + T_{|\phi b|^2},$$

which is the Toeplitz operator with symbol $(1 - |\phi|^2)(1 - |b|^2)$, hence positive semidefinite, as desired. The case of $H(b)$ follows immediately from the case of $H(\bar{b})$ in conjunction with Lemma 2.2.

3. Cauchy integrals

For μ a finite complex Borel measure on ∂D, we let $K\mu$ denote its Cauchy integral, that is, the holomorphic function in $C \backslash \partial D$ defined by

$$(K\mu)(z) = \int_{\partial D} \frac{1}{1 - e^{-i\theta} z} \, d\mu(e^{i\theta}).$$

If μ is absolutely continuous and σ is its Radon-Nikodym derivative with respect to normalized Lebesgue measure, we write $K\sigma$ in place of $K\mu$. (What we are calling Cauchy integrals are often referred to as integrals of Cauchy-Stieltjes type.)

If the measure μ is positive, we define the transformation K_μ on $L^2(\mu)$ by $K_\mu q = K(q\mu)$. The function $K_\mu q$ vanishes identically in D if and only if q is orthogonal to $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$. We denote by

$K^2(\mu)$ the space of all functions $K_\mu q$ with q in $L^2(\mu)$ and give it the Hilbert space structure that makes K_μ an isometry of $H^2(\mu)$ onto it. As before, if μ is absolutely continuous with Radon-Nikodym derivative σ, we write K_σ and $K^2(\sigma)$ in place of K_μ and $K^2(\mu)$.

We let μ_b denote the measure on $\partial\mathrm{D}$ whose Poisson integral is the real part of the function $(1 + b)/(1 - b)$. For q in $L^2(\mu_b)$, we define the function $V_b q$ in D by

$$(V_b q)(z) = (1 - b(z))(K_{\mu_b} q)(z).$$

A proof of the following representation for $H(b)$ can be found in [20].

Lemma 3.1. *The transformation V_b is an isometry of $H^2(\mu_b)$ onto $H(b)$. It maps the function k_w ($|w| < 1$) to the function $(1 - \overline{b(w)})^{-1}k_w^b$.*

Thus, the problem of finding the multipliers of $H(b)$ is the same as the problem of finding the multipliers of $K^2(\mu_b)|\mathrm{D}$. We note for future reference that the equality $H^2(\mu_b) = L^2(\mu_b)$ holds if and only if the Radon-Nikodym derivative with respect to Lebesgue measure of the absolutely continuous component of μ_b fails to be log-integrable [14, p. 50]. That Radon-Nikodym derivative equals $(1 - |b|^2)/|1 - b|^2$ and so is not log-integrable if and only if b is an extreme point of $B(H^\infty)$.

The operator on $H^2(\mu_b)$ of multiplication by $e^{i\theta}$ will be denoted by Z_b.

Lemma 3.2. *The transformation $V_b|H^2(\mu_b)$ intertwines the operators $Z_b^*\,[1 - (1 - b(0))(1 \otimes 1)]$ and S^*.*

In the proof, we shall denote the inner product in $L^2(\mu_b)$ by $\langle\cdot,\cdot\rangle_{\mu_b}$. Let q be any function in $L^2(\mu_b)$, and let $g = K_{\mu_b} q$, so that $V_b q = (1 - b)g$. Since $(K_{\mu_b} Z_b^* q)(z) = \langle Z_b^* q, k_z\rangle_{\mu_b} = \langle q, Z_b k_z\rangle_{\mu_b}$, we have, for $z \neq 0$,

$$\begin{aligned}(K_{\mu_b} Z_b^* q)(z) &= \int_{\partial\mathrm{D}} \frac{e^{-i\theta} q(e^{i\theta})}{1 - ze^{-i\theta}}\, d\mu_b(e^{i\theta}) \\ &= \frac{1}{z}\int_{\partial\mathrm{D}} \left(\frac{1}{1 - ze^{-i\theta}} - 1\right) q(e^{i\theta})d\mu_b(e^{i\theta}) \\ &= \frac{g(z) - g(0)}{z}.\end{aligned}$$

Therefore,

$$(V_b Z_b^* q)(z) = (1 - b(z))\,\frac{g(z) - g(0)}{z}$$

$$= \frac{(1 - b(z))g(z) - (1 - b(0))g(0)}{z} + g(0)\frac{b(z) - b(0)}{z}$$

$$= \frac{(V_bq)(z) - (V_bq)(0)}{Z} + \frac{(V_bq)(0)}{1 - b(0)}(S^*b)(z)$$

$$= (S^*V_bq)(z) + \langle V_bq, V_b1\rangle_{\mu_b}(S^*b)(z).$$

In the last line we have used the equality $V_b1 = (1 - \overline{b(0)})^{-1}k_0^b$ from Lemma 3.1. The same equality shows that, when q is the constant function 1, the function g equals $(1 - \overline{b(0)}b)/(1 - \overline{b(0)})(1 - b)$. Inserting these expressions into the equality above (the one that gives $K_{\mu_b}Z_b^*q$ in terms of g) one obtains, after a few lines of calculation, the formula

$$(K_{\mu_b}Z_b^*1)(z) = \frac{1}{(1 - b(0))(1 - b(z))}\left(\frac{b(z) - b(0)}{z}\right),$$

implying that

$$V_bZ_b^*1 = (1 - b(0))^{-1}S^*b.$$

The expression for $V_bZ_b^*q$ can thus be rewritten as

$$V_bZ_b^*q = S^*V_bq + (1 - b(0))\langle V_bq, V_b1\rangle_b V_bZ_b^*1,$$

or as

$$S^*V_bq = V_bZ_b^*[q - (1 - b(0))\langle q, 1\rangle_b 1],$$

which is the desired conclusion.

For $H(\bar{b})$, the situation is simpler than for $H(b)$. We let ϱ denote the function $1 - |b|^2$ on $\partial\mathrm{D}$.

Lemma 3.3. *The transformation* K_ϱ is an isometry of $\mathrm{H}^2(\varrho)$ onto $H(\bar{b})$. *It* maps k_w *to* $k_w^{\bar{b}}$. *Hence* $H(\bar{b}) = K^2(\varrho)|\mathrm{D}$.

We denote the inner product in $L^2(\varrho)$ by $\langle\cdot,\cdot\rangle_\varrho$. For any points z and w of D we have

$$\begin{aligned}\langle k_w, k_z\rangle_\varrho &= \langle(1 - |b|^2)k_w, k_z\rangle\\ &= \langle(1 - T_{\bar{b}}T_b)k_w, k_z\rangle\\ &= \langle k_w^{\bar{b}}, k_z\rangle\\ &= k_w^{\bar{b}}(z)\\ &= \langle k_w^{\bar{b}}, k_z^{\bar{b}}\rangle_b.\end{aligned}$$

But $(K_\varrho k_w)(z) = \langle k_w, k_z \rangle_\varrho$, so it follows that $K_\varrho k_w = k_w^{\bar{b}}$ and that $\langle K_\varrho k_w, K_\varrho k_z \rangle_{\bar{b}} = \langle k_w, k_z \rangle_\varrho$. Thus K_ϱ maps the linear manifold in $L^2(\varrho)$ spanned by the functions k_w isometrically onto a dense linear manifold in $H(b)$ (the one spanned by the functions $k_w^{\bar{b}}$). One can now complete the proof by a standard limit argument.

The operator on $H^2(\varrho)$ of multiplication by $e^{i\theta}$ will be denoted by Z_ϱ.

Lemma 3.4. *The transformation $K_\varrho | H^2(\varrho)$ intertwines the operators Z_ϱ^* and S^*.*

This is a standard property of Cauchy integrals. It is established, except for a difference in notation, as the first step in the proof of Lemma 3.2.

Corollary 3.5. *If ϕ is a function in H^∞, then the transformation $K_\varrho | H^2(\varrho)$ intertwines the operators $\phi(Z_\varrho)^*$ and $T_{\bar{\phi}}$.*

In fact, the case where ϕ is a polynomial follows immediately from Lemma 3.4. To handle the general case one takes a sequence of polynomials that is uniformly bounded on ∂D and converges almost everywhere on ∂D to ϕ. The obvious limit argument yields the conclusion.

Our first theorem implies, in virtue of Corollary 2.4, that any multiplier of $H(b)$ differs by a constant from a function in $H(\bar{b})$. (If b is not an extreme point of $B(H^\infty)$ then $H(\bar{b})$ contains the constants, so one gets the stronger conclusion that the multipliers of $H(b)$ lie in $H(\bar{b})$. The theorem itself is trivial in that case.)

Theorem 3.6. *If b is not an inner function, then every multiplier of $H(\bar{b})$ differs by a constant from a function in $H(\bar{b})$.*

As noted above, the theorem is trivial if b is not an extreme point of $B(H^\infty)$, so we assume it is an extreme point. Let m be a multiplier of $H(\bar{b})$ and let h be any function in $H(\bar{b})$ such that $h(0) \neq 0$. By Lemma 2.6, the functions mS^*h and $S^*(mh)$ belong to $H(\bar{b})$. Since $S^*(mh) = mS^*h + h(0)S^*m$, it follows that S^*m is in $H(\bar{b})$. Because b is an extreme point, the function ϱ is not log-integrable, which implies that $H^2(\varrho) = L^2(\varrho)$ [14, p. 50]. Therefore, by Lemma 3.4, the operator $S^*|H(\bar{b})$ is unitary, so in particular $S^*H(\bar{b}) = H(\bar{b})$. The function in $H(\bar{b})$ sent to S^*m by S^* differs from m by a constant.

4. Vinogradov's theorem

If μ is a finite complex Borel measure on ∂D then its Cauchy integral, $K\mu$, as a function in D, belongs to H^p for $0 < p < 1$ and so has an inner-outer factorization [7, p. 39]. The theorem of Vinogradov [24] states that if the inner function u divides the inner factor of $K\mu$, then the quotient $K\mu/u$ is a Cauchy integral; in fact, it is the Cauchy integral of a measure whose norm does not exceed that of μ. A simple and natural proof of this can be based on Lemmas 2.6 and 3.1.

For simplicity we assume $\|\mu\| = 1$, and we choose b so that $|\mu| = \mu_b$. By Lemma 3.1 the function $(1 - b)K\mu$ is in $H(b)$ and has norm at most 1. Thus, by Lemma 2.6, if u is an inner function, then

$$T_{\bar{u}}[(1 - b)K\mu] = (1 - b)K(q\mu_b),$$

where q is in $H^2(\mu_b)$ and has norm at most 1. But if u divides the inner factor of $K\mu$ then it divides the inner factor of $(1 - b)K\mu$, so that

$$T_{u}[(1 - b)K\mu] = (1 - b)K\mu/u.$$

In that case $K\mu/u = K(q\mu_b)$, which proves Vinogradov's theorem since the measure $q\mu_b$ has norm at most 1.

