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Domains with Strong Barrier

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Dedicated to the Memory of J. L. Rubio de Francia

Introduction

The level sets of any Riemann mapping f can not be arbitrarily long. More precisely, there exists an absolute constant P so that if Ω is a plane simply connected domain, f a Riemann mapping onto Ω and L is an straight line then

$$\text{length}\,(f^{-1}(\Omega \cap L)) \leqslant P.$$

This beautiful result was first proved by Hayman and Wu [HW], and a bit later by Garnett, Gehring and Jones, [GGJ]. See [FHM] for a simple proof, where it is shown that one can take $P = 4\pi^2$ and a conjecture as to the correct value of P is offered.

One wonders as to what is the role of simple connectivity in the Hayman-Wu theorem. Let us call a domain Ω in the plane a *Hayman-Wu domain* if there exists a constant $C(\Omega)$ so that

$$\text{length}\,(F^{-1}(\Omega \cap L)) \leqslant C(\Omega) \tag{0.1}$$

for any straight line L and universal cover F from the unit disk Δ onto Ω.

It was shown in [FH] that domains of finite connectivity with no complementary components consisting of a single point are Hayman-Wu domains. A word of caution: in [FH] one is not concerned with the dependence of the constant of (0.1) upon F, but the argument applies. Moreover, it is easy to see that the punctured disk, Δ^*, is not a Hayman-Wu domain, so that the non-degeneracy condition on the complementary components is essential.

Let Γ denote a covering group of the domain Ω, *i.e.*, a fix-point free discrete group of Möbius transformations of Δ with quotient Δ/Γ conformally equivalent to Ω. Any two covering groups of Ω are conjugate, and conversely.

With Γ we associate the invariant function

$$U_{\Gamma,t}(z) = U_t(z) = \sum_{t \in \Gamma} (1 - [z, Tz]^2)^t$$

where

$$[a, b] = \left| \frac{a - b}{1 - \bar{a}b} \right| .$$

In [FH] it was shown (see Section 6 for the proof).

Theorem A. *If $U_{1/2}$ is bounded in Δ then Ω is Hayman-Wu.*

As a consequence of Theorem 2 one also has that if Ω is Hayman-Wu then U_1 is bounded. Notice that $U_1 \leqslant U_{1/2}^2$.

The exponent of Γ is defined as the exponent of convergence of the Dirichlet series

$$\sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0)))$$

i.e. the smallest number s which makes the series convergent. Here, and hereafter, $\rho(a, b)$ denotes the Poincaré distance between the points a and b on the unit disk; namely,

$$\rho(a, b) = h([a, b]),$$

with

$$h(t) = \log \frac{1 + t}{1 - t}, \qquad 0 \leqslant t \leqslant 1.$$

Since conjugate groups have the same exponent we may also speak of the exponent of Ω. We shall use the notation $\delta(\Gamma)$, $\delta(\Omega)$ to denote exponents.

It is an elementary fact that $\delta(\Omega) \leqslant 1$. Also, $\delta(\Omega) \geqslant 1/2$, if Γ contains parabolic elements, or, equivalently if $\partial\Omega$ has isolated points.

Notice that the groups satisfying the hypothesis of Theorem A have exponent at most $1/2$.

Here we shall show the following somehow surprising result.

Theorem 1. *If Ω is a Hayman-Wu domain then $\delta(\Omega) < 1$.*

Since there are domains of finite connectivity (with no point-boundary components) with exponent arbitrarily close to 1 we see that Theorem 1 is in a certain sense sharp. The exponent of the domain

$$\Omega_\epsilon = \Delta\left(0, \frac{1}{\epsilon}\right) \setminus \bar{\Delta}(0, \epsilon) \setminus \bar{\Delta}(1, \epsilon), \qquad \epsilon \in \left(0, \frac{1}{2}\right)$$

increases to 1 as ϵ decreases to 0.

