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On the Angular Boundedness of Bloch Functions

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1. Introduction

Let D be the unit disk and $T = \partial D$. A Bloch function [12, p. 268] is a function f analytic in D such that

$$\|f\|_* = |f(0)| + \sup (1 - |z|^2)|f'(z)| < +\infty .$$

With this norm the Bloch functions form a Banach space $\mathcal{B}$. The closure in $\mathcal{B}$ of the polynomials is a subspace $\mathcal{B}_0$ that consists of all $f \in \mathcal{B}$ such that

$$(1 - |z|^2)|f'(z)| \to 0 \quad \text{as} \quad |z| \to 1 .$$

For Bloch functions radial and angular limits are identical. Furthermore, a Bloch function is radially bounded at a point of T if and only if it is angularly bounded at this point [12, p. 269].

In this paper we deal with the size of the set

$$B_f = \{ \zeta \in T : \overline{\lim}_{r \to 1} |f(r\zeta)| < +\infty \} .$$

There are Bloch functions [11] that do not have a radial limit at any point of T, but it is known [6] that for each Bloch function f, the set B_f is an uncountable dense set.

It was asked in [4] whether all $f \in \mathcal{B}$ satisfy $\dim B_f = 1$, where dim denotes the Hausdorff dimension and, as far as we know, this question remains open. One has to remark that it is not possible to replace dim 1 by positive (Lebesgue) measure, since there are Bloch functions f such that the corresponding set B_f has zero measure (see the comment after Theorem 4).

In this paper we develop a simplified version of a method of Noshiro [10] based in the Ahlfors'theory of covering surfaces, that may go further than used here, and apply it to prove that the set B_f has positive logarithmic capacity when $f \in \mathcal{B}$. In fact, using a sort of localization of Bloch functions we prove that $\operatorname{Cap}(B_f \cap I) > 0$ for every arc I, $I \subset T$. This is done in Sections 2 and 3.

The localization that we use, when applied to many functions in $\mathcal{B}_0$, gives a new way to obtain inner functions in $\mathcal{B}_0$. In connection with these functions T. Wolff asked [3] if the singular set of each inner function in $\mathcal{B}_0$ has Hausdorff dimension one and proved [16], by means of Noshiro's method, that this singular set has positive capacity. † In fact, the present paper is a development of Wolff's ideas.

In Section 4 we give a modification of Noshiro's method, using the equilibrium potential rather than the Evan's potential, that leads to a lower bound for the capacity of B_f. More precisely we prove that if $f \in \mathcal{B}$ and $f(z_0) = 0$, then there is a set $A_f \subset T$ such that

$$\overline{\lim}\, |f(r\zeta)| \leqslant k\|f\|_* \quad \text{for each} \quad \zeta \in A_f,$$

and

$$\operatorname{Cap} A_f \geqslant (1 - |z_0|^2) \exp\left[-\frac{k\|f\|_*^2}{(1 - |z_0|^2)^2 |f'(z_0)|^2} \right],$$

where k is an absolute constant.

2. The use of Ahlfors' Covering Theorem

Let f be a non-constant analytic function in D. For $z_0 \in D$ and $r > 0$ let $\Omega(z_0, r)$ denote the component of $\{z \in D: |f(z) - f(z_0)| < r\}$ that contains z_0. As a first result we can improve Theorem 1 of [6] by means of the following.

Theorem 1. *Let $f \in \mathcal{B}$. If $r > e\|f\|_*$, then* $\operatorname{Cap}(\partial\Omega(z_0, r) \cap T) > 0$ *for each* $z_0 \in D$.

To prove this theorem we use a consequence of the Ahlfors' Covering Theorem that we state in the following form [15, p. 255].

† After this paper was submitted for publication we received the preprint «Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch class» by G. Hungerford in which Wolff's conjecture is proved. (Thesis, California Institute of Technology, Pasadena, Cal., 1988.)