5. Nonmultipliers

Our concern from now on will be with the case where b is an extreme point of $B(H^\infty)$. In this section we obtain two negative results about multipliers.

It was mentioned in Section 1 that, if b is not an extreme point of $B(H^\infty)$, then every function holomorphic in a neighborhood of $\bar{\mathrm{D}}$ is a multiplier of $H(b)$. If b is an extreme point, exactly the opposite is true: no nonconstant multiplier can be continued analytically across all of $\partial\mathrm{D}$. This is a consequence of the next theorem together with Theorem 3.6 and Corollary 2.4.

Theorem 5.1. *If b is an extreme point of $B(H^\infty)$, then no nonzero function in $H(\bar{b})$ can be continued analytically across all of* $\partial\mathrm{D}$.

In fact, suppose the function h in $H(\bar{b})$ can be continued analytically across all of $\partial\mathrm{D}$. By Lemma 3.3 we can write $h = K(q\varrho)$ with q in $L^2(\varrho)$. The function $q\varrho$ is in L^2, being the product of the L^2 function $q\varrho^{1/2}$ and the bounded function $\varrho^{1/2}$. This enables us to write $h = P_+(q\varrho)$. (Recall that P_+ is the orthogonal projection in L^2 with range H^2.) Because b is an extreme point of $B(H^\infty)$, the function ϱ is not log-integrable, and therefore neither is $q\varrho$, because

$$\log|q\varrho| \leq \log^+|q\varrho^{1/2}| + \frac{1}{2}\log\varrho.$$

But the forward Fourier coefficients of $q\varrho$ coincide with the Taylor coefficients of h, which tend to zero exponentially since h is holomorphic across $\partial\mathsf{D}$. It is known [16, p. 12] that a function on $\partial\mathsf{D}$ whose forward Fourier coefficients tend to 0 exponentially is log-integrable unless it vanishes identically. Hence $q\varrho = 0$, which means $h = 0$, as desired.

Corollary 5.2. *If b is an extreme point of $B(H^\infty)$ and an outer function, then no nonzero function in $H(b)$ can be continued analytically across all of* $\partial\mathsf{D}$.

In fact, suppose the function h in $H(b)$ can be continued analytically across all of $\partial\mathsf{D}$. Its Taylor coefficients then tend to 0 exponentially, and a simple estimate shows that then the forward Fourier coefficients of $\bar{b}h$ exhibit the same behavior. Hence $T_{\bar{b}}h$ can be continued analytically across all of $\partial\bar{\mathsf{D}}$. By Lemma 2.2, $T_{\bar{b}}h$ is in $H(\bar{b})$, so it is 0 by Theorem 5.1. Since b is outer it follows that $h = 0$, as desired.

The noncyclic vectors of the backward shift operator, S^*, have been characterized by R. G. Douglas, H. S. Shapiro, and A. L. Shields [6] as the functions in H^2 that possess pseudocontinuations to the complement of $\bar{\mathsf{D}}$. Our next theorem implies that, if b is an extreme point of $B(H^\infty)$ then, just as is the case with an ordinary continuation, the possession of a pseudocontinuation disqualifies a nonconstant function from being a multiplier of $H(b)$. (Davis and McCarthy [1] have obtained this independently.) In particular, if b is an extreme point, then no nonconstant inner function is a multiplier of $H(b)$, a result from [15].

Theorem 5.3. *If b is an extreme point of $B(H^\infty)$, then the nonzero functions in $H(\bar{b})$ are cyclic vectors of S^*.*

This theorem is nearly disjoint from Theorem 5.1: the only functions in H^2 that possess both ordinary continuations across $\partial\mathsf{D}$ and pseudocontinuations to the complement of $\bar{\mathsf{D}}$ are the rational functions [6].

To prove the theorem, let h be a nonzero function in $H(\bar{b})$. As in the proof of Theorem 5.1, we have $h = P_+(q\varrho)$ where q is a function in $L^2(\varrho)$. Also as in the proof of Theorem 5.1, the function $q\varrho$ is not log-integrable.

Let M be the invariant subspace of S^* generated by h and let $N = M + (H^2)^\perp$. Then N is an invariant subspace of the adjoint of the bilateral shift operator on L^2. By the known structure of these subspaces [14, p. 111], either $N = \chi_E L^2$ with E a measurable subset of $\partial\mathsf{D}$ or $N = v\overline{H^2}$ with v a unimodular function in L^∞. The latter possibility is precluded because N contains the function $q\varrho$, which fails to be log-integrable (and is not the zero function). Thus N is of the form $\chi_E L^2$, and since it contains the function h, which is nonzero almost everywhere, it must actually be all of L^2. That means $M = H^2$, so h is a cyclic vector of S^*, as desired.

Corollary 5.4. *If b is an extreme point of $B(H^\infty)$ and an outer function, then the nonzero functions in $H(b)$ are cyclic vectors of S^*.*

In fact, suppose h is a nonzero function in $H(b)$. Then $T_{\bar{b}}h$ is in $H(\bar{b})$ by Lemma 2.2 and is nonzero because b is outer. By Theorem 5.3, then, $T_{\bar{b}}h$ is a cyclic vector of S^*. But $T_{\bar{b}}h$ lies in the S^*-invariant subspace generated by h, so h also is a cyclic vector of S^*.

6. Decompositions of $H(b)$

Let u_0 be the inner part and b_0 the outer part of the function b. Then, as de Branges and Rovnyak [4, p. 32] first pointed out, the space $H(b)$ is the orthogonal direct sum of the two subspaces $H(u_0)$ and $u_0H(b_0)$. Moreover, the inclusion map of $H(u_0)$ into $H(b)$ is an isometry, and T_{u_0} acts as an isometry of $H(b_0)$ into $H(b)$. To verify these statements, it suffices to rewrite the equality

$$1 - T_bT_{\bar{b}} = 1 - T_{u_0}T_{\bar{u}_0} + T_{u_0}(1 - T_{b_0}T_{\bar{b}_0})T_{\bar{u}_0}$$

as $1 - T_bT_{\bar{b}} = AA^*$, where $A = (A_1\ A_2)$, an operator from $H^2 \oplus H^2$ to H^2, with

$$A_1 = (1 - T_{u_0}T_{\bar{u}_0})^{1/2} \quad \text{and} \quad A_2 = T_{u_0}(1 - T_{b_0}T_{\bar{b}_0})^{1/2}.$$

The equality tells us that $H(b) = M(A_1) + M(A_2)$, and this is an orthogonal direct sum, with the inclusion map of each summand into $H(b)$ isometric, because $\ker A = \ker A_1 \oplus \{0\}$. Since $M(A_1) = H(u_0)$ and $M(A_2) = u_0H(b_0)$, the decomposition of $H(b)$ follows. One immediate consequence of the decomposition is that every multiplier of $H(b)$ is a multiplier of $H(b_0)$. More generally, the same reasoning shows that if u is any inner function, then $H(ub)$ is the orthogonal direct sum of $H(u)$ and $uH(b)$. Thus, every multiplier of $H(ub)$ is a multiplier of $H(b)$.

When b is an extreme point of $B(H^\infty)$, there is a companion orthogonal decomposition of $H(b)$.

Theorem 6.1. *Let b be an extreme point of $B(H^\infty)$. Then $H(b)$ is the orthogonal direct sum of $H(b_0)$ and $b_0H(u_0)$. The inclusion map of $H(b_0)$ into $H(b)$ is an isometry, and the operator T_{b_0} acts as an isometry from $H(u_0)$ into $H(b)$.*

The situation when b is not an extreme point is completely different. In that case, $H(b_0)$ is dense in $H(b)$ [18, p. 87].

Theorem 6.1 can be established by a slight modification of the argument in the discussion preceding it. We suppose that b is not inner, since otherwise the theorem reduces to a triviality. This time we use the factorization $1 - T_b T_{\bar{b}} = AA^*$, where $A = (A_1\ A_2)$, but with

$$A_1 = (1 - T_{b_0} T_{\bar{b}_0})^{1/2} \quad \text{and} \quad A_2 = T_{b_0}(1 - T_{u_0} T_{\bar{u}_0})^{1/2}.$$

We assert that $\ker A = \{0\} \oplus \ker A_2$. Clearly, once this has been verified, the previous reasoning applies. To establish the assertion, let $f_1 \oplus f_2$ belong to $\ker A$, and write $g = (1 - T_{b_0} T_{\bar{b}_0})^{1/2} f_1$ and $h = (1 - T_{u_0} T_{\bar{u}_0}) f_2$. Then g is in $H(b_0)$, while h is in $H(u_0)$, and $g = -b_0 h$, implying by Lemma 2.3 that h is in $H(\bar{b}_0)$ (which is the same as $H(\bar{b})$). Since b is an extreme point, Theorem 5.3 implies that $h = 0$, and hence also that $g = 0$. It follows that f_2 is in $\ker A_2$ and $f_1 = 0$, the latter because $\ker (1 - T_{b_0} T_{\bar{b}_0})$ is trivial, b_0 being a nonconstant outer function. This concludes the proof of the theorem.

The next theorem clarifies the relation of $H(b)$ and $H(\bar{b})$ in the extreme point case.

Theorem 6.2. *Let b be an extreme point of $B(H^\infty)$. Then the othogonal complement of $H(\bar{b})$ in $H(b)$ is $b_0 H(u_0)$. The closure of $H(\bar{b})$ in $H(b)$ is $H(b_0)$.*

Again, the situation is completely different when b is not an extreme point. In that case $H(\bar{b})$ is always dense in $H(b)$ [18, p. 87].

The second assertion in Theorem 6.2 follows immediately from the first assertion together with Theorem 6.1. It will thus be enough to establish the first assertion. Some new notations are needed.

As in Section 3, we let ϱ denote the function $1 - |b|^2$ on ∂D and $\langle \cdot, \cdot \rangle_\varrho$ the inner product in $L^2(\varrho)$. Let J_ϱ denote the natural injection of H^2 into $L^2(\varrho)$. One easily verifies that K_ϱ is the adjoint of J_ϱ and that $K_\varrho J_\varrho = 1 - T_{\bar{b}} T_b$. If h is a function in $H(b)$ then $T_b h$ belongs to $H(\bar{b})$, by Lemma 2.2 and so is the image under K_ϱ of a function in $L^2(\varrho)$, by Lemma 3.3. The latter function is unique (also by Lemma 3.3, since $H^2(\varrho) = L^2(\varrho)$); we denote it by $W_\varrho h$.

