

Variational Characterization of Equations of Motion in Bundles of Embeddings

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1. Introduction

In this paper we study variational principles for a general situation which includes free boundary problems with surface tension. Following [2], our main result concerns a variational principle in a infinite dimensional principal bundle of embeddings of a compact region D in a manifold M having the same dimension as D . By considering arbitrary variations, free boundary problems are included, while variations parallel to the boundary permit to consider fluid motion or flow of Hamiltonian vector fields in non compact regions, generalizing [3], [4].

In Section 2 the main result is stated and proved. The Lagrangian includes a boundary term allowing us to include surface tension [5], or to remove it. Section 3 applies our result to Hamiltonian vector fields, while Section 4 concerns free boundary problems.

2. Variational Principle in Bundles of Embeddings

Let M be an n -manifold and Ω a given volume on M . Let $D \subseteq M$ a submanifold having dimension n with boundary $\partial D \subseteq M$.

The set

$$P = \{\eta: D \rightarrow M: \eta \text{ is an embedding}\}$$

is a principal bundle having structure group

$$G = \{g: D \rightarrow D: g \text{ is a diffeomorphism}\}$$

acting on P on the right, by composition of maps.

Similarly, we define $P_{\text{vol}}, G_{\text{vol}}$, by adding the further incompressibility condition, namely

$$\eta * \Omega = \Omega, \quad g * \Omega = \Omega,$$

where the star means the pull-back operation.

A typical example of this situation to be considered afterwards with some more detail, is the liquid drop D moving freely in $M = \mathbb{R}^3$ with

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3.$$

Thus, at each instant of time t , the element $\eta_t: D \rightarrow \mathbb{R}^3$ of P represents the position of the fluid particles at that instant namely, if $X = (X^1, X^2, X^3)$ are the coordinates of the position of a given particle at time $t = 0$, and $x = (x^1, x^2, x^3)$ the position of the same particle after the interval of time $[0, t]$ has passed, then $x = \eta_t(X)$. If the fluid is incompressible, then for each $X \in D$ and each t , we have

$$J_{\eta_t}(X) = 1$$

where J_{η_t} is the Jacobian of η_t . An equivalent condition is that $\eta_t * \Omega = \Omega$.

Now, back to the general situation, let $L: TM \rightarrow \mathbb{R}$ be a given lagrangian. This induces a Lagrangian $\mathcal{L}: TP \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(\eta, \dot{\eta}) = \int_D L(\eta(X), \dot{\eta}(X)) \Omega(X).$$

It is sometimes useful to think of $(\eta, \dot{\eta})$ as the derivative with respect to t of a curve $x = \eta_t(X)$, $X \in M$. Thus

$$\dot{\eta}_t[\eta_t^{-1}(x)] = \frac{\partial \eta_t(\eta_t^{-1}(x))}{\partial t}$$

represents a vector field on $D_t = \eta_t(D) \subseteq M$.

Of course, \mathcal{L} has a restriction

$$\mathcal{L}: TP_{\text{vol}} \rightarrow \mathbb{R}.$$

Let $\eta_t; t \in [t_0, t_1]$ be a curve in $P_{\text{vol}} \subseteq P$. Thus, for each $t \in [t_0, t_1]$, $\eta_t: D \rightarrow M$ is a volume preserving diffeomorphism. Now consider the following functional with $\eta_{t_0} = \eta_0, \eta_{t_1} = \eta_1$ fixed, defined on the curves (η_t, λ_t) where λ_t is a curve on the set $\mathfrak{F}(D)$ of real valued C^∞ functions on D

$$\begin{aligned} \mathcal{Q}(\eta, \lambda) &= \int_{t_0}^{t_1} \left[\mathcal{L}(\eta, \dot{\eta}) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt \\ &= \int_{t_0}^{t_1} dt \int_D \left[L(\eta_t(X), \dot{\eta}_t(X)) + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] \Omega(X). \end{aligned}$$

The constraint $J_{\eta_t} = \text{constant}$, or equivalently $\frac{dJ_{\eta_t}}{dt} = 0$ with the Lagrange multiplier $\lambda_t \in \mathfrak{F}(D)$ together with the condition $J_{\eta_0} = 1$ gives the end this does not imply any loss of generality. Likewise, we can assume that

$$\Omega(X) = dX^1 \wedge \cdots \wedge dX^n.$$

This is because variational principles are essentially local in nature.