We shall deduce Theorem 1 from combining two results about domains with strong barriers.

Definition. *Let Ω be a plane domain. A non-constant positive superharmonic function U of Ω is called a **strong barrier** if there exists a positive number ϵ such that*

$$\Delta U + \frac{\epsilon \cdot U}{\operatorname{dist}(\cdot, \partial\Omega)^2} \leqslant 0,$$

(*where this inequality is meant in the weak sense*).

If Ω has a strong barrier then Ω has a Green's function and moreover every boundary point is regular for the Dirichlet problem, and thus Ω has no point-boundary components.

Domains with strong barriers can be characterized in a variety of ways, and we shall use the rich knowledge about them to prove the following two results which yield Theorem 1 immediately.

Theorem 2. *If Ω is a Hayman-Wu domain then Ω posseses a strong barrier.*

The reciprocal of Theorem 2 does not hold. This follows from Theorem 4 below.

Theorem 3. *If Ω posseses a strong barrier then $\delta(\Omega) < 1$.*

In this case it is easy to see that the reciprocal does not hold; simply take $\Omega = \Delta^*$, then $\delta(\Omega) = 1/2$, but Ω does not have a strong barrier.

It should be remarked that in [Po2] an example is offered of a domain with a strong barrier but $\delta = 1$. There is an error in the calculations there.

A *Denjoy domain* is a domain in the sphere whose complement is a compact set of the real line. Thus $\Omega = \hat{\mathbb{C}} \setminus E$, $E \subset \mathbb{R}$, E compact. Denjoy domains have been recently studied by several authors in connection mostly with the Corona problem. See [RR], [C], [JM], [GJ]. They provide a test case for problems about multiply connected domains.

A compact set $E \subset \mathbb{R}$ is called *homogeneous* if there exists a constant c_E so that if $x \in \mathbb{R}$ and $\delta > 0$.

$$\frac{|(x-\delta, x+\delta) \cap E|}{\delta} \geqslant c_E.$$

Carleson introduced this condition in [C] where he showed that the associated Denjoy domain satisfies the Corona theorem.

Garnett and Jones [GJ] later showed this with no restriction on the set E. More recently, Zinsmeister has shown that E is homogeneous if and only if $H^1(E) = H^1(\mathbb{R})$ (see [Z] for definitions and results).

If E is homogeneous then $\hat{\mathbb{C}} \backslash E$ has an strong barrier.

For Denjoy domains the homogeneity of the boundary is the key for being a Hayman-Wu domain.

Theorem 4. *If Ω is a Denjoy domain, then Ω is a Hayman-Wu domain if and only if $\partial\Omega$ is homogeneous.*

The proof of Theorem i is in Section i, $i = 2, 3, 4$. In Section 5 we consider another notion of a domain being almost simply connected and relate that to the results above. In Section 6 we give the proof of Theorem A for the sake of completeness.

I wish to thank A. Ancona for pointing out the example in the Remark in Section 7. I am most grateful to Juha Heinonen for very stimulating conversations which motivated this paper.

1. Domains with Strong Barrier

Here we collect the relevant features of domains with strong barrier.

Let Ω be a plane domain other than the plane or a punctured plane. The universal covering Riemann surface is the unit disk. Consider the Poincaré metric in the unit disk. Via the universal covering map, π, it can be projected onto a metric in Ω so that π is a local isometry. This projected metric is conformal with the euclidean metric and the scale factor, denoted by λ_Ω, is determined by the equation

$$\lambda_\Omega(\pi(z))|\pi'(z)| = \frac{2}{1-|z|^2}, \qquad z \in \Delta.$$

The volume form of this metric will be denoted by ω_Ω; it is simply

$$\omega_\Omega = \lambda_\Omega^2 \, dx \wedge dy.$$

It is always the case and follows from Schwarz's lemma that

$$\lambda_\Omega \leqslant \frac{2}{\operatorname{dist}(\cdot, \partial\Omega)}.$$