Lemma 1. *Let G be a simply connected domain, let H be a disk and let H_1, H_2 be disks contained in H with disjoint closures. Let f be an analytic function in G with $f(G) \subset H$. For $k = 1, 2$ let N_k be the number of domains $\Omega \subset G$ that satisfy* (i) $\bar{\Omega} \subset G$, (ii) $f(\Omega) = H_k$. *Then, for each domain $U \subset G$ such that ∂U is piecewise analytic and f is analytic on ∂U, one has*

$$N_1 + N_2 \geqslant \int_U |f'(z)|^2 \, dm(z) - k_0 \int_G |f'(z)| \, |dz|,$$

where k_0 is a constant that depends only on H, H_1 and H_2 and C is the relative boundary,

$$C = \{z \in \partial U : f(z) \in H\}.$$

PROOF OF THEOREM 1. We can suppose that $z_0 = 0$, $f(0) = 0$ and $\|f\|_* = 1$. Put $E = \partial\Omega(0, r) \cap T$ with $r > e$, suppose that $\operatorname{Cap} E = 0$ and let u be the Evan's potential for E [15, p. 75]. For each $\rho < \infty$ set

$$U_\rho = \{z \in \Omega(0, r) : u(z) < \rho\}, \qquad C_\rho = \{z \in \Omega(0, r) : u(z) = \rho\}.$$

We shall apply Lemma 1 with $G = \Omega(0, r)$ and $U = U_\rho$. We claim now that

$$\liminf_{\rho \to \infty} \frac{L_\rho}{S_\rho} = 0,$$

where

$$L_\rho = \int_{C_\rho} |f'(z)| \, |dz| \quad \text{and} \quad S_\rho = \int_{U_\rho} |f'(z)|^2 \, dm(z).$$

The claim is proved in [10], [15] but for later use we do the necessary calculations in a more direct way.

Assume that the claim is false, so that

$$L_\rho \geqslant c S_\rho \quad \text{for} \quad \rho \geqslant \rho_0, \quad c > 0.$$

The Schwarz inequality yields

$$\begin{aligned} L_\rho^2 &= \left(\int_{C_\rho} \frac{|f'(z)|}{|\nabla u(z)|^{1/2}} |\nabla u(z)|^{1/2} |dz| \right)^2 \\ &\leqslant \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz| \int_{C_\rho} |\nabla u(z)| \, |dz| \\ &\leqslant \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz| \cdot \int_{u=\rho} \frac{\partial u}{\partial n}(z) \, |dz| = 2\pi \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz|, \end{aligned}$$

where $\partial u/\partial n$ is the derivative of u with respect to the normal to $\{u=\rho\}$. The crucial fact that the integral of this derivative along $\{u=\rho\}$ equals 2π was used. For S_ρ we can use the co-area formula [9, p. 37]

$$S_\rho - S_{\rho'} = \int_{\rho'}^{\rho} dr \int_{C_r} \frac{|f'(z)|^2}{|\nabla u(z)|}\,|dz|,$$

for adequate $\rho' > 0$.

This shows that S_ρ is an absolutely continuous function of ρ and

$$S'_\rho = \frac{dS_\rho}{d\rho} = \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|}\,|dz| \quad \text{a.e. } (\rho).$$

So we get

$$L_\rho^2 \leqslant 2\pi S'_\rho \quad \text{a.e. for} \quad \rho \geqslant \rho_0$$

and

$$S_\rho^2 \leqslant \frac{1}{c^2} L_\rho^2 \leqslant c_1 S'_\rho \quad \text{a.e.}$$

Integrating from ρ_0 to ρ one gets

$$\frac{1}{c_1}(\rho - \rho_0) \leqslant \int_{\rho_0}^{\rho} \frac{S'_\rho}{S_\rho^2}\,d\rho \leqslant \frac{1}{S_{\rho_0}} \quad \text{for} \quad \rho \geqslant \rho_0,$$

which is impossible, and the claim is proved.