That $b_0 H(u_0)$ is contained in the orthogonal complement of $H(\bar{b})$ in $H(b)$ is an immediate consequence of Theorem 6.1. To establish the opposite containment, let h be any function in $H(b)$ that is orthogonal to $H(\bar{b})$. Let g be any function in $H(\bar{b})$ and, using Lemma 3.3, write $g = K_\varrho q$ with q in $L^2(\varrho)$. Corollary 3.5 tells us that $T_{\bar{b}} g = K_\varrho(\bar{b} q)$, showing that $W_\varrho g = \bar{b} q$. Thus, by Lemma 2.2,

$$\begin{aligned} 0 &= \langle h, g \rangle_b \\ &= \langle h, g \rangle + \langle T_{\bar{b}} h, T_{\bar{b}} g \rangle_{\bar{b}} \end{aligned}$$

$$= \langle h, g\rangle + \langle W_\varrho h, W_\varrho g\rangle_\varrho$$
$$= \langle h, K_\varrho q\rangle + \langle W_\varrho h, \bar{b}q\rangle_\varrho$$
$$= \langle J_\varrho h + bW_\varrho h, q\rangle_\varrho.$$

This equality holds for all q in $L^2(\varrho)$, so $J_\varrho h + bW_\varrho h$ is the zero function in $L^2(\varrho)$, in other words, $W_\varrho h = -h/b$ on the set where $1 - |b|^2$ is nonzero. Multiplying the last equality by $1 - |b|^2$, we conclude that

$$(1 - |b|^2)W_\varrho h = -\frac{h}{b} + \bar{b}h \ ;$$

in particular, the function h/b belongs to L^2. Projecting both sides of the preceding equality onto H^2, we obtain

$$K_\varrho W_\varrho h = P_+(-h/b) + T_{\bar{b}}h.$$

But $T_{\bar{b}}h = K_\varrho W_\varrho h$ by the definition of W_ϱ, so the function h/b is orthogonal to H^2. However, the function h/b_0 is in H^2 since it is in L^2 and b_0 is an outer function. Since $h/b = \bar{u}_0 h/b_0$, we conclude that h/b_0 is in $H(u_0)$, which means that h is in $b_0H(u_0)$, as desired.

7. Consequences of invertibility

Theorem 7.1. *If b is an extreme point of $B(H^\infty)$, then the following conditions are equivalent*

(i) *b is invertible in H^∞,*
(ii) *$H(b) = H(\bar{b})$,*
(iii) *b is a multiplier of $H(b)$,*
(iv) *$S^*|H(b)$ is similar to a unitary operator.*

An analogous result for the case where b is not an extreme point of $B(\mathrm{H}^\infty)$ can be found in [18]. In condition (ii), by the equality $H(b) = H(\bar{b})$ we mean to say that the two spaces are equal as vector spaces but not that their Hilbert space structures coincide. If they are equal as vector spaces then their norms are equivalent, by the closed graph theorem.

The equivalence of conditions (ii) and (iii) in the theorem is an immediate consequence of the equality $M(b) \cap H(b) = bH(\bar{b})$ from Lemma 2.3. To see that (i) implies (ii), assume b is invertible and write $H(b) = T_{\bar{b}^{-1}}T_{\bar{b}}H(b)$. By Lemma 2.2, $T_{\bar{b}}H(b) \subset H(\bar{b})$, and, by Lemma 2.6, $T_{\bar{b}^{-1}}H(\bar{b}) \subset H(\bar{b})$, so it follows that $H(b) = H(\bar{b})$, as desired. This much does not involve the hypothesis that b is an extreme point.

We complete the proof by showing that (ii) implies (iv) and (iv) implies (i). Actually, the first of these implications follows immediately from Lemma 3.4, which says that the operator $S^*|H(\bar{b})$ is unitarily equivalent to the operator Z_ϱ^*; the last operator is unitary when b is an extreme point (since then $H^2(\varrho) = L^2(\varrho)$). It only remains to prove that (iv) implies (i).

Assume that b is not invertible in H^∞. We shall show that then $S^*|H(b)$ is not similar to a unitary operator. The noninvertibility of b implies the noninvertibility of $T_{\bar{b}}$. Hence, given $\epsilon > 0$, there is an f in H^2 with $\|f\|_2 = 1$ and $\|T_{\bar{b}}f\|_2 < \epsilon$. Let $h = (1 - T_bT_{\bar{b}})^{1/2}f$. Then h is in $H(b)$ with $\|h\|_b \leq 1$, and

$$\begin{aligned}\|h\|_b^2 \geq \|h\|_2^2 &= \langle (1 - T_bT_b)f,f\rangle \\ &= \|f\|_2^2 - \|T_bf\|_2^2 \\ &\geq 1 - \epsilon^2.\end{aligned}$$

One consequence of the assumption that b is an extreme point is (in the terminology of de Branges and Rovnyak) the identity for difference quotients:

$$\|S^*g\|_b^2 = \|g\|_b^2 - |g(0)|^2 \qquad (g \in H(b))$$

[19, p. 162]. From this it follows that

$$\lim_{n\to\infty} \|S^{*n}h\|_b^2 = \|h\|_b^2 - \|h\|_2^2 \leq 1 - (1 - \epsilon^2) = \epsilon^2.$$

As ϵ is arbitrary, the desired conclusion, that $S^*|H(b)$ is not similar to a unitary operator, follows, and the proof of Theorem 7.1 is complete.

Later, in Section 11, we shall see that the condition that b be a multiplier of $H(\bar{b})$ is equivalent to the conditions of Theorem 7.1.

Corollary 7.2. *If b is an extreme point of $B(H^\infty)$ and is invertible in H^∞ then b^{-1} is a multiplier of H(b).*

In fact, if b is invertible, then Lemma 2.3 and Theorem 7.1 combine to give $bH(\bar{b}) = H(b)$. (The same result holds, and the same reasoning applies, when b is not an extreme point. The corollary uses only the implication (i) implies (ii) from Theorem 7.1, which, as noted in the proof, holds for nonextreme points as well.)

Corollary 7.3. *If b is not an inner function then H(b) has nonconstant multipliers.*

We need only to treat the case where b is an extreme point. Under the assumption that b is not an inner function, there is a factorization $b = b_1b_2$, where b_1 and b_2 are in $B(H^\infty)$ and b_1 is nonconstant and invertible in H^∞. (For example, one can take b_1 to be the outer function whose modulus on $\partial\mathrm{D}$ is the maximum of $|b|$ and $1/2$). Using the reasoning at the beginning of Section 6 we obtain the decompositions

$$H(b) = H(b_1) + b_1 H(b_2) = H(b_2) + b_2 H(b_1).$$

Thus $b_1 H(b_2) \subset H(b)$ and $b_2 H(b_1) \subset H(b)$, and

$$b_1 H(b) = b_1 H(b_2) + b_2 b_1 H(b_1).$$

By Theorem 7.1 $b_1 H(b_1) \subset H(b_1)$, and hence b_1 is a multiplier of $H(b)$.

As in the last section, we let u_0 denote the inner part and b_0 the outer part of b.

Theorem 7.4. *If b is an extreme point of $B(H^\infty)$ and b_0 is invertible in H^∞, then b_0 and $1/b_0$ are multipliers of $H(b)$ and one has the decompositions*

$$H(b) = H(u_0) + H(b_0) = b_0 H(u_0) + u_0 H(b_0).$$

The proof depends on the decompositions

$$H(b) = H(u_0) + u_0 H(b_0) = H(b_0) + b_0 H(u_0)$$

from Section 6. If b_0 is invertible then, as seen above, we have $b_0 H(b_0) = H(b_0)$, so that

$$\begin{aligned} b_0 H(b) &= b_0 H(u_0) + u_0 b_0 H(b_0) \\ &= b_0 H(u_0) + u_0 H(b_0) \\ &\subset H(b), \end{aligned}$$

and

$$\begin{aligned} b_0^{-1} H(b) &= b_0^{-1} H(b_0) + H(u_0) \\ &= H(b_0) + H(u_0) \\ &\subset H(b), \end{aligned}$$

Thus b_0 and $1/b_0$ are multipliers of $H(b)$, so the preceding inclusions must actually be equalities, and the desired decompositions of $H(b)$ follow.

8. Multiplication operators

For m a multiplier of $H(b)$, we let M_m denote the corresponding multiplication operator on $H(b)$. For w in D, the kernel function k_w^b is an eigenvector of M_m^*

with eigenvalue $\overline{m(w)}$ (since it is orthogonal to the range of $M_m - m(w)$). Conversely, if M is an operator on $H(b)$ such that each kernel function k_w^b is an eigenvector of M_m^*, then M is a multiplication operator. This well-known property of reproducing kernel Hilbert spaces can be found in [23].

It will be convenient to denote the operator $S^*|H(b)$ by X; it is a contraction by Lemma 2.6. The adjoint X^* is given by

$$X^*h = Sh - \langle h,S^*b\rangle_b b$$

[4], [19]. If b is an extreme point of $B(H^\infty)$, then b is not in $H(b)$, and one can draw the following conclusion.

Lemma 8.1. *If b is an extreme point of $B(H^\infty)$ and h is in $H(b)$, then Sh is in $H(b)$ if and only if $\langle h,S^*b\rangle_b = 0$.*

When b is an extreme point and m is a multiplier of $H(b)$, the operator M_m^* has unexpected eigenvectors.

Theorem 8.2. *Let b be an extreme point of $B(H^\infty)$ and m a multiplier of $H(b)$. Then S^*b is an eigenvector of M_m^*. If $\bar{\alpha}$ is the corresponding eigenvalue, then $(m - \alpha)b$ belongs to $H(b)$ and $m - \alpha$ belongs to $H(\bar{b})$, and the commutation relation*

$$M_m^*X - XM_m^* = S^*b \otimes (m - \alpha)b$$

holds.

In fact, Lemma 8.1 implies that the orthogonal complement of S^*b in $H(b)$ is in variant under M_m, so that S^*b is an eigenvector of M_m^*. To obtain the commutation relation, consider a point w in D and the corresponding kernel function k_w^b. From the expresion $k_w^b = (1 - \overline{b(w)}b)k_w$ one easily obtains the equality $Xk_w^b = wk_w^b - \overline{b(w)}S^*b$. Thus

$$\begin{aligned}(M_m^*X - XM_m^*)k_w^b &= \bar{w}\,\overline{m(w)}k_w^b - \bar{\alpha}\overline{b(w)}S^*b - \overline{m(w)}(wk_w^b - \overline{b(w)}S^*b)\\ &= (\overline{m(w)} - \alpha)\overline{b(w)}S^*b.\end{aligned}$$

As the functions k_w^b span $H(b)$ it follows that $M_m^*X - XM_m^*$ is an operator of rank 1 with range spanned by S^*b:

$$M_m^*X - XM_m^* = S^*b \otimes \phi,$$

where ϕ is some function in $H(b)$. But by the preceding equality,

$$\overline{\phi(w)} = \langle k_w^b,\phi\rangle_b = (\overline{m(w)} - \alpha)\overline{b(w)},$$

in other words, $\phi = (m - \alpha)b$. In particular, $(m - \alpha)b$ is in $H(b)$. It now follows from Lemma 2.3 than $m - \alpha$ is in $H(\bar{b})$, and the proof of the theorem is complete.