To have a reversed inequality, *i.e.*, to have $0 < \inf_{z\in\Omega} \lambda_\Omega(z) \operatorname{dist}(z, \partial\Omega)$ is equivalent to the existence of a strong barrier, [BP], [Po1]. Also in terms of the group Γ we have that Ω has a strong barrier if and only if there exists $\tau_0 > 0$ so that the translation length of every element of Γ is at least τ_0, [P1]. (The translation length of a parabolic element is defined to be zero.) In geometric terms this translates into having no punctures plus the existence of a positive lower bound for the length of closed simple Poincaré geodesics of Ω.

We shall need another characterization. A domain Ω has a strong barrier if and only if $\partial\Omega$ verifies the following capacity condition: there exists a constant $C_0 > 0$

$$\operatorname{cap}(\Delta(b, r) \cap \partial\Omega) > C_0 r \tag{1.1}$$

for every $b \in \partial\Omega$, and r, $0 < r \leqslant \operatorname{diam}(\partial\Omega)$.

The strong barrier condition is also equivalent to $U_{\Gamma,1}$ being bounded [Po2]. Recall that the condition appearing in Theorem A is that $U_{\Gamma,1/2}$ is bounded.

All this can be found in [A], [BP], [Po1], [Po2].

2. Proof of Theorem 2

We will check that if Ω is a Hayman-Wu domain then ***(a)***, there is a constant $\tau_0 > 0$ so that all closed simply geodesic have Poincaré length at least τ_0, and ***(b)***, there are no punctures. We need a simple lemma:

Lemma. *Let T be hyperbolic Möbius transformation of the unit disk onto itself whose axis passes through* 0. *Then*

$$\frac{1}{|T(0)|} \leqslant \sum_k (1 - |T^k(0)|^2) \leqslant \frac{2}{|T(0)|^2}.$$

PROOF. We may assume that the fixed points of T are -1 and 1, and that $T(0) = a \in (0, 1)$. Let $b_n = T^n(0)$, $n \geqslant 0$. Then

$$1 - |b_n|^2 = 1 - |T(b_{n-1})|^2 = \frac{(1 - |b_{n-1}|^2)(1 - |a|^2)}{|1 + b_{n-1}a|^2}, \qquad n \geqslant 1.$$

For $n \geqslant 1$ we have:

$$1 \leqslant |1 + b_{n-1}a| \leqslant 1 + a,$$

and so

$$(1-a^2)^n \geqslant 1-|b_n|^2 \geqslant \left(\frac{1-a}{1+a}\right)^n, \qquad n \geqslant 1.$$

Therefore

$$\frac{2}{a^2} > \sum_{k\in\mathbb{Z}} (1-|T^k(0)|^2) = 1 + 2\sum_{n=1}^{\infty} (1-|b_n|^2) \geqslant \frac{1}{a}\cdot$$

(*a*) Let σ be a closed simple geodesic in Ω.

The Jordan curve σ contains points of $\partial\Omega$ in its Jordan interior. Let $s \in \sigma$ and $b \in \partial\Omega$ be such that

$$|b-s| = \operatorname{dist}(\sigma, \partial\Omega \cap \operatorname{interior}(\sigma)).$$

Let F be a universal covering map which takes 0 to s.

Lift σ to a geodesic segment in Δ through 0. The lift is part of a diameter $\tilde{\sigma}$ of Δ. Let T be the Möbius covering transformation ($F \circ T = F$) corresponding to σ. Then the axis of T is $\tilde{\sigma}$ since σ is smooth. Moreover the length L of σ safisfies

$$\frac{1}{\tanh\left(\frac{L}{2}\right)} \leqslant \sum_{k\in\mathbb{Z}} 1-|T^k(0)|^2. \tag{2.1}$$

The segment from s to b is contained in Ω and its preimage under F contains a collection of curves each one of them emanates from a point of the orbit of 0 and goes all the way to $\partial\Delta$, therefore the total length of these curves is at least $\sum_{\gamma\in\Gamma} 1-|\gamma(0)|$. And consequently we have that

$$\sum_{\gamma\in\Gamma} (1-|\gamma(0)|) \leqslant c_\Omega\,. \tag{2.2}$$

Therefore,

$$\tanh\left(\frac{L}{2}\right) \geqslant \frac{1}{2c_\Omega}\cdot$$

and so L is bounded below by a constant depending only on c_Ω.