Sometimes we will write $\Omega(X) = d^3X$, whenever computations are simpler in the case $M = \mathbb{R}^3$.

Now think of a variation

$$\delta\eta_t = \left. \frac{d}{d\epsilon} \eta_{t\epsilon} \right|_{\epsilon=0}$$

such that

$$\delta\eta_{t_0} = 0, \quad \delta\eta_{t_1} = 0,$$

and $\eta_{t\epsilon}$ is a curve on P for each $\epsilon \in (-\epsilon_1, \epsilon_2)$. On the other hand assume that $\eta_t \equiv \eta_{t_0}$ is a curve on P_{vol} . If (η_t, λ_t) is a critical point of \mathcal{Q} , then we have

$$\delta\mathcal{Q} = \left. \frac{d}{d\epsilon} \mathcal{Q} \right|_{\epsilon=0} = 0.$$

This means that

$$\begin{aligned} 0 &= \delta \int_0^{t_1} dt \int_D \left\{ L[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] + \frac{d}{dt} J_{t\epsilon}(X) \lambda_t(X) \right\} d^3X \\ &= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial x} [\eta_t(X), \dot{\eta}_t(X)] \delta\eta_t + \frac{\partial L}{\partial \dot{x}} [\eta_t(X), \dot{\eta}_t(X)] \delta\dot{\eta}_t \right. \\ &\quad \left. + \frac{d}{dt} [\nabla \cdot (\partial\eta_t \circ \eta_t^{-1}) \circ \eta_t(X)] \lambda_t(X) \right\} d^3X. \quad (\delta) \end{aligned}$$

Since

$$\left. \frac{d}{d\epsilon} J_{\eta_{t\epsilon}}(X) \right|_{\epsilon=0} = \nabla \cdot (\delta\eta_t \circ \eta_t^{-1}) \circ \eta_t(X).$$

(To check this, let

$$v = \left. \frac{d\eta_\epsilon}{d\epsilon} \right|_{\epsilon=0} \circ \eta^{-1}(x).$$

Then

$$(\nabla \cdot v) \circ \eta = \left. \frac{dJ_{\eta_\epsilon}}{d\epsilon} \right|_{\epsilon=0}.$$

In fact we can assume without loss of generality that $\eta = \eta_0 = \text{identity}$. Thus

$$\left. \frac{d}{d\epsilon} \eta_\epsilon * (d^3x) \right|_{\epsilon=0} = \left. \frac{dJ_{\eta_\epsilon}}{d\epsilon} d^3X \right|_{\epsilon=0} = L_v(d^3x) = \nabla \cdot v d^3x.$$

Thus by applying integration by parts to (δ) , we get

$$0 = \int_{t_0}^{t_1} dt \int_D \left\{ \left[\frac{\partial L}{\partial x}(\eta_t, \dot{\eta}_t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(\eta_t, \dot{\eta}_t) \right] \delta\eta_t - \right. \\ \left. - [\nabla \cdot (\delta\eta_t \circ \eta_t^{-1})] \circ \eta_t \frac{d}{dt} \lambda_t \right\} (X) d^3X.$$

Since η_t is volume preserving we have $J_{\eta_t} = 1$, and then, by the change of variables formula for a multiple integral, we get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\left(\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) \right) \delta x_t - (\nabla \cdot \delta x_t) \eta_t(x) \right] d^3x.$$

where

$$\mu_t(x) = \frac{d}{dt} (\lambda_t \circ \eta_t^{-1})(x).$$

But from Gauss' divergence theorem we have

$$\int_\Omega (\nabla \cdot Y)(x) \mu(x) d^3x = \int_\Omega [\nabla \cdot (\mu Y) - \nabla \mu \cdot Y] d^3x = \\ = \int_{\partial\Omega} \mu(Y, \bar{n}) - \int_\Omega \nabla \mu \cdot Y.$$

Applying this formula we finally get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) + \nabla \mu_t(x) \right] \delta x_t d^3x - \int_{t_0}^{t_1} dt \int_{\partial \eta_t(D)} \mu(\delta x_t, \bar{n}).$$

At any event, we need the two separate integrals to be zero. From the first integral we have that

$$\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) = -\nabla \mu_t(x).$$

We consider two possibilities now.

(a) If the variations δx_t are arbitrary on the boundary, we need

$$\mu_t|_{\partial D} = 0.$$

(b) If the δx_t are parallel to the boundary, the second integral is automatically zero and there is no additional condition on μ_t .