For w in D we define the operator Q_w on H^2 by

$$(Q_w f)(z) = \frac{f(z) - f(w)}{z - w}.$$

A simple argument produces the alternative expression $Q_\omega = (1 - wS^*)^{-1}S^*$. In particular, $Q_w b = (1 - wX)^{-1}S^*b$, showing that $Q_w b$ is in $H(b)$.

Corollary 8.3. *If b is an extreme point of $B(H^\infty)$ and m is a multiplier of $H(b)$, then each function $Q_w b$ is an eigenvector of M_m^*.*

The case $w = 0$ is given by Theorem 8.2 so, for the proof, assume $w \neq 0$. The communication relation gives

$$(1 - wX)M_m^* - M_m^*(1 - wX) = wS^*b \otimes (m - \alpha)b.$$

Applying both sides to $Q_w b$, we obtain

$$(1 - wX)M_m^* Q_w b - \bar{\alpha}S^*b = \omega\langle Q_w b, (m - \alpha)b\rangle_b S^*b.$$

It follows that

$$M_m^* Q_w b = [\bar{\alpha} + w\langle Q_w b, (m - \alpha)b\rangle_b] Q_w b,$$

the desired conclusion.

The properties of M_m given by Theorem 8.2 characterize multiplication operators in the extreme point case.

Theorem 8.4. *If b is an extreme point of $B(H^\infty)$ and if M is an operator on $H(b)$ such that $M^*S^*b = \bar{\alpha}S^*b$ and*

$$M^*X - XM^* = S^*b \oplus \phi,$$

then $M = M_m$ for a multiplier m of $H(b)$.

To prove this it will suffice to show that the hypotheses imply k_w^b is an eigenvector of M^* whenever $b(w) \neq 0$. Assuming the last condition and applying the commutation relation to k_w^b, we obtain

$$\begin{aligned}\overline{\phi(w)}S^*b &= M^*(\bar{w}k_w^b - \overline{b(w)}S^*b) - XM^*k_w^b \\ &= (\bar{w} - X)M^*k_w^b - \bar{\alpha}\overline{b(w)}S^*b,\end{aligned}$$

so that

$$(\bar{w} - X)M^*k_w^b = (\overline{\phi(w)} + \bar{\alpha}\overline{b(w)})S^*b.$$

But also $(\bar{w} - X)k_w^b = \overline{b(w)}S^*b$, and the operator $\bar{w} - X$ is injective because of the assumption that $b(w) \neq 0$ (which implies that k_w is not in $H(b)$ [19]). We can conclude that

$$M^*k_w^b = \frac{\overline{\phi(w)} + \bar{\alpha}\overline{b(w)}}{\overline{b(w)}} k_w^b,$$

and the proof is complete.

9. Conjugations

We assume throughout this section that b is an extreme point of $B(H^\infty)$. As we mentioned earlier, in Section 3, one consequence of this assumption is the equality $H^2(\mu_b) = L^2(\mu_b)$, which enables us to define a conjugation on $H(b)$ by transferring via the map V_b a conjugation on $L^2(\mu_b)$. This conjugation and another on a space related to $H(\bar{b})$ that we shall introduce a little later are intimately connected with the structure of the multipliers of $H(b)$.

The conjugation on $H(b)$ that turns out to be useful is the one that corresponds to the conjugation $q \to e^{-i\theta}\bar{q}$ on $L^2(\mu_b)$. We denote it by C:

$$Ch = V_b(Z_b^*\overline{V_b^{-1}h}) \qquad (h \in H(b)).$$

That C is a conjugation (an anti-unitary involution) is obvious.

Lemma 9.1. *For w in $\mathbb{D}$, $Ck_w^b = Q_w b$.*

This is a straightforward calculation. By the way μ_b is defined, the function $(1 + b)/(1 - b)$ differs by an imaginary constant from the Herglotz integral of μ_b. Using the equality $k_w^b = (1 - \overline{b(w)})V_b k_w$ from Lemma 3.1, we obtain

$$\begin{aligned}
(Ck_w^b)(z) &= (1 - b(w))(1 - b(z)) \int_{\partial\mathbb{D}} \frac{e^{-i\theta}}{(1 - e^{-i\theta}w)(1 - e^{-i\theta}z)} d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{z - w} \int_{\partial\mathbb{D}} \left[\frac{1}{1 - e^{-i\theta}z} - \frac{1}{1 - e^{-i\theta}w}\right] d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{2(z - w)} \int_{\partial\mathbb{D}} \left[\frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{e^{i\theta} + w}{e^{i\theta} - w}\right] d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{2(z - w)} \left[\frac{1 + b(z)}{1 - b(z)} - \frac{1 + b(w)}{1 - b(w)}\right]
\end{aligned}$$

$$= \frac{b(z) - b(w)}{z - w},$$

as desired.

The conjugation C intertwines the operator X ($= S^*|H(b)$) and its adjoint.

Lemma 9.2. $CXC = X^*$

It will suffice to show that $CXk_w^b = X^*Ck_w^b$ for all w. We transform both sides with the aid of Lemma 9.1. First,

$$\begin{aligned} CXk_w^b &= C(wk_w^b - \overline{b(w)}S^*b) \\ &= wQ_wb - b(w)k_0^b. \end{aligned}$$

Next, by the formula for X^* mentioned in Section 8,

$$\begin{aligned} X^*Ck_w^b &= X^*Q_wb \\ &= SQ_wb - \langle Q_wb, S^*b\rangle_b b \\ &= SQ_wb - \langle k_0^b, k_w^b\rangle_b b \\ &= SQ_wb - (1 - \overline{b(0)}b(w))b. \end{aligned}$$

Now

$$\begin{aligned} SQ_w &= (S - w)S^*(1 - wS^*)^{-1} + wQ_w \\ &= -(1 - SS^*)(1 - wS^*)^{-1} + 1 + wQ_w. \end{aligned}$$

The operator $(1 - SS^*)(1 - wS^*)^{-1}$ is easily seen to equal $1 \otimes k_w$; in fact, its adjoint applied to the H^2 function f gives

$$(1 - \overline{w}S)^{-1}(1 - SS^*)f = f(0)(1 - \overline{w}S)^{-1}1 = f(0)k_w.$$

Thus $SQ_wb = -b(w) + b + wQ_wb$. Inserting this into the expression above for $X^*Ck_w^b$, we get

$$\begin{aligned} X^*Ck_w^b &= -b(w) + b + wQ_wb - (1 - \overline{b(0)}b(w))b \\ &= wQ_wb - b(w)(1 - \overline{bh(0)}b), \end{aligned}$$

as desired.

We now introduce our second conjugation. It will act on the space $K_+^2(\varrho)$, by which we mean the space of functions that are sums of functions in $K^2(\varrho)$ and constant functions. The functions in $K^2(\varrho)$, and here those in $K_+^2(\varrho)$, are defined in the complement of $\partial\mathrm{D}$, and we define them at ∞ in the obvious way (namely, those in $K^2(\varrho)$ are assigned the value 0 at infinity). For f in $K^2_+(\varrho)$ we define its conjugate, f_*, by

$$f_*(z) = \overline{f(1/\bar{z})}.$$

Straightforward calculations show that, if $f = K_p q + c$ with q in $L^2(\varrho)$ and c a constant, then $f_* = -SK_\varrho(Z_\varrho^* \bar{q}) + \bar{c}$, and also $f_* = -K_\varrho \bar{q} + (K_\varrho \bar{q})(0) + \bar{c}$. The latter expression shows that f_* is in $K^2_+(\varrho)$.

We let $K^\infty(\varrho)$ denote the space of bounded functions in $K^2_+(\varrho)$. (Here, by bounded we mean bounded in the entire complement of ∂D, not merely in D.) It is obvious that the conjugation on $K^2_+(\varrho)$ maps $K^\infty(\varrho)$ into itself.

The next lemma gives a relation between our two conjugations.

Lemma 9.3. *If f is in $H(\bar{b})$, then* $C[(1 - b)f] = (b - 1)S^*f_*$.

To prove this, let q be the function in $L^2(\varrho)$ such that $f = K_\varrho q$. Because $(1 - |b|^2)/|1 - b|^2$ is the Radon-Nikodym derivative with respect to normalized Lebesgue measure of the absolutely continuous component of μ_b, we can also write $f = K_{\mu_b}(|1 - b|^2 q)$, provided we regard $|1 - b|^2 q$ as vanishing on the singular component of μ_b, if there is one. Thus

$$(1 - b)f = V_b(|1 - b|^2 q),$$

and we obtain

$$\begin{aligned} C[(1 - b)f] &= V_b Z_b^*(|1 - b|^2 \bar{q}) \\ &= (1 - b)K_{\mu_b} Z_b^*(|1 - b|^2 \bar{q}) \\ &= (1 - b)K_\varrho(Z_\varrho^* \bar{q}). \end{aligned}$$

As mentioned above, $f_* = -SK_\varrho(Z_\varrho^* \bar{q})$, so that $K_\varrho(Z_\varrho^* \bar{q}) = -S^*f_*$, and the desired equality follows.

We are now able to determine the effect of conjugation on multiplication operators.

Theorem 9.4. *If m is a multiplier of $H(b)$, then CM_mC is a multiplication operator, namely, it equals M_{m_*}.*

Corollary 9.5. *The multipliers of $H(b)$ are in $K^\infty(\varrho)$.*

The corollary follows immediately from the theorem. To prove the theorem we note first that, because C is a conjugation, $(CM_mC)^* = CM_m^*C$. This in conjunction with Lemma 9.1 and Corollary 8.3 implies that if m is a multiplier of $H(b)$ then each of the functions $\dot{k}_w^b$ is an eigenvector of $(CM_mC)^*$ and hence that CM_mC is a multiplication operator. It remains to determine the corresponding multiplier. Let it be denoted by m'.