Let us take now for H_1, H_2 two discs of disjoint closures and radii s, $e < 2s < r$, contained in $\{|w| < r\}$. Acording to Lemma 1 we get

$$N_1 + N_2 \geqslant S_\rho - k_0 L_\rho = S_\rho\left(1 - k_0 \frac{L_\rho}{S_\rho}\right) > 0 \quad \text{for} \quad \rho \geqslant \rho_1.$$

Applying [6, Lemma 2] we obtain

$$1 \geqslant \sup\{(1 - |z|^2)|f'(z)| : z \in \Omega(0, r)\} \geqslant 2e^{-1}s,$$

a contradiction. □

Corallary. *Let $f \in \mathcal{B}$ and $f(0) = 0$. Then there is a set $A_f \subset T$ such that* $\operatorname{Cap} A_f > 0$ *and*

$$\overline{\lim}\,|f(r\zeta)| \leqslant k\|f\|_* \quad \textit{for each} \quad \zeta \in A_f,$$

where k is an absolute constant.

Proof. We can suppose that $\|f\|_* = 1$. Applying Theorem 1 with $r = 3$ and $A_f = \partial\Omega(0, r) \cap T$ we get $\operatorname{Cap} A_f > 0$. Moreover, for $\zeta \in A_f$ there is a curve γ in $\Omega(0, r)$ ending at ζ. So $|f(z)| \leqslant 3$, $z \in \gamma$ and, since $f \in \mathcal{B}$, we get $\overline{\lim}_{r\to 1} |f(r\zeta)| \leqslant k$, for each $\zeta \in A_f$, according to [1, Theorem 4.2]. □

3. The Local Results

In order to prove that the set B_f has locally positive capacity when $f \in \mathcal{B}$, we improve the conclusion of Theorem 1. We follow the notation of the beginning of Section 2.

Lemma 2. *Let $f \in \mathcal{B}$ and $r > 0$. Let I be an open arc in T and $z_0 \in D$. If $\partial\Omega(z_0, r) \cap I \neq \varnothing$, then* $\operatorname{Cap}(\partial\Omega(z_0, r') \cap I) > 0$ *for each $r' > r$.*

Proof. Write $G_s = \Omega(z_0, s)$ and $E_s = \partial G_s$ for each $s \geqslant r$. Let $I \subset T$ be an arc and suppose that $I \cap E_r \neq \varnothing$. We can assume that $I \cap E_r$ has zero Lebesgue measure. The boundary of G_r is formed by a sequence of Jordan arcs that end at two points of T [8, p. 10]. Since $I \cap E_r$ has no interior points we can take an arc $I_0 \subset I$ whose extreme points A_0, B_0 are the mid points of arcs in $T \backslash E_r$. We can join A_0 and B_0 to points A_r, B_r on E_r by means of arcs contained in $D \backslash G_r$. Take now a Jordan arc Γ_r from A_r to B_r with $\Gamma_r \subset G_r$. This arc separates G_r into two parts. Let F_r be the part of G_r with $\bar{F}_r \cap I_0 \neq \varnothing$. So $\partial F_r \cap T \subset I_0 \cap E_r$.

Let us assume that $\operatorname{Cap}(\partial F_r \cap T) = 0$. Using the notation of Theorem 1 with F_r instead of G we see that $\{z \in \partial U_\rho : |f(z) - f(z_0)| < r\}$ is formed by $C_\rho = \{z \in G_0 : u(z) = \rho\}$ plus the arc Γ_r for large enough ρ. The argument used in this proof shows that

$$\lim_{\rho\to\infty} \frac{L_\rho + \int_\Gamma |f'(z)|\,|dz|}{S_\rho} = 0$$

and Lemma 1 leads to a contradiction.