Before we state our results, let us introduce some notation. The Euler-Lagrange operator will be denoted by $[L]_x$. In local coordinates

$$[L]_x = \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) \right] dx.$$

This is a well defined 1-form on M , once a curve $\eta_t(X) = x$ on P has been chosen. Here

$$\dot{x} = \dot{\eta}_t(X).$$

Summarizing the previous calculations we have

Lemma 1. *Let η_t be a curve on P_{vol} with $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed, and λ_t a curve on $\mathcal{F}(D)$, and let*

$$\mu_t = \frac{d\lambda_t}{dt} \circ \eta_t^{-1}.$$

Then the following statements are equivalent.

- (i) (η_t, λ_t) is a critical point of $\mathcal{G}(\eta, \lambda)$ in the set of curves (η_t, λ_t) such that $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$, and ∂D_t fixed (i.e. $\delta \eta_t \parallel \partial D_t$).
- (ii) $[L]_x = d\mu_t(x)$, $x \in \eta_t(D)$.

We will need now the following lemma, which gives a particular version of the Lagrange Multipliers Theorem.

Lemma 2. *Let η_t be a curve on P_{vol} . The following statements are equivalent*

- (i) η_t is a critical point of $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t) dt$ on curves η_t belonging to P_{vol} with fixed end points $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ and ∂D_t fixed (i.e. $\partial\eta_t \parallel \partial D_t$).
- (ii) There exists a curve $\lambda_t \in \mathcal{F}(D)$ such that (η_t, λ_t) is a critical point of the functional \mathcal{Q} on curves $\eta_t \in P$, $\lambda_t \in \mathcal{F}(D)$ with the conditions $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$.

PROOF. That (ii) implies (i) is easy to check.

To prove that (i) implies (ii), we must show the global existence of λ_t .

Let η_t be a curve on P_{vol} satisfying (i). A variation $\eta_{t\epsilon}$ of η_t on P_{vol} can be constructed as follows.

Let Z be a vector field on D which is divergence-free ($\text{div } Z = 0$) and parallel to the boundary ($Z \parallel \partial D$). Then for each t , $\eta_t * Z = Z_t$ is a vector field on $D_t = \eta_t(D)$ such that $\text{div } Z_t = 0$ and $Z_t \parallel \partial D_t$. Let $\varphi(t, \epsilon)$ be any real valued function defined for $t \in [t_0, t_1]$ and $\epsilon > 0$. For our particular purposes, φ will be taken to be a bump function in the variable t for each ϵ , approximating the Dirac Delta function at $T \in [t_0, t_1]$ and satisfying $\varphi(t_0, \epsilon) = \varphi(t_1, \epsilon) = 0$.

Define $Z_{t\epsilon} = \varphi(t, \epsilon)Z_t$. Thus for each t , $Z_{t\epsilon}$ satisfies $\text{div } Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$. For each t , let $F_{t\epsilon}$ be the flow of $Z_{t\epsilon}$ for $\epsilon > 0$. So for each t and ϵ , $F_{t\epsilon}: D_t \rightarrow D_t$ is a diffeomorphism. Define

$$\eta_{t\epsilon} = F_{t\epsilon} \circ \eta_t,$$

then $\eta_{t\epsilon}$ is a variation of η_t on P_{vol} satisfying

$$\eta_{t_0\epsilon} = \eta_0, \quad \eta_{t_1\epsilon} = \eta_1$$

and

$$\left(\frac{d\eta_{t\epsilon}}{d\epsilon}(X) \right) \parallel \partial D_t \quad \text{for all } \epsilon > 0.$$

In general, if we are given a vector field $Z_{t\epsilon}$ for each t, ϵ such that $\text{div } Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$ depending smoothly on the parameters, then we can construct in a similar way, a variation $\eta_{t\epsilon}$ as before.