(*b*) It remains to deal with the possible isolated points of the boundary of Ω. We may assume that $0 \in \partial\Omega$ and $\Delta^* \subset \Omega$. Let F be a universal covering map which takes 0 to 1/2. The circle $|z| = 1/2$ is lifted to a curve joining 0 to $T(0)$ where $T \in \Gamma$ is parabolic. We may assume that the unique fixed point of T is 1. Now the segment σ from 1/2 to 0 lifts to a curve $\tilde{\sigma}$ in

Δ which joins 0 to 1. Notice that $F\left(\bigcup_{k\in\mathbb{Z}} T^k(\tilde{\sigma})\right) \subset (0, 1/2]$, and therefore since $T(\tilde{\sigma})$ joins $\gamma(0)$ to 1, we see that

$$\sum_{k\in\mathbb{Z}} |1 - T^k(0)| \leqslant \text{length}\,(F^{-1}(\Omega \cap \mathbb{R})) \leqslant c(\Omega). \tag{2.3}$$

But it is easy to see that $|1 - T^k(0)||k| \to t_0$ as $|k| \to \infty$ where t_0 is a positive number. Therefore the sum on the left is actually infinite. Thus we have shown that $\partial\Omega$ has no isolated points and so the proof is complete.

3. Proof of Theorem 3

Our proof of Theorem 3 is actually a combination of results which appear in papers by Ancona [A] and Sullivan [S1]. Ancona shows that in domains with strong barrier the following form of Hardy's inequality holds: there exists a constant c_1 so that for every smooth function φ compactly supported in Ω

$$\iint_\Omega |\varphi(z)|^2 \frac{dx\,dy}{\text{dist}\,(z, \partial\Omega)^2} \leqslant c_1 \iint_\Omega |\nabla\varphi(z)|^2\, dx\,dy, \qquad (z = x + iy). \tag{3.1}$$

The constant c_1 depends only on the ϵ in the definition of strong barrier. As a matter of fact the existence of strong barrier is equivalent to (3.1).

Recall that the density of the Poincaré metric is denoted by λ_Ω, while its volume form is denoted by ω_Ω.

The Dirichlet integral is a conformal invariant. Therefore the integral on the right hand side of the inequality (3.1) equals

$$\iint_\Omega |\nabla_\Omega \varphi|_\Omega^2\, \omega_\Omega \tag{3.2}$$

where ∇_Ω denotes the gradient with respect to the Poincaré metric of Ω, and $|\ \ |_\Omega$ denotes length in the tangent space with respect to the Poincaré metric of Ω.

Moreover, it is always the case that

$$\lambda_\Omega(z) \leqslant \frac{2}{\text{dist}\,(z, \partial\Omega)}, \quad \text{for every} \quad z \in \Omega. \tag{3.3}$$

Using (3.2) and (3.3) we see that inequality (3.1) implies that

$$\iint_\Omega |\varphi|^2 \omega_\Omega \leqslant c_1 \iint_\Omega |\nabla_\Omega \varphi|^2 \omega_\Omega, \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega). \tag{3.4}$$

But this means that the Poincaré inequality holds in the Riemannian manifold Ω and therefore the spectrum of the Laplace-Beltrami operator of Ω is contained in $(-\infty, -1/C_1)$.