If g_r has no singularity in γ_r then for $r' > r$, close enough to r, we can repeat the argument with $F_{r'}$ instead of F_r. Now $g_{r'}$ has some singularity in $\gamma_{r'}$ because, if this is not the case, $g_{r'}$ would be analytic through γ'_r. Then the pre-image by f of an arc contained in $|z| = r$ would be a compact set in $F_{r'}$. But it contains some point of $T \cap \partial F_r \neq \varnothing$ and this is a contradiction. □

It is possible that $\operatorname{Cap}(\partial\Omega(z, r) \cap I) = 0$ even with the additional hypothesis $r > e\|f\|_*$. We thank the referee because his observation about the previous statement of Lemma 2 allowed us to correct it.

Theorem 2. *If $f \in \mathcal{B}$, there is a set $E_f \subset T$ such that* $\operatorname{Cap}(I \cap E_f) > 0$ *for each arc $I \subset T$ and $\overline{\lim}_{r\to 1} |f(r\zeta)| < +\infty$ for $\zeta \in E_f$.*

PROOF. Suppose $f(0) = 0$ and $\|f\|_* = 1$. Consider the components of $H = f^{-1}(\{|w| < 3\})$. Each of them touches T; see [13] or Theorem 1 in [6]. Put $E_f = E_1 \cup E_2$ where

$$E_1 = \{\zeta \in T: \zeta \text{ is in the closure of some component of } H\},$$
$$E_2 = \{\zeta \in T: \lim_{r\to 1} f(r\zeta) \neq \infty \text{ exists}\}.$$

Since $f \in \mathcal{B}$ we have $\lim |f(r\zeta)| < \infty$ at each point of E_f. Given an arc $I \subset T$, if $|I \cap E_2| = |I|$ it is clear that $\operatorname{Cap}(E_f \cap I) > 0$. So, let us assume that $|I \cap E_2| < |I|$. In this case there is some point $\zeta \in T$ which is a Plessner point for f [12, p. 324]. So there are points $z_n \to \zeta$, $z_n \in D$ with $f(z_n) \to 0$. If there are only a finite number of components of H some one has to touch T on I. If there are infinitely many, by a result of McLane [8, p. 10], it is not possible that all of these components do not touch I, because then their diameter would not go to zero. So, in this case some component has to touch T on I. Now we apply Lemma 2. □

Remark. If the function f has angular limits almost nowhere on T the above proof shows that, in this case, there is a set $E_f \subset T$ with $\operatorname{Cap}(I \cap E_f) > 0$ for each arc $I \subset T$ and

$$\overline{\lim}\, |f(r\zeta) - f(0)| < k\|f\|_*,$$

for all $\zeta \in E_f$, where k is an absolute constant.

Let f be an inner function and let $S(f)$ denote its singular set, the set of points of T at which f has no analytic continuation. We can improve the result of Wolff [16], which was the starting point of the present paper.

Theorem 3. *Let f be an inner function and suppose that $\|f\|_* < e^{-1}$ or that $f \in \mathcal{B}_0$. Then for each arc $I \subset T$ one has $I \cap S(f) = \varnothing$ or* $\operatorname{Cap}(I \cap S(f)) > 0$.

PROOF. Assume first $\|f\|_* < e^{-1}$ and $f(0) = 0$. Take r with $e\|f\|_* r < 1$ and consider the components of $f^{-1}(\{|w| < r\})$. It is clear that $\partial\Omega(z, r) \cap T \subset S(f)$. Now given an arc I with $I \cap S(f) \neq \varnothing$, there are points $\zeta \in I \cap S(f)$ and $z_n \to \zeta$ with $f(z_n) \to 0$ and we can follow the same argument as in the proof of Theorem 2. Consider now that $f \in \mathcal{B}_0$, by use of Frostman's Theorem we may assume without loss of generality that f is an infinite Blaschke product. Given an arc $I \subset T$ with $I \cap S(f) \neq \varnothing$ we can find components $\Omega(z, r)$ with $r < 1$ such that $\partial\Omega(z, r) \subset I \cap S(f)$. Now the domains U_ρ used in the proof of Theorem 1 satisfy also $S_\rho \to \infty$ as $\rho \to \infty$ and the assumption $\operatorname{Cap}(I \cap S(f)) = 0$ leads to