Now we must compute

$$\begin{aligned} & \frac{d}{d\epsilon} \int_{t_0}^{t_1} \mathcal{L}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] dt = \frac{d}{d\epsilon} \int_{t_0}^{t_1} dt \int_D L[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] d^3X \\ &= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial X} [\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] - \frac{d}{dt} \frac{\partial L}{\partial \dot{X}} [\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] \right\} \frac{d}{d\epsilon} \eta_{t\epsilon}(X) d^3X. \end{aligned}$$

Since $\eta_{t\epsilon}$ is volume preserving we can change variables $x = \eta_{t\epsilon}(X)$, so that the right hand side equals

$$\int_{t_0}^{t_1} dt \int_{D_t} [L]_x Z_{t\epsilon}(x) d^3x.$$

Now choose $\varphi(t, \epsilon)$ such that $\varphi(t, \epsilon) \rightarrow \delta(t - T)$ for $\epsilon \rightarrow 0$. Then we finally get the condition

$$\int_{D_T} [L]_x Z_T(x) d^3x = 0.$$

At this point, we should remark that $Z_T = \eta_T * Z$ can be chosen to be an arbitrary vector field on D_T except for the conditions $\operatorname{div} Z_T = 0$, $Z_T \parallel \partial D_T$. By Hodge theorem, we can conclude that there exists μ_T globally defined on D_T and such that

$$[L]_x = d\mu_T(x), \quad x \in D_T.$$

We leave to the reader to check that even though $\mu_T(x)$ is determined up to a constant, however we can choose $\mu_T(x)$ to be a C^∞ function of x , T and satisfying the previous condition.

To finish the proof, define λ_t by

$$\lambda_t(X) = \int^t \mu_t \circ \eta_t(X) dt$$

and apply Lemma 1. \square

In order to state our main result, we need some notation. Let $\eta_t: M \rightarrow M$ be a curve on $\operatorname{Diff}(M)$ such that $J_{\eta_t} = 1$, i.e. η_t is volume preserving. For each $D \subseteq M$, a compact submanifold of M with boundary ∂D , define

$$\eta_t^D = \eta_t|_D$$

and denote by P^D the principal bundle of embeddings of D into M and by $P_{\text{vol}}^D \subseteq P^D$ the principal bundle of volume preserving embeddings. Define $\mathcal{L}_{\text{vol}}^D: TP_{\text{vol}}^D \rightarrow \mathbb{R}$ by

$$\mathcal{L}_{\text{vol}}^D(\eta_t^D, \dot{\eta}_t^D) = \int_D L[\eta_t^D(X), \dot{\eta}_t^D(X)] d^3X.$$

We also define for a given C^∞ curve λ_t on $\mathcal{F}(M)$, $\lambda_t^D = \lambda_t|_D$ and

$$\mathcal{L}^D[\eta_t^D, \dot{\eta}_t^D, \lambda_t^D, \dot{\lambda}_t^D] = \int_D \left[L(\eta_t^D(X), \dot{\eta}_t^D(X)) + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] d^3X.$$

Theorem. *The following conditions on a curve $\eta_t \in \operatorname{Diff}_{\text{vol}}(M)$ are equivalent.*

(i) *There exists $\mu_t: M \rightarrow \mathbb{R}$, a C^∞ curve on $\mathfrak{F}(M)$ such that*

$$[L]_{\eta_t(X)} = d\mu_t(\eta_t(X)).$$

(ii) *For each D , η_t^D is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X), \dot{\eta}_t(X)] dt$$

on the set of curves η_t on P_{vol}^D with fixed end point conditions $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ and ∂D_t fixed (i.e. $\delta\eta_t \parallel \partial D_t$).

(iii) *There exists a C^∞ curve $\lambda_t \in \mathfrak{F}(M)$ such that for each D , (η_t^D, λ_t^D) is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D(\eta_t, \dot{\eta}_t, \lambda_t, \dot{\lambda}_t) dt$$

on the set of curves $(\eta_t, \lambda_t) \in P^D \times \mathfrak{F}(D)$ with conditions $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ and ∂D_t fixed (i.e. $\delta\eta_t \parallel \partial D_t$).

Notice that η_t and λ_t are related by

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

PROOF. We first prove that (i) implies (ii). Let $D \subseteq M$ as before,

$$\mu_t^D = \mu_t|_{\eta_t(D)}$$

and

$$\lambda_t^D = \int^t \mu_t^D \circ \eta_t^D.$$

Thus by Lemma 1, (η_t^D, λ_t^D) is a critical point of $\mathcal{Q}(\eta^D, \lambda^D)$. By Lemma 2, we conclude that (ii) holds.