And now the theorem of Elstrodt-Patterson-Sullivan (see [S1, p. 333]) provides the final stroke because if $\delta = \delta(\Gamma)$ then it claims in our case that

$$\delta(1-\delta) \geqslant \frac{1}{C_1},$$

if $\delta \geqslant 1/2$. In particular,

$$\delta \leqslant \max\left\{1 - \frac{1}{C_1}, \frac{1}{2}\right\} < 1.$$

Remark. One can use the argument of Lemma 1 of [Su] to show directly that if a domain posseses strong barrier then the isoperimetric inequality, $A < cL$, holds (for its Poincaré metric), and combine this with Cheeger's inequality to give the result.

4. Proof of Theorem 4

Sufficiency

Here we assume that $\partial\Omega$ is homogeneous.

First of all we reduce the proof to the case $L = \mathbb{R}$. Let a universal covering map F be given and assume that we have seen that

$$\text{(4.1)} \qquad \operatorname{length}(F^{-1}(\Omega \cap \mathbb{R})) \leqslant M$$

where M depends on Ω but not on F. Let L be any other straight line and L^+ be the part of L above $\mathbb{R}$. Let G be any branch of F^{-1} defined on the upper half plane. By the Hayman-Wu theorem (see [GGJ]) we have that

$$\text{(4.2)} \qquad \operatorname{length}(G(L^+)) \leqslant \tilde{P} \operatorname{length}(\partial G(U))$$

where $\tilde{P}$ is an absolute constant. Adding up (4.2) over all branches G and using (4.1) we see that

$$\operatorname{length}(F^{-1}(L^+)) \leqslant \tilde{M},$$

where $\tilde{M}$ depends only on Ω. Similarly, length $(F^{-1}(L^-)) \leqslant \tilde{M}$ and so

$$\operatorname{length}(F^{-1}(L)) \leqslant 2\tilde{M}$$

Choose now a universal covering map F. We will check that (4.1) holds. Let us denote by I_j the complementary intervals of E in $\mathbb{R}$.

In each I_j we select points $z_k^{(j)}$ as follows: if $I_j = (a, b)$, with a, b finite then

$$z_0^{(j)} = \frac{a+b}{2},$$

and

$$z_k^{(j)} = z_0^{(j)} + \operatorname{sign}(k)\frac{|I_j|}{2}(1 - 1/2^k), \qquad k \in \mathbb{Z}.$$

If the interval contains ∞, we select ∞ as $z_0^{(j)}$ and, if $q = \sup E$ and $p = \inf E$,
we let

$$z_k^{(j)} = q + \frac{1}{2^k}\operatorname{diam}(E), \qquad k \geqslant 1,$$

$$z_k^{(j)} = p - 2^k \operatorname{diam}(E), \qquad k \leqslant -1.$$

Let $Z = \{z_k^{(j)}: j, k\}$. We shall check that $F^{-1}(Z)$ is an interpolating sequence whose constants are independent of the choice of the universal covering F of Ω. Assume this for the moment and let us show how to finish the proof.

Denote by $I_{j,k}$ the interval $(z_k^{(j)}, z_{k+1}^{(j)})$, $k \in \mathbb{Z}$. Let G be any branch of F^{-1} defined on the whole interval $3I_{j,k}$ (which is the interval with same center and triple the length). Then $G(I_{j,k})$ is a curve in Δ whose Poincaré diameter is bounded by an absolute constant (log 4); this follows from Schwarz' Lemma. In particular if $x \in I_{j,k}$ we have

$$1 - |G(x)|^2 \leqslant A(1 - |G(z_k^{(j)})|^2), \tag{4.3}$$

where A is an absolute constant. Thus, if $x \in I_{j,k}$,

$$|G'(x)| = (1 - |G(x)|^2)\lambda_\Omega(x) \leqslant \frac{1 - |G(x)|^2}{\operatorname{dist}(x, \partial\Omega)} \leqslant \frac{1 - |G(x)|^2}{|I_{j,k}|} \leqslant A\frac{1 - |G(