$N_1 + N_2 = +\infty$ in Lemma 1 for two discs H_1, H_2 contained in $\{|w| < r\}$. This contradicts the geometric characterization of $\mathcal{B}_0$ functions given by Theorem 1, (i) of [13]. □

Let f be a non constant analytic function in $\boldsymbol{D}$ with $f(0) = 0$ and let $G = \Omega(z, r)$ be a component of $f^{-1}(\{|w| < r\})$. If φ is a conformal mapping from $\boldsymbol{D}$ onto G, the function $g = f \circ \varphi$ reproduces in the unit disk the local behaviour of f. This localization has been used in the proof of Lemma 2 and we will show now that it can be used to produce inner functions in $\mathcal{B}_0$ with some additional properties.

Theorem 4. *Suppose that f is a function in $\mathcal{B}_0$ that has angular limits almost nowhere. If φ maps $\boldsymbol{D}$ conformally onto a component of $f^{-1}(\boldsymbol{D})$ then $f \circ \varphi$ is an inner function in $\mathcal{B}_0$. Furthermore, if f has Hadamard gaps then $f \circ \varphi$ assumes every value in $\boldsymbol{D}$ infinitely often.*

A (lacunary) series with Hadamard gaps has the form

$$f(z) = \sum_{k=0}^{\infty} b_k z^{n_k}, \qquad \frac{n_{k+1}}{n_k} \geqslant \lambda > 1 \quad (k = 0, 1, \ldots).$$

This function belongs to $\mathcal{B}_0$ if and only if $b_k \to 0$ as $k \to \infty$, [1], and is radially bounded on a set of positive measure if and only if $\sum |b_k|^2 < +\infty$ [17, vol. 1, p. 203]. Hence

$$f_0 = \sum_{k=1}^{\infty} k^{-1/2} z^{2^k}$$

is an example that has all the properties required in the theorem; note that the angular limit ∞ occurs almost nowhere by the Privalov Uniqueness Theorem.

PROOF. We have

$$(1 - |z|^2)\left|\frac{d}{dz} f(\varphi(z))\right| = \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |\varphi(z)|^2)|f'(\varphi(z))|.$$

If $|\varphi(z_n)| \to 1$ then the last factor tends to 0 because $f \in \mathcal{B}_0$ while the quotient is bounded by 1. If however $\limsup |\varphi(z_n)| < 1$ then the quotient tends to 0 because φ is univalent while the last factor is bounded. The fact that g is inner is a consequence of Loewner's Lemma as used in the proof of Lemma 2, (see also [15, p. 323]).

Suppose now that f has Hadamard gaps. Let w_0 be a point with $|w_0| = r$ and $f(z) \neq w_0$ for each z with $f'(z) = 0$. We claim that there are infinitely many points $z_n \in \partial\varphi(D)$ with $f(z_n) = w_0$. If the number of these points were finite we could then draw a curve $\gamma \subset \partial\varphi(D)$ with $\bar{\gamma} \cap T \neq \emptyset$ and f would map γ in a finite-to-one manner on an arc of finite length in $|w| = r$. Since $\sum_{n=1}^{\infty} |b_n| = +\infty$ this would contradict Theorem 1 of [5].

Now take a point w, $|w| < r$, and let Γ be a rectifiable Jordan arc from w_0 to w, with $f'(z) \neq 0$ when $f(z) \in \Gamma \setminus \{w\}$. Let us consider the components Γ_n of $f^{-1}(\Gamma)$ with $z_n \in \Gamma_n$ since Γ has finite length it follows from [5] that Γ_n is a Jordan arc and $f(\Gamma_n) = \Gamma$. We conclude that there are distinct points $z'_n \in \Gamma_n$ with $f(z'_n) = w$. $\square$

Remark. Previously, the known ways to construct inner functions in $\mathcal{B}_0$ were the use of a singular measure whose primitive is in the Zygmund class λ_* or the more geometric one of [14], by means of the Riemann surface of the function. T. Wolff asked if $\dim S(f) = 1$ when $f \in \mathcal{B}_0$ is inner. The corresponding result for inner functions omitting some values is true [3]. Theorem 4 shows that this gives no aid in order to answer Wolff's question. Theorem 4 shows that there is a close relation between the size of $\partial\Omega(z, r) \cap T$ for a function in $\mathcal{B}_0$ and the answer to Wolff's conjecture.