We now prove that (ii) implies (i). Using Lemma 2 and Lemma 1 we get for each D a function $\mu_t^D: \eta_t^D(D) \rightarrow \mathbb{R}$ such that

$$[L]_{\eta_t^D(X)} = d\mu_t^D[\eta_t^D(X)].$$

This shows that the Euler-Lagrange operator $[L]_{\eta_t^D(X)}$ is exact on $\eta_t^D(D)$. Since $\eta_t^D(D)$ can be chosen to be any given compact submanifold with boundary of M (having the same dimension as M), this immediately implies that μ_t^D can be taken as being the restriction to D of a globally defined 0-form μ_t .

Similarly, we can easily prove the equivalence between (iii) and (i) or (ii) by using Lemmas 1 and 2 if we define λ_t by

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

3. Hamiltonian Vector Fields

Let us consider a symplectic manifold M with volume element $\Omega = \omega^n$ where $\omega = d\alpha$ is its canonical 2-form and ω^n is the exterior power of order n . This problem was studied by Lacomba and Losco [4] for the case where M is a compact manifold with boundary.

We construct the principal fiber bundles P with structure group G and P_{vol} with structure group G_{vol} as in the general theory.

In this case we define, for a given curve η_t in $P_{\text{vol}} \subseteq P$ and the corresponding curve $Z_t = \dot{\eta}_t \cdot \eta_t^{-1}$ of vector fields on M , the Lagrangian

$$L[\eta_t(X), \dot{\eta}_t(X)] = i_{Z_t(X)}\alpha = \alpha[Z_t(X)].$$

For any compact submanifold with boundary $D \subseteq M$ as in Section 2, this induces a Lagrangian $\mathcal{L}_{\text{vol}}^D: TP_{\text{vol}}^D \rightarrow \mathbb{R}$ by

$$\mathcal{L}_{\text{vol}}^D(\eta_t^D, \dot{\eta}_t^D) = \int_D L[\eta_t^D(X), \dot{\eta}_t^D(X)] d^{2n}x.$$

If η_t^D is a critical point of $\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X), \dot{\eta}_t(X)] dt$ on the set of curves η_t on P_{vol}^D with fixed endpoint conditions $\eta_{t_0} = \eta_{t_0}^D, \eta_{t_1} = \eta_{t_1}^D$ and ∂D_t fixed, the main result implies the existence of a C^∞ curve $\mu_t: M \rightarrow \mathbb{R}$. From [4] and considering each $D \subseteq M$, we conclude that $\mu_t = H_t$ is a Hamiltonian function and Z_t is the associated Hamiltonian vector field. This means that the critical curves preserve not only the volume Ω , but also the symplectic form ω .

Notice that the arbitrariness of H_t permits to get any given Hamiltonian vector field.

We remark that this construction is still valid if (M, ω) is a non exact symplectic manifold. Since ω is closed, consider two different local expressions $\omega = d\alpha$ and $\omega = d\tilde{\alpha}$. Hence, $\tilde{\alpha} = \alpha + \gamma$ where γ is a closed form.

They produce two different but equivalent Lagrangians L, \tilde{L} such that $\tilde{L} = L + \gamma$. It can be proved that the corresponding integrals

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X), \dot{\eta}_t(X)] dt$$

give the same variational principle.

Indeed, we can write in local coordinates

$$\begin{aligned} \delta \int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X), \dot{\eta}_t(X)] dt &= \int_{t_0}^{t_1} dt \int_D \{ \omega[\dot{\eta}_t(X), \delta\eta_t(X)] - d\mu[\dot{\eta}_t(X), \delta\eta_t(X)] \} d^{2n}x \\ &= \int_{t_0}^{t_1} dt \int_{D_t} [\omega(\dot{x}, \delta x) - dH(\dot{x}, \delta x)] d^{2n}x. \end{aligned}$$

A related result for non exact symplectic manifolds appears in [1].

4. Free Boundary Problems

Free boundary problems, like a liquid incompressible homogeneous drop with surface tension, or a free elastic body, can be studied by using methods like those described in the previous sections. In this paper we will concentrate on the example of the liquid drop. A setting for this, from the Hamiltonian point of view can be found in [5]. However we may use part of that framework for our purposes, within the variational approach.

Let us denote P_{vol} the principal bundle of embeddings of the unit closed ball $D \subseteq \mathbb{R}^3$ into \mathbb{R}^3 . Thus a curve $\eta_t \in P_{\text{vol}}$ represents a motion of the liquid drop. Note that the base B of the bundle P_{vol} consists of the set of all $\Sigma \subseteq \mathbb{R}^3$ where Σ is a 2-submanifold of \mathbb{R}^3 diffeomorphic to ∂D . Obviously every $\Sigma \in B$ can be written $\Sigma = \eta(\partial D)$ for some $\eta \in P_{\text{vol}}$. The group acting on P_{vol} on the right, is $G_{\text{vol}} = \text{Diff}_{\text{vol}}(D)$.