From the proof of Corollary 8.3 we have

$$M_m^*Q_wb = [\bar{\alpha} + w\langle Q_wb, (m - \alpha)b\rangle_b]Q_wb,$$

where a is the eigenvalue of S^*b as an eigenvector of M_m^*. Consequently

$$CM_m^*Ck_w^b = [\alpha + \bar{w}\langle (m - \alpha)b, Q_wb\rangle_b]k_w^b,$$

from which we conclude that

$$\begin{aligned} m^1(z) &= \bar{\alpha} + z\langle Q_zb, (m - \alpha)b\rangle_b \\ &= \bar{\alpha} + z\langle C[(m - \alpha)b], k_z^b\rangle_b \\ &= \bar{\alpha} + zC[(m - \alpha)b](z). \end{aligned}$$

Hence $m^1(0) = \bar{\alpha}$, and

$$S^*m^1 = C[(m - \alpha)b].$$

By Lemma 9.3,

$$C[(m - \alpha)(b - 1)] = (1 - b)S^*m_*.$$

(The lemma applies because $m - \alpha$ belongs to $H(\bar{b})$, by Theorem 8.2) In view of the last two equalities, we seek an expression for $C(m - \alpha)$ in terms of m^1.

Let $\bar{\beta}$ be the eigenvalue of S^*b as an eigenvector of $M_{m^1}^*$. Because m and m^1 play symmetric roles, we have $\beta = \overline{m(0)}$ and $C[(m^1 - \beta)b] = S^*m$. We can rewrite the last equality as

$$X(m - \alpha) = C[(m^1 - \beta)b].$$

Since $CX = X^*C$, it follows that

$$X^*C(m - \alpha) = (m^1 - \beta)b.$$

Using the formula for X^* mentioned in Section 8, we can rewrite the left side here as

$$\begin{aligned} SC(m - \alpha) - \langle C(m - \alpha), S^*b\rangle_b b &= SC(m - \alpha) - \langle k_0^b, m - \alpha\rangle_b b \\ &= SC(m - \alpha) - (\overline{m(0)} - \bar{\alpha})b \\ &= SC(m - \alpha) - (\bar{\alpha} - \beta)b. \end{aligned}$$

Applying S^*, we find that

$$C(m - \alpha) + (\bar{\alpha} - \beta)S^*b = S^*[(m^1 - \beta)b]$$

$$= bS^*m^1 + (m^1(0) - \beta)S^*b$$
$$= bS^*m^1 + (\bar{\alpha} - \beta)S^*b.$$

Hence $C(m - \alpha) = bS^*m^1$.

Combining the last equality with the previously obtained expressions for $C[(m-\alpha)b]$ and $C[(m-\alpha)(b-1)]$, we find that $S^*m^1 = S^*m_*$, so m^1 and m_* differ by at most a constant. But from the way m_* is defined one easily sees that $m_* - \beta\ (= m_* - \overline{m(0)})$ belongs to $H(\bar{b})$. Since $m^1 - \beta$ also belongs to $H(\bar{b})$, but the constant functions do not, we must have $m^1 = m_*$, and the proof of the theorem is complete.

10. Lemmas on Cauchy integrals

We need two simple facts about Cauchy integrals. For μ a finite complex Borel measure on ∂D, we let $P{*}\mu$ denote the Poisson integral of μ and $Q{*}\mu$ the conjugate Poisson integral of μ.

Lemma 10.1. *If μ is a finite complex Borel measure on ∂D and $f = K\mu$, then, in D,*

$$f(z) - f(1/\bar{z}) = (P * \mu)(z)$$

$$f(z) = \frac{1}{2}[(P * \mu)(z) + i(Q * \mu)(z) + (P * \mu)(0)]$$

Lemma 10.2. *If μ is a finite complex Borel measure on ∂D, and if f and g are holomorphic functions in D such that $f - \bar{g} = P * \mu$, then, in D,*

$$f(z) = (K\mu)(z) + \overline{g(0)}$$
$$\overline{g(z)} = (K\mu)(1/\bar{z}) + \overline{g(0)}.$$

Lemma 10.1 is a straightforward consequence of the relation between the Cauchy kernel and the Poisson and conjugate Poisson kernels, and Lemma 10.2 follows easily from Lemma 10.1.

To illustrate the use of these lemmas we show here that, when b is an extreme point of $B(H^\infty)$, the space $K^\infty(\varrho)$ is closed under multiplication. Let f and g be functions in $K^\infty(\varrho)$. The function $fg - \bar{f}_*\bar{g}_*$ is then bounded and harmonic in D, so it is the Poisson integral of its boundary function. (The function is defined in $\bar{\mathbb{C}}\backslash\partial D$, but by its boundary function we mean the interior boundary function, that is, the boundary function from D.) By Lemma 10.2, to prove fg is in $K^\infty(\varrho)$ it will suffice to prove that the interior boundary function of $fg - \bar{f}_* \bar{g}_*$ has the form $q\varrho$ with q in $L^2(\varrho)$. Let q_1 and q_2 be the functions in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho(q_1)$ and $g = g(\infty) + K_\varrho(q_2)$. Then, by Lemma 10.1, the interior boundary function of $f - \bar{f}_*$ is $q_1\varrho$ and the interior boundary function of $g - \bar{g}_*$ is $q_2\,\varrho$. Writing

$$fg - \bar{f}_*\bar{g}_* = (f - \bar{f}_*)g + \bar{f}_*(g - \bar{g}_*),$$

we see that the interior boundary function of $fg - \bar{f}_*\bar{g}_*$ is $(q_1 g + q_2\bar{f}_*)\varrho$. (In the last expression, of course, g and $\bar{f}_*$ denote interior boundary functions.) Since g and $\bar{f}_*$ are bounded, the function $q_1 g + q_2 \bar{f}_*$ is in $L^2(\varrho)$, the desired conclusion.

Thus, $K^\infty(\varrho)$ is an algebra, and by reasoning like that above one easily sees that the spectrum of a function f in this algebra equals the closure of $f(\mathbf{C}\backslash\partial\mathbf{D})$. In fact, that the spectrum of f contains the closure of $f(\mathbf{C}\backslash\partial\mathbf{D})$ is obvious, so one only needs to show that f is invertible in $K^\infty(\varrho)$ if it is bounded away from 0 in $\mathbf{C}\backslash\partial\mathbf{D}$. If the latter happens, and if q_1 is the function in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho q_1$, then by Lemma 10.1 the interior boundary function of $f^{-1} - \bar{f}_*^{-1}$ is $-q_1\,\varrho/f\bar{f}_*$, which is of the form $q\varrho$ with q in $L^2(\varrho)$. Lemma 10.2 thus guarantees that f^{-1} is in $K^\infty(\varrho)$.

11. More on $H(\bar{b})$

We return in this section to the assumption that b is an extreme point of $B(H^\infty)$. The functions in $H(\bar{b})$ are restrictions to $\mathbf{D}$ of functions in $K^2(\varrho)$, so they have natural extensions to the exterior of $\partial\mathbf{D}$. The next lemma states the process of extension preserves multiplication, to the extent that it can. (This fails when b is not an extreme point, except in the trivial case where ϱ is constant.) For f in $H(\bar{b})$, we let f_* denote the restriction to $\mathbf{D}$ of the conjugate of the extension of f. (It differs by a constant from a function in $H(\bar{b})$.)

Lemma 11.1. *If the function f, g, and fg belong to $H(\bar{b})$, then $(fg)_* = f_* g_*$.*

For the case where f and g are in $K^\infty(\varrho)$ this is established at the end of the preceding section. The argument for the general case is similar but slightly more elaborate.

Let q_1, q_2, and q be the functions in $L^2(\varrho)$ such that $f = K_\varrho q_1$, $g = K_\varrho q_2$, and $fg = K_\varrho q$. By Lemma 10.1, the boundary functions of $f - \bar{f}_*$ and $\bar{g} - \bar{g}_*$ are $q_1\,\varrho$ and $g_2\,\varrho$ respectively. The function $fg - \bar{f}_*\bar{g}_*$ is the sum of an H^2 function and the conjugate of an H^1 function, so it is the Poisson integral of its boundary function. Writing

$$fg - \bar{f}_*\bar{g}_* = (f - \bar{f}_*)g + \bar{f}_*(g - \bar{g}_*),$$

we see that its boundary function is $(gq_1 + \bar{f}_* q_2)\varrho$. (In the usual way, we are identifying functions in D with their boundary functions.) Therefore, by Lemma 10.2,

$$fg = K[(gq_1 + \bar{f}_* q_2)\varrho].$$

Hence

$$K[(gq_1 + \bar{f}_* q_2 - q)\varrho] = 0.$$

As the functions $q_1\varrho^{1/2}$, $q_2\,\varrho^{1/2}$, and $q\varrho^{1/2}$ are in L^2, the function $(gq_1 + \bar{f}_* q_2 - q)\varrho^{1/2}$ is in L^1, which implies that the function $(gq_1 + \bar{f}_* q_2 - q)\varrho$ fails to be log-integrable (the reasoning can be found in Section 5). Since the Cauchy integral of the latter function vanishes so do its forward Fourier coefficients, and hence it is the zero function. Thus $fg - \bar{f}_*\bar{g}_*$ is actually the Poisson integral of $q\varrho$, and we can conclude by Lemma 10.2 that $(fg)_* = f_* g_*$, as desired.

Lemma 11.1 enables us to obtain the analogue of Theorem 9.4 and its corollary for multipliers of $H(\bar{b})$.

Theorem 11.2. *If m is a multiplier of $H(\bar{b})$ then m is in $K^\infty(\varrho)$ and m_* is a multiplier of $H(\bar{b})$.*

To prove this we use the conjugation on $H(\bar{b})$ that corresponds, under the transformation K_ϱ, to the conjugation $q \to -Z_\varrho^* \bar{q}$ on $L^2(\varrho)$. We shall not introduce a special notation for it because we shall not have occasion to use it again. A straightforward calculation shows that it is given by $f \to S^* f_*$. The important property for us is that the preceding map sends $H(\bar{b})$ onto itself, which also follows from the unitarity of $S^*|H(\bar{b})$ (used before in the proof of Theorem 3.6).

Let m be a multiplier of $H(\bar{b})$, and let f be any function in $H(\bar{b})$. By Theorem 3.6, m is in $K_+^2(\varrho)|\mathrm{D}$, so Lemma 11.1 can be applied to give $(mf)_* = m_* f_*$. Also $f_*(0) = 0$, so $S^*(mf)_* = m_* S^* f_*$. In view of the remark at the end of the last paragraph we can conclude that m_* is a multiplier of $H(\bar{b})$. In particular, m_* is bounded in D, and thus m is in $K^\infty(\varrho)$.

The next result enables us to supplement Theorem 7.1.