4. A Lower Bound for Capacities

In this Section we present a modification of the idea of Noshiro using the equilibrium potential rather than the Evans's potential. It leads to a lower bound for the capacity of the set B_f.

The following Lemma contains the basic estimate.

Lemma 3. *Let f be analytic in D and continuous in $\bar{D}$ and let $f(D) \subset D$, $f(0) = 0$, $f'(0) \neq 0$. We assume that $\|f\|_* < 1/3$ and that the set*

$$B = \{\zeta \in T : |f(\zeta)| < 1\}$$

consists of a finite number of arcs. Then

$$\operatorname{cap} B \geqslant \exp\left(-\frac{k}{|f'(0)|^2}\right),$$

k being an absolute constant.

PROOF. The set $A = \bar{B}$ is regular for the Dirichlet problem with respect to $C \setminus A$. If we take the equilibrium potential v, we have $v(z) \leqslant V$ and $v(z) = V$ if $z \in A$, where $\operatorname{Cap} A = \exp(-V)$.

For $\rho < V$ let

$$G_\rho = \{z \in D: v(z) < \rho\},$$
$$C_\rho = \{z \in \partial G_\rho: |f(z)| < 1\} = D \cap \partial G_\rho.$$

Also put

$$L_\rho = \int_{C_\rho} |f'(z)|\,|ds|, \qquad S_\rho = \int_{G\rho} |f'(z)|^2\,dm(z).$$

Take now two disks $H_1, H_2 \subset D$ with disjoint closures and radii 5/11. Then, by Lemma 2 of [6] and since $\|f\|_* < 1/3$, we see that here is no domain $\Omega, \bar{\Omega} \subset D$ with $f(\Omega) = H_1$ or $f(\Omega) = H_2$. Now, by the reflection principle, f has an analytic extension to $T \setminus B$, so we can apply Lemma 1, and we get $S_\rho \leqslant k_0 L_\rho$ for $\rho < V$, k_0 being some constant. Moreover the same calculation performed in the proof of theorem 1 yields

$$L_\rho^2 \leqslant 2\pi S'_\rho \quad \text{a.e.} \quad (\rho)$$

and so $S_\rho^2 \leqslant k_1 S'_\rho$ a.e. $\rho < V$. Now we remark that there is a number ρ_0, independent of f, such that $D(0, 1/2) \subset G_{\rho_0}$. To see this we can suppose $\operatorname{Cap} A \leqslant 1/2$ or $V \geqslant \log 2$. In this case for $|z| < 1/2$ and $\zeta \in A$ one has $|z - \zeta| \geqslant 1/2$ and

$$v(z) = \int_A \log \frac{1}{|z - \zeta|}\,d\mu(\zeta) \leqslant \log 2.$$

Then we can take $\rho_0 = \log 2$ to guarantee that G_{ρ_0} contains $D(0, 1/2)$.