The surface tension coefficient being τ and the density being 1 and assuming that gravitational forces are absent, we can write the Lagrangian for the liquid drop as follows

$$\mathcal{L}(\eta_t, \dot{\eta}_t) = \int_D \frac{1}{2} [\dot{\eta}_t(X)]^2 d^3X - \tau \int_{\Sigma_t} d\Sigma_t$$

where $d\Sigma_t$ represents the area element on $\Sigma_t = \eta_t(\partial D)$.

Now suppose that η_t is a curve on P_{vol} which is a critical point of the functional $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t) dt$ on the set of curves η_t such that $\eta_{t_0} = \eta_0, \eta_{t_1} = \eta_1$ fixed (note that we are not imposing here the condition ∂D_t fixed; thus variations δv_t are allowed such that they are not necessarily assumed to be parallel to ∂D_t).

As we did in Section 2 where we assumed the condition $\delta \eta_t \parallel \partial D_t$, we can show that the problem of finding a critical curve η_t as stated above is equivalent to the problem at finding a critical curve (η_t, λ_t) of the functional

$$\mathcal{Q}(\eta, \lambda) = \int_{t_0}^{t_1} \left[\mathcal{L}(\eta, \dot{\eta}) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt$$

on curves (η_t, λ_t) with $\lambda \in \mathcal{F}(D), \eta_t \in P, \eta_{t_0} = \eta_0, \eta_{t_1} = \eta_1$ fixed.

By a similar computation to the one performed before the statement of Lemma 1 in Section 2, we can find that for a variation $\delta \eta_t$ with $\delta \eta_{t_0} = 0, \delta \eta_{t_1} = 0$, we have

$$0 = \int_{t_0}^{t_1} \int \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) + \nabla \mu_t(x) \right] \delta x_t d^3x - \int_{t_0}^{t_1} dt \int_{\partial D_t} \mu_t(x)(\delta x_t, \bar{n}) d\Sigma - \tau \int_{t_0}^{t_1} dt \int_{\partial D_t} K(x)(\delta x_t, \bar{n}) d\Sigma$$

where K is the mean curvature of ∂D_t and integrals on ∂D_t are both surface integrals (here, we are implicitly assuming the standard Riemannian metric given on \mathbb{R}^3).

The last term comes out as follows (see [5] for more details). A given variation $\eta_{t\epsilon}$ induces a variation $\eta_{t\epsilon}(\partial D) = D_{t\epsilon}$. Thus

$$\frac{d}{d\epsilon} \int_{D_{t\epsilon}} d\Sigma \Big|_{\epsilon=0} = \int_{D_t} K(\delta x_t, \bar{n}) d\Sigma.$$

A simple argument shows that equality to 0 for arbitrary variations δx_t will imply

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= -\nabla \mu_t, \quad \text{for } x \in D_t, \\ \mu_t(x) &= -\tau K(x), \quad \text{for } x \in \partial D_t, \end{aligned}$$

the incompressibility condition $J_{\eta_t} = 1$ comes out after variations $\delta \lambda_t$ are considered, as usual. Putting all this together and taking into account that

$$L(x, \dot{x}) = \frac{1}{2} \dot{x}^2,$$

we finally get

$$\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= -\nabla \mu_t \circ \eta_t, \quad \text{on } D \\ \mu_t &= \tau K, \quad \text{on } \partial D_t \\ J_{\eta_t} &= 1, \quad \text{on } D. \end{aligned}$$

We can write these equations in Eulerian (rather than Lagrangian) variables. Namely let

$$v = \frac{\partial x}{\partial t} \circ \eta_t^{-1}$$

be the Eulerian velocity.

Then we get

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v$$

and $J_{\eta_t} = 1$ implies $\nabla \cdot v = 0$. Thus

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla \mu, \quad \text{on } D_t,$$

$$\begin{aligned}\nabla \cdot v &= 0, & \text{on } D_t, \\ \mu_t &= 2K, & \text{on } \partial D_t.\end{aligned}$$

These are precisely the equations of motion of the liquid drop with surface tension.

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