Theorem 11.3. *The function b belongs to $K_+^2(\varrho)|\mathrm{D}$ if and only if $1/b$ is in H^2. In that case $b_* = 1/b$.*

For the proof, suppose first that $1/b$ is in H^2. Then $1/b$ is also in $L^2(\varrho)$, and we have

$$K_\varrho(1/\bar{b}) = P_+\left(\frac{1-|b|^2}{\bar{b}}\right) = 1/\overline{b(0)} - b,$$

showing that $1/\overline{b(0)} - b$ is in $K^2(\varrho)|\mathrm{D}$ and hence that b is in $K^2_+(p)|\mathrm{D}$. Moreover, because of the way the transformation K_ϱ interacts with the conjugation on $K^2_+(\varrho)$ (as was pointed out in Section 9), we have

$$\begin{aligned}(1/\overline{b(0)} - b)_* &= -SK_\varrho(Z^*_\varrho(1/b)) \\ &= -SS^*K_\varrho(1/b) \\ &= -SS^*P_+\left(\frac{1-|b|^2}{\bar{b}}\right) \\ &= -SS^*\left(\frac{1}{b} - \overline{b(0)}\right) \\ &= \frac{1}{b(0)} - \frac{1}{b},\end{aligned}$$

which gives $b_* = 1/b$.

Suppose, conversely, that b is in $K^2_+(\varrho)|\mathrm{D}$, in other words, that $b - c$ is in $H(\bar{b})$, where c is a constant. Then $c \neq 0$, since b is not in $H(b)$. Also, because $1 - \overline{b(0)}b$ ($= k^b_0$) is in $H(b)$, we must have $b(0) \neq 0$ and $c = 1/\overline{b(0)}$. Let q be the function in $L^2(\varrho)$ that maps to $b - 1/\overline{b(0)}$ under K_ϱ. Then

$$\begin{aligned}K_\varrho(\bar{b}q) &= T_{\bar{b}}K_\varrho q = T_{\bar{b}}(b - 1/\overline{b(0)}) \\ &= T_{\bar{b}}b - 1 = -P_+(1-|b|^2) \\ &= K_\varrho(-1).\end{aligned}$$

Since K_ϱ has a trivial kernel, it follows that $q = -1/\bar{b}$ (modulo the measure $\varrho d\theta$). Therefore $(1-|b|^2)/|b|^2$ is in L^1, implying that $1/b$ is in L^2. In addition,

$$1/\overline{b(0)} - b = K_\varrho(1/\bar{b}) = P_+\left(\frac{1-|b|^2}{\bar{b}}\right) = P_+(1/\bar{b}) - b,$$

so $P_+(1/\bar{b}) = 1/\overline{b(0)}$. Therefore $1/\bar{b}$ is in $\overline{H^2}$, in other words, $1/b$ is in H^2, and the proof is complete.

Corollary 11.4. *If b is a multiplier of $H(\bar{b})$ then b is invertible in H^∞.*

The corollary is an immediate consequence of Theorems 11.2. and 11.3.

12. Construction of multipliers

We retain the assumption that b is an extreme point of $B(H^\infty)$. Our next main result, Theorem 12.2, is a criterion for a function in $K^\infty(\varrho)$ to be a multiplier of $H(b)$. The criterion enables us to show that $H(b)$ has an abundance of multipliers; in particular, the multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.

Lemma 12.1. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho(q)$. Let g be a function in H^2 such that $g(0) = 0$. Then $T_{\bar{g}}m = K(\bar{g}q\varrho)|\mathrm{D}$. The function $T_{\bar{g}}m$ is in $H(b)$ if and only if gq is in $L^2(\varrho)$.*

In fact, by Lemma 10.1, the interior boundary function of $m - \overline{m}_*$ is $q\varrho$, so

$$T_{\bar{g}}m = P_+(\bar{g}\overline{m}_*) + P_+(\bar{g}q\varrho).$$

The first term on the right is 0 because $g(0) = 0$, and the second term is $K(\bar{g}q\varrho)|\mathrm{D}$. This proves the first assertion in the lemma. It is obvious that $T_{\bar{g}}m$ is in $H(\bar{b})$ if $g\varrho$ is in $L^2(\varrho)$, which is one direction in the second assertion. For the other direction, suppose $T_{\bar{g}}m$ is in $H(\bar{b})$, say $T_{\bar{g}}m = K_\varrho(q_1)|\mathrm{D}$ with q_1 in $L^2(\varrho)$. Then $K((\bar{g}q - q_1)\varrho)|\mathrm{D} = 0$. But $(\bar{g}q - q_1)\varrho$ is not log-integrable since it is the product of the L^1 function $gq\varrho^{1/2} - q_1\varrho^{1/2}$ and the function $\varrho^{1/2}$, which is not log-integrable. It follows that $q_1 = \bar{g}q$, and the proof is complete.

Theorem 12.2. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$.*

(i) *The function m is a multiplier of $H(\bar{b})$ if and only if fq is in $L^2(\varrho)$ for every f in $H(\bar{b})$.*

(ii) *The function m is a multiplier of $H(b)$ if and only if hq is in $L^2(\varrho)$ for every h in $H(b)$.*

To prove (i), let f be any function in $H(\bar{b})$, and let q_1 be the function in $L^2(\varrho)$ such that $f = K_\varrho q_1|\mathrm{D}$. By Lemma 11.1, if mf is in $H(\bar{b})$ then $(mf)_* = m_*f_*$. This in conjunction with Lemmas 10.1 and 10.2 implies that mf is in $H(\bar{b})$ if and only if the boundary function of $mf - \overline{m}_*\bar{f}_*$ has the form $q_2\varrho$ with q_2 in $L^2(\varrho)$. On $\partial\mathrm{D}$ we have

$$\begin{aligned} mf - \overline{m}_*\bar{f}_* &= (m - \overline{m}_*)f + \overline{m}_*(f - \bar{f}_*) \\ &= fq\varrho + \overline{m}_*q_1\varrho. \end{aligned}$$

Since m_* is bounded the function $\overline{m}_* q_1$ is in $L^2(\varrho)$. Hence $mf - \overline{m}_* \bar{f}_*$ has the required form if and only if fq is in $L^2(\varrho)$, which proves (i).

Because of (i), in proving (ii) we can assume, without loss of generality, that m is a multiplier of $H(\bar{b})$. Let h be any function in $H(b)$. By Lemma 2.2, mh is in $H(b)$ if and only if $T_{\bar{b}}(mh)$ is in $H(\bar{b})$. We have

$$\begin{aligned} T_{\bar{b}}(mh) &= mT_{\bar{b}}h + P_+((\bar{b}h - P_+\bar{b}h)m) \\ &= mT_{\bar{b}}h + T_{\bar{g}}m, \end{aligned}$$

where $g = \overline{(1 - P_+)(\bar{b}h)}$. The first term on the right is in $H(\bar{b})$ since we have assumed that m is a multiplier of $H(\bar{b})$. Hence mh is in $H(b)$ if and only if the second term on the right, $T_{\bar{g}}m$, is in $H(\bar{b})$. By Lemma 12.1, that happens if and only if gq is in $L^2(\varrho)$. The function $qP_+(\bar{b}h)$ $(= qT_{\bar{b}}h)$ is in $L^2(\varrho)$ by (i) (since $T_{\bar{b}}h$ is in $H(\bar{b})$ and m is a multiplier of $H(\bar{b})$). Hence mh is in $H(b)$ if and only if bhq is in $L^2(\varrho)$, in other words, if and only if $|b|^2 |h|^2|q|^2\varrho$ is in L^1. But $(1 - |b|^2)|h|^2|q|^2\varrho$ $(= |h|^2|q|^2\varrho^2)$ is in L^1 since $q\varrho$ $(= m - \overline{m}_*)$ is bounded. Hence mh is in $H(b)$ if and only if $|h|^2|q|^2\varrho$ is in L^1, in other words, if and only if hq is in $L^2(\varrho)$. This proves (ii).

Corollary 12.3. *If m is a multiplier of $H(b)$ then the spectrum of M_m is the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$.*

As shown in Section 10, the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$ equals the spectrum of m in the algebra $K^\infty(\varrho)$, and this set is obviously contained in the spectrum of M_m. To establish the opposite containment it will suffice to show that the invertibility of m in $K^\infty(\varrho)$ implies that $1/m$ is a multiplier of $H(b)$. Assume m is invertible in $K^\infty(\varrho)$, and let q and q_1 be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$ and $1/m = 1/m(\infty) + K_\varrho q_1$. Since

$$\frac{1}{m} - \frac{1}{\overline{m}_*} = \frac{\overline{m}_* - m}{m\overline{m}_*}$$

we conclude by Lemmas 10.1 and 11.1 that $q_1 = -q/m\overline{m}_*$. If h is in $H(b)$ then Theorem 12.2(ii) tells us that hq is in $L^2(\varrho)$, and therefore so is hq_1, since $1/m$ and $1/m_*$ are bounded. Theorem 12.2.(ii) now implies that $1/m$ is a multiplier of $H(b)$, as desired.

Corollary 12.4. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. If $q\varrho^{1/2}$ is bounded, then m is a multiplier of $H(b)$.*

This corollary is an immediate consequence of Theorem 12.2.

Corollary 12.5. *If m is an invertible function in H^∞ such that $(1 - |m|^2)^2/\varrho$ is bounded on $\partial\mathbb{D}$, then m is a multiplier of $H(b)$, and $m_* = 1/m$.*

We remark that if b is not an inner function, in other words, if ϱ does not vanish identically, then nonconstant functions satisfying the hypotheses of Corollary 12.5 can be constructed by standard means. (One example is the outer function with modulus $\max\{|b|, 1/2\}$ on ∂D.)

To establish Corollary 12.5 it suffices to note that the bounded harmonic function $m - 1/\overline{m}$ is the Poisson integral of its boundary function, which equals $(|m|^2 - 1)/\overline{m}$. From Lemma 10.2. it follows that m is in $K^\infty(\varrho)$ with $m - 1/\overline{m(0)} = K_\varrho((|m|^2 - 1)/\varrho\overline{m})$ in D, and $m_* = 1/m$. That m is a multiplier of $H(b)$ is now immediate from Corollary 12.4.

Corollary 12.6. *If m is a function in H^∞ such that $|Re\ m|/\varrho^{1/2}$ is bounded on ∂D, then m is a multiplier of $H(b)$ and $m_* = -m$.*

It is not completely obvious that there are nonzero functions satisfying the hypotheses of the corollary in all cases where b is not an inner function. That there are will be pointed out below in connection with the proof of Corollary 12.8.

To establish Corollary 12.6, it suffices to use Lemma 10.2 in the same way as in the preceding proof and earlier ones to obtain

$$m + \overline{m(0)} = K_\varrho\left(\frac{2Re\ m}{\varrho}\right)$$

in D and $m_* = -m$. Corollary 12.4 now applies to show that m is a multiplier of $H(b)$.

Corollary 12.7. *If the outer factor, b_0, of b is invertible in H^∞, then all of the functions $k_w^{b_0}$ and $Q_w b_0$ are multipliers of $H(b)$, and $(Q_w b_0)_* = Sk_w^{b_0}/b_0$ and $(k_w^{b_0})_* = -b_0(w)SQ_w(1/b_0)$.*

To simplify the notation slightly in the proof, we shall assume that b itself is invertible. This is not a genuine loss of generality, because the criterion in Corollary 12.4, upon which the proof of Corollary 12.7 is based, is insensitive to the inner factor of b.