Integrating now the inequality $S_\rho / S_\rho^2 \geqslant k_2$ a.e. from ρ_0 to V we get

$$\frac{1}{S_{\rho_0}} \geqslant \frac{1}{S_{\rho_0}} - \frac{1}{S_V} \geqslant k_2(V - \rho_0) \quad \text{or} \quad V \leqslant \rho_0 + \frac{k_3}{S_{\rho_0}}.$$

Taking into account that S_{ρ_0} is the area (counting multiplicities) of the image through f of G_{ρ_0} and the inclusion $G_{\rho_0} \supset D(0, 1/2)$, we get by means of Bloch's Theorem [15, p. 262] that

$$S_{\rho_0} \geqslant k_4 |f'(0)|^2.$$

This inequality and the previous one give the lemma. □

Lemma 4. *Let f be analytic in D with $f(z_0) = 0$ and let G be a component of $f^{-1}(\{|w| < 1\})$ containing z_0. We write $\delta(z) = \operatorname{dist}(z, \partial G)$ and assume*

$$\sup_{z \in G} \delta(z) |f'(z)| < \frac{1}{12}.$$

Then

$$\operatorname{Cap}(\partial G \cap T) \geqslant (1 - |z_0|) \exp\left(-\frac{k}{\delta(z_0)^2 |f'(z_0)|^2}\right),$$

where k is an absolute constant.

PROOF. For each r, $|z_0| < r < 1$ let $G(r)$ be the component of $G \cap \{|z| < r\}$ containing z_0. Since $G(r)$ is simply connected we can take a conformal mapping φ_r from D onto $G(r)$ with $\varphi_r(0) = z_0$. Writing $\tilde{f}_r = f \circ \varphi_r$ we see that $\tilde{f}_r$ is analytic in D, continuous on $\bar{D}$ and the set

$$B_r = \{z \in D: |\tilde{f}_r(z)| < 1\}$$

consists of a finite number of arcs. Moreover

$$\tilde{f}_r(0) = 0, \qquad \tilde{f}'_r(0) = f'(z_0)\varphi'_r(0).$$

Also

$$\|\tilde{f}_r\|_* = \sup_{w \in D} (1 - |w|^2)|\varphi'_r(w)|\,|f'(\varphi_r(w))|.$$

The Koebe distorsion Theorem [12, p. 22] gives

$$(1 - |w|^2)|\varphi'_r(w)| \leqslant 4 \operatorname{dist}(\varphi_r(w), \partial G(r)) \leqslant 4\delta(\varphi_r(w)),$$

and we conclude that $\|\tilde{f}_r\|_* < 1/3$. Furthermore

$$|\tilde{f}'_r(0)| \geqslant \operatorname{dist}(z_0, \partial G(r))|f'(z_0)| \geqslant \frac{1}{2}\delta(z_0)|f'(z_0)|,$$

if r is big enough. So Lemma 3 implies

$$\operatorname{Cap} B_r \geqslant \exp\left(-\frac{k}{\delta(z_0)^2 |f'(z_0)|^2}\right).$$

If we write $L(r) = \partial G(r) \cap \{|z| = r\}$, one has $L(r) \supset \varphi_r(B_r)$. Furthermore the mapping $h = \tau_{z_0/r} \circ (\varphi_r/r)$ where $\tau_{z_0/r}$ is the automorphism of D sending z_0/r to 0, satisfies $h(B_r) \subset T$ and $h(0) = 0$. Then applying [12, p. 348] and Schwarz's Lemma we obtain $\operatorname{Cap} h(B_r) \geqslant \operatorname{Cap} B_r$.

Moreover, [7, p. 138] gives

$$\operatorname{Cap} \varphi_r(B_r) \geqslant \frac{1}{2}\left(1 - \frac{|z_0|}{r}\right) \operatorname{Cap} h(B_r) \geqslant \frac{1}{2} r \left(1 - \frac{|z_0|}{r}\right) \operatorname{Cap} B_r.$$

So

$$\operatorname{Cap} L(r) \geqslant \operatorname{Cap} \varphi_r(B_r) \geqslant \frac{1}{2} r \left(1 - \frac{|z_0|}{r}\right) \exp\left(-\frac{k}{\delta(z_0)^2 |f'(z_0)|^2}\right).$$

Considering now the compact sets $E_r = \{z \in \bar{G}: |z| \geqslant r\}$ we have $\operatorname{Cap} E_r \geqslant \operatorname{Cap} L(r)$ and since E_r decreases to $\partial G \cap T$ when $r \to 1$, we get the estimate of the lemma. □

The announced lower bound for the capacity of the set B_f is the following.

Theorem 5. *Let f be a Bloch function and $z_0 \in D$. Then there is a set $A_f \subset T$ such that*

$$\overline{\lim_{r\to 1}}\, |f(r\zeta) - f(z_0)| \leqslant k\|f\|_* \quad \textit{for each} \quad \zeta \in A_f,$$

and

$$\operatorname{Cap} A_f \geqslant (1 - |z_0|) \exp\left(-\frac{k\|f\|_*^2}{(1-|z_0|^2)^2|f'(z_0)|^2}\right),$$

where k in an absolute constant.

PROOF. We can assume $\|f\|_* = 1$ and $f(z_0) = 0$. Consider the analytic function $\tilde{f} = \lambda f \circ \tau$, where $\lambda = 1/12$ and $\tau(z) = (z + z_0)/(1 + z_0 z)$.

We have

$$\tilde{f}(0) = 0, \qquad \|\tilde{f}\|_* \leqslant \lambda \quad \text{and} \quad |\tilde{f}'(0)| = \lambda(1 - |z_0|^2)|f'(z_0)|.$$

Let G be the component of $\tilde{f}^{-1}(\{|w| < 1\})$ that contains 0. Then

$$\delta(z)|\tilde{f}'(z)| \leqslant (1 - |z|)|f'(z)| \leqslant \|f\|_* < \frac{1}{12},$$

where, as before, we write $\delta(z) = \operatorname{dist}(z, \partial G)$.

Furthermore $|z| < 1/2$ implies

$$|\tilde{f}(z)| \leqslant \|\tilde{f}\|_* \log\frac{1 + 1/2}{1 - 1/2} < 2\lambda < 1.$$

This means that $\{|z| < 1/2\} \subset G$ and so $\delta(0) \geqslant 1/2$ and

$$\delta(0)|\tilde{f}'(0)| \geqslant \frac{1}{24}(1 - |z_0|^2)|f'(z_0)|.$$

Applying Lemma 4 we get

$$\operatorname{Cap}(\partial G \cap T) \geqslant \exp\left(-\frac{k}{(1-|z_0|^2)^2|f'(z_0)|^2}\right),$$

and

$$\operatorname{Cap}(\tau(\partial G) \cap T) \geqslant (1 - |z_0|) \exp\left(- \frac{k}{(1 - |z_0|^2)^2 |f'(z_0)|^2} \right).$$

If we use the fact that f is a Bloch function then we get the theorem with $A_f = \tau(\partial G)$. □

Theorem 6. *Let f be an inner function with $\|f\|_* < 1/2e$ and $f(z_0) = 0$. Then*

$$\operatorname{Cap} S(f) \geqslant (1 - |z_0|^2) \exp\left(- \frac{k}{(1 - |z_0|^2)^2 |f'(z_0)|^2} \right),$$

k being an absolute constant.

PROOF. Assume that $f(0) = 0$ and let G be a component of $f^{-1}(\{|w| < 1/2\})$ containg 0, so that $\varnothing \neq \partial G \cap T \subset S(f)$.

Since $|z| < 1/2$ implies $|f(z)| \leqslant 2\|f\|_* < 1/2$ we see that $\{|z| < 1/2\} \subset G$ and so $\delta(0) \geqslant 1/2$. Now by Lemma 4 we get

$$\operatorname{Cap} S(f) \geqslant \operatorname{Cap}(\partial G \cap T) \geqslant \exp\left(- \frac{k}{|f'(0)|^2} \right).$$

If $z_0 \neq 0$ we deal with the function

$$\tilde{f}(z) = f\left(\frac{z + z_0}{1 + z_0 z} \right). \quad \square$$

Remark. Makarov (written communication) has proved the conjecture that $\dim B_f = 1$ for every Bloch function. His method is more direct and completely different.

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