Assuming then that b is invertible, we note that

$$\begin{aligned} K_\varrho(Z_\varrho^*\bar{k}_w/\bar{b}) &= S^*K_\varrho(\bar{k}_w/\bar{b}) \\ &= S^*P_+\left(\frac{(1-|b|^2)\bar{k}_w}{b}\right) \\ &= -S^*P_+(b\bar{k}_w) \\ &= -S^*(1-wS^*)^{-1}b \\ &= -Q_w b. \end{aligned}$$

Therefore, by the relation between the transformation K_ϱ and the conjugation on $K_+^2(\varrho)$ (noted in Section 9),

$$\begin{aligned}(Q_w b)_* &= SK_\varrho(k_w/b)\\ &= SP_+(k_w/b - \bar{b}k_w)\\ &= S(k_w/b - T_{\bar{b}}k_w)\\ &= S(1 - \overline{b(0)}b)k_w/b\\ &= Sk_w^b/b.\end{aligned}$$

Thus $Q_w b$ is in $K^\infty(\varrho)$, and Corollary 12.4 implies that it is a multiplier of $H(b)$. We see also that Sk_w^b/b is a multiplier of $H(b)$. Since the space of multipliers is invariant under S^* (by Lemma 2.6), and since b is a multiplier of $H(b)$ (Theorem 7.1), it follows that k_w^b is a multiplier of $H(b)$. To determine $(k_w^b)_*$ one verifies that $k_w^b = \overline{b(w)}K_\varrho(k_w/b)$, which gives the formula

$$(k_w^b)_* = -b(w)SS^*K_\varrho(\bar{k}_w/b).$$

The right side is easily reduced to the desired expression. The details are similar to those above and we omit them.

Corollary 12.8. *The multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.*

To prove this, let q be a real function in $L^\infty(\varrho)$ and let $f = K_\varrho q$. As such functions f clearly span $H(\bar{b})$, it will suffice to show that f can be approximated in the norm of $H(\bar{b})$ by multipliers of $H(b)$. From Lemma 10.1 one sees that the real part of f is bounded in modulus by $\|q\varrho\|_\infty$ in $\mathbf{C}\backslash\partial\mathbf{D}$. For ϵ a positive number smaller than $1/\|q\varrho\|_\infty$, the functions $f/(1+\epsilon f)$ and $f_*/(1+\epsilon f_*)$ are then in H^∞, and we have

$$\frac{f}{1+\epsilon f} - \frac{\bar{f}_*}{1+\epsilon\bar{f}_*} = \frac{f-\bar{f}_*}{(1+\epsilon f)(1+\epsilon\bar{f}_*)}.$$

The interior boundary function of $f - \bar{f}_*$ is by Lemma 10.1 equal to $q\varrho$, so the interior boundary function of the preceding function is $q_\epsilon\varrho$, where $q_\epsilon = q/(1+\epsilon f)(1+\epsilon f_*)$. By Lemma 10.2 we conclude that the function $m_\epsilon = f/(1+\epsilon f)$ equals $K_\varrho(q_\epsilon)$ in $\mathbf{D}$ and that $(m_\epsilon)_* = f_*/(1+\epsilon f_*)$. Thus m_ϵ is in $K^2(\varrho)$ and in $K^\infty(\varrho)$. It now follows immediately from Corollary 12.4 that m_ϵ is a multiplier of $H(b)$. Finally, since $q_\epsilon \to q$ in $L^2(\varrho)$ as $\epsilon \to 0$, we have $\|f - m_\epsilon\|_{\bar{b}} \to 0$ as $\epsilon \to 0$, completing the proof.

A comment on the preceding proof: Suppose for simplicity that $\|q\varrho\|_\infty = 1$, and let $g = f - f(0)/2$ (which makes $g_* = -g$). The functions

$$m_\epsilon^1 = \frac{1}{2}\left(\frac{g}{1 + \epsilon g} + \frac{g}{1 - \epsilon g}\right)$$

are then in H^∞ for $0 < \epsilon < 1$, and a simple estimate shows that $|Re\, m_\epsilon^1| \leq |q|\varrho/(1 - \epsilon)^2$ on ∂D. The functions m_ϵ^1 thus satisfy the hypothesis of Corollary 12.6, and the functions $m_\epsilon^1 - m_\epsilon^1(\infty)$ could have been used in the proof of Corollary 12.8 in place of the functions m_ϵ. One can also deduce Corollary 12.8 by combining Corollary 12.6 with the following nice lemma of A. M. Gleason and H. Whitney [9, Lemma 3.1] (slightly rephrased): If k is a non-negative function in L^∞, then there is a sequence in H^∞ whose real parts lie between 0 and k on ∂D and converge almost everywhere to k.

13. The effect of the inner factor

We continue to asume that b is an extreme point of $B(H^\infty)$. As we observed earlier, the multiplication criterion in Corollary 12.4 is insensitive to the inner factor of b: if a function m passes that test then it is a multiplier not only of $H(b)$ but of $H(ub)$ for every inner function u. We shall show that the condition of Corollary 12.4 characterizes multipliers of the preceding kind.

Lemma 13.1. *Let m be a multiplier of $H(b)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. Let u be an inner function. Then m is a multiplier of $H(ub)$ if and only if gq is in $L^2(\varrho)$ for every g in $H(u)$.*

This lemma follows immediately from Theorem 12.2 and the decomposition $H(ub) = H(u) + uH(b)$ (explained in Section 6).

Corollary 13.2. *If u is a finite Blaschke product, then every multiplier of $H(b)$ is a multiplier of $H(ub)$.*

Indeed, if u is a finite Blaschke product, then the functions in $H(u)$ are bounded (in fact, they are rational functions), so the condition in Lemma 13.1 is satisfied.

Theorem 13.3. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. Then m is a multiplier of $H(ub)$ for every inner function u if and only if $q\varrho^{1/2}$ is bounded.*

The «if» part is Corollary 12.4. The «only if» part is an immediate consequence of the preceding lemma and the following one.

Lemma 13.4. *Let σ be a nonnegative essentially unbounded measurable func-*

tion on ∂D. *Then there is a function* g *in* H^2 *that is noncyclic for* S^* *such that* $g\sigma$ *is not in* L^2.

The function g that we shall produce lies in $H(u)$ for an interpolating Blaschke product u. We shall let $|E|$ stand for the unnormalized Lebesgue measure of the measurable subset E of ∂D.

Since σ is unbounded there is a sequence $\{t_n\}_1^\infty$ of positive numbers such that $t_{n+1} > 2t_n$ for all n and such that each set $E_n = \{t_n \leq \sigma \leq 2t_n\}$ has positive measure. For each n let λ_n be a point of density of E_n. The points λ_n are distinct so, passing to a subsequence, we can assume they converge to a point distinct from all of them. That being the case, we can find disjoint arcs $I_1, I_2, \ldots$ such that I_n has center λ_n for each n. Shrinking these arcs successively, if need be, we can assume $|I_{n+1}| < |I_n|/2$ and $|I_n \cap E_n| > |I_n|/2$ for each n.

Let w_n be the point in D such that $w_n/|w_n| = \lambda_n$ and $1 - |w_n| = |I_n|/2$, and let $g_n = (1 - |w_n|^2)^{1/2} k_{w_n}$, the normalized kernel function for the point w_n. Since $1 - |w_{n+1}| < (1 - |w_n|)/2$, the sequence $\{w_n\}_1^\infty$ is an interpolating sequence. Therefore, by a theorem of H. S. Shapiro and A. L. Shields [22], the functions g_n form a Riesz basis for their span in H^2, that span being $H(u)$, where u is the Blaschke product with zero sequence $\{w_n\}_1^\infty$.

We need to estimate the size of g_n on I_n. For that, fix an n and let $r = |w_n|$. We have

$$g_n(\lambda_n e^{i\theta}) = \frac{(1 - r^2)^{1/2}}{1 - re^{i\theta}}$$

$$= \frac{(1 - r^2)^{1/2}(1 - r\cos\theta + ir\sin\theta)}{(1 - r)^2 + 4r\sin^2(\theta/2)}.$$

Thus $Re g_n > 0$ on ∂D, and for $\lambda_n e^{i\theta}$ in I_n, that is, for $|\theta| < 1 - r$,

$$Re g_n(\lambda_n e^{i\theta}) \geq \frac{(1 - r^2)^{1/2}(1 - r)}{(1 - r)^2 + 4r\sin^2((1 - r)/2)}$$

$$\geq \frac{(1 + r)^{1/2}(1 - r)^{3/2}}{2(1 - r)^2}$$

$$\geq \frac{1}{2|I_n|^{1/2}}.$$

Since $t_n \to \infty$ we can find a sequence $\{c_n\}_1^\infty$ of positive numbers such that $\Sigma c_n^2 < \infty$ but $\Sigma c_n^2 t_n^2 = 2$. Let $g = \Sigma c_n g_n$. By the theorem of Shapiro and Shields mentioned above, g is in H^2 and is not a cyclic vector of S^*. On I_n we have

$$|g| \geq Reg \geq Reg_n \geq \frac{c_n}{2|I_n|^{1/2}}.$$

Hence

$$\begin{aligned}\int_{\partial D} |g\sigma|^2 \, d\theta &\geq \sum \int_{I_n \subset E_n} |g\sigma|^2 \, d\theta \\ &\geq \sum \frac{c_n^2}{4|I_n|} t_n^2 |I_n \cap E_n| \\ &\geq \frac{1}{8} \sum c_n^2 t_n^2 = \infty,\end{aligned}$$

which proves the lemma.

Up to now we have not given an example of a multiplier that fails to satisfy the criterion in Corollary 12.4. That will come in the next section. In the other direction, one sees from Theorem 13.3 that if ϱ is bounded away from 0 on the set where it is nonzero, then $H(b)$ and $H(ub)$ have the same multipliers for all inner functions u.

The next result, which identifies a class of inner functions u for which $H(ub)$ and $H(b)$ have the same multipliers, does not require the assumption that b is an extreme point.

Theorem 13.5. *If u is an inner function such that dist$(b, uH^\infty) < 1$, then every multiplier of $H(b)$ is a multiplier of $H(ub)$.*

We first show that the distance inequality is equivalent to the equality $H(u) = (1 - T_u T_{\bar{u}})H(b)$, or, what amounts to the same thing, to the inclusion $H(u) \subset (1 - T_u T_{\bar{u}})H(b)$. By the criterion of Douglas we used earlier (in Section 2), the inclusion is equivalent to the operator inequality

$$1 - T_u T_{\bar{u}} \leq c\,(1 - T_u T_{\bar{u}})(1 - T_b T_{\bar{b}})(1 - T_u T_{\bar{u}})$$

for some $c \geq 1$. The operator inequality means that

$$\|h\|_2^2 \leq c\,(\|h\|_2^2 - \|T_{\bar{b}} h\|_2^2)$$

for all h in $H(u)$, in other words, that

$$\|T_{\bar{b}} h\|_2^2 \leq \frac{c-1}{c} \|h\|_2^2$$

for all h in $H(u)$, in other words, that $\|T_{\bar{b}}|H(u)\| < 1$. Since it is known [17] that $\|T_{\bar{b}}|H(u)\| = \text{dist}(b, uH^\infty)$, the equivalence is established.

Thus, assuming u satisfies the condition in the theorem, we have $H(u) = (1 - T_u T_{\bar{u}})H(b)$. Suppose m is a multiplier of $H(b)$. Then, because $H(ub) = H(u) + uH(b)$, to show m is a multiplier of $H(ub)$ we need only show $mH(u) \subset H(ub)$. Let g be any function in $H(u)$. Then, because $H(u) = (1 - T_u T_{\bar{u}}) H(b)$, there is a function h in $H(b)$ whose projection onto $H(u)$ is g. The difference $h - g$ is then in $H(ub)$ and in uH^2, so it is in $uH(b)$. Hence $m(h - g)$ is in $uH(b)$ and thus in $H(ub)$. Since also mh is obviously in $H(ub)$, it follows that mg is in $H(ub)$, and the theorem is established.

14. Helson-Szegö weights

For certain extreme points b of $B(H^\infty)$, those for which the conjugation operator behaves in a decent manner relative to μ_b, we are able to describe the multipliers of $H(b)$ completely. By a Helson-Szegö weight we shall mean a nonnegative function σ on ∂D that has the form $\sigma = \exp(\phi + \tilde{\psi})$, where ϕ and ψ are real functions in L^∞ with $\|\psi\|_\infty < \pi/2$, and $\tilde{\psi}$ denotes the conjugate function of ψ. The following properties hold.

1. If σ is a Helson-Szegö weight then so is $1/\sigma$.
2. A Helson-Szegö weight is in $L^{1+\epsilon}$ for sufficiently small positive numbers ϵ.
3. If σ is a Helson-Szegö weight then the conjugation operator is bounded on $L^2(\sigma)$. This property characterizes Helson-Szegö weights.

Property 1 is trivial and property 2 is a well-known result of V. I. Smirnov [7, p. 34]. Property 3 is the basic theorem of H. Helson and G. Szegö [13]. A thorough discussion of these and related matters can be found in the book [8].

If μ_b is absolutely continuous and its Radon-Nikodym derivative is a Helson-Szegö weight then, as Davis and McCarthy show [1] (on the basis of the Helson-Szegö theorem), every function in H^∞ is a multiplier of $H(b)$. (They prove the converse also.) Such a b of course is not an extreme point of $B(H^\infty)$. The next theorem says that an analogous result holds for extreme points whose corresponding measures are made in a simple way from Helson-Szegö weights.

Theorem 14.1. *If μ_b is absolutely continuous with Radon-Nikodym derivative $\chi_E \sigma$, where σ is a Helson-Szegö weight and E is a subset of ∂D of positive measure whose complement has positive measure, then the following spaces coincide:*

1. The space of multipliers of $H(b)$, *2.* $K^\infty(\varrho)$,

3. $K^\infty(\chi_E)$.

A lemma is needed.

Lemma 14.2. *Let σ be a Helson-Szegö weight and let q be a function in $L^2(\sigma)$. Then, in* D*, the Cauchy integral $K_\sigma q$ belongs to H^1, and its interior boundary function has the form $q_1\sigma$ with q_1 in $L^2(\sigma)$.*

To see that $K_\sigma q$ is in H^1, choose a positive number ϵ such that σ is in $L^{1+\epsilon}$. Then $q\sigma$ is the product of the L^2 function $q\sigma^{1/2}$ and the $L^{2+2\epsilon}$ function $\sigma^{1/2}$, so it is in $L^{(2+2\epsilon)/(2+\epsilon)}$, by Hölder's inequality. By M. Riesz's theorem, the conjugate function of $q\sigma$ lies in the same space. Hence (by Lemma 10.1), the Cauchy integral $K_\sigma q$ is in $H^{(2+2\epsilon)/(2+\epsilon)}$ and a fortiori in H^1.

To see that the interior boundary function of $K_\sigma q$ has the required form we note that, because $q\sigma$ is in $L^2(1/\sigma)$, the Helson-Szegö theorem implies that the interior boundary function of $K_\sigma q$, its Cauchy integral, is in $L^2(1/\sigma)$. Thus, if q_2 is that boundary function, then the function $q_1 = q_2/\sigma$ is in $L^2(\sigma)$, which is the desired conclusion.

As for the theorem, we already know that every multiplier of $H(b)$ is in $K^\infty(\varrho)$, and one easily sees that $K^\infty(\varrho)$ is contained in $K^\infty(\chi_E)$. It only remains to show that every function in $K^\infty(\chi_E)$ is a multiplier of $H(b)$, or, equivalently, of $K^2(\chi_E\sigma)$.

The argument is similar to several we have already given. Let f be a function in $K^2(\chi_E\sigma)$, say $f = K(q\chi_E\sigma)$, where q is in $L^2(\chi_E\sigma)$. Let m be a function in $K^\infty(\chi_E)$. By Lemma 14.2 the functions f and f_*, in D, belong to H^1. Hence mf and m_*f_* are in H^1, implying that the harmonic function $mf - \overline{m}_*\overline{f}_*$ is the Poisson integral of its boundary function. By Lemma 10.2, to prove mf is in $K^2(\chi_E\sigma)$ it will suffice to prove that the boundary function of $mf - \overline{m}_*\overline{f}_*$ is of the form $q_1\chi_E\sigma$ with q_1 in $L^2(\chi_E\sigma)$. For this we write, as usual,

$$mf - \overline{m}_*\overline{f}_* = (m - \overline{m}_*)f + \overline{m}_*(f - \overline{f}_*).$$

In the first summand on the right, the boundary function of the first factor, $m - \overline{m}_*$, is bounded and vanishes off E (Lemma 10.1), while the boundary function of the second factor is in $L^2(\sigma)$, by Lemma 14.2. The boundary function of the first summand is thus of the required form. In the second summand, the boundary function of the first factor, $\overline{m}_*$, is bounded, and the boundary function of the second factor, $f - \overline{f}_*$, is $q\chi_E\sigma$. The boundary function of the second summand thus also has the required form, and the proof is complete.

We are now able to given an example of an extreme point b, a multiplier m of $H(b)$, and an inner function u, such that m is not a multiplier of $H(ub)$. Fix δ in (0,1), and let the function σ on ∂D be defined by $\sigma(e^{i\theta}) = |\theta|^{\delta}$, $(-\pi \leq \theta \leq \pi)$. This is a Helson-Szegö weight by a result of G. H. Hardy and J. E. Littlewood [11]. One can prove that nowadays by verifying that σ satisfies B. Muckenhoupt's condition (A_2), which characterizes Helson-Szegö weights. (Details are in [8].) Let E be the right half of ∂D, and let b be the function such that $(1 + b)/(1 - b)$ is the Herglotz integral of $\chi_E\sigma$. Theorem 14.1 applies, telling us that $K^{\infty}(\chi_E)$ is the space of multipliers of $H(b)$.

Let q_0 be a C^1 function on D that vanishes off E and is nonzero at the point 1. Since q_0 is of class C^1 its conjugate function is continuous, and this implies by Lemma 10.1. that the Cauchy integral $m = Kq_0$ is bounded in $C\backslash\partial D$ and hence belongs to $K^{\infty}(\chi_E)$. Thus m is a multiplier of $H(b)$.

We also have $m = K_{\varrho}q$ where $q = q_0/\varrho$. The function $q\varrho^{1/2}$ $(= q_0\,\varrho^{-1/2})$ is unbounded because $q_0\sigma^{-1/2}$ is and $\varrho = |1 - b|^2\chi_E\sigma \leq 4\chi_E\,\sigma$. Hence Theorem 13.3 guarantees the existence of an inner function u such that m is not a multiplier of $H(ub)$. The proof of Lemma 13.4 provides an explicit example of such a u, a certain interpolating Blaschke product. By using estimates similar to those in the proof of Lemma 13.4 it is not hard to show that the Blaschke product with zero sequence $\{1 - 2^{-n}\}_1^{\infty}$ also has the required property.

15. Questions

Many questions puzzle us.

1. If b is an extreme point of $B(H^{\infty})$, must every function in $K^{\infty}(\varrho)$ be a multiplier of $H(\bar{b})$? An answer most likely will involve subtleties of the conjugation operator (although we may be overlooking something simple).
2. If b is an outer function, must $H(b)$ and $H(\bar{b})$ have the same multipliers? Results in [18] show that the answer can be negative when b is not an extreme point. What about the extreme point case?
3. To understand better the multipliers of $H(b)$, one needs examples, in addition to those given by Theorem 14.1, where they can be described completely. As a very special query: Suppose in the example in Section 14 one lets $\delta = 1$, thus passing beyond the realm of Helson-Szegö weights. What are the multipliers of $H(b)$ for the corresponding b?
4. Davis and McCarthy [1] prove that if μ is a finite positive Borel measure on ∂D and μ_a is its absolutely continuous component, then every multiplier of $K^2(\mu)$ is a multiplier of $K^2(\mu_a)$. In case the singular component

of μ is a finite sum of point masses and the Radon-Nikodym derivative of μ_a is log-integrable, they are able to specify precisely which multipliers of $K^2(\mu_a)$ are also multipliers of $K^2(\mu)$. Can one describe in more general cases, or perhaps even in general, how the singular component of μ influences the space of multipliers of $K^2(\mu)$? Progress on this will undoubtedly lead to a better understanding of the structure of the corresponding space $H(b)$.

5. In case b is an extreme point, the algebra $K^\infty(\varrho)$ appears to be an interesting object of study. It becomes a Banach algebra when equipped with the norm $\|f\| = \|f\|_\infty + \|q\|_{L^2(\varrho)}$, where q is the function in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho q$, and $\|f\|_\infty$ stands for the supremum of $|f|$ over $\mathbb{C}\backslash\partial\mathbb{D}$. As shown in Section 10, the spectrum of a function f in $K^\infty(\varrho)$ is the closure of $f(\mathbb{C}\backslash\partial\mathbb{D})$. What can one say about the maximal ideal space of $K^\infty(\varrho)$? Is $\mathbb{C}\backslash\partial\mathbb{D}$ dense in it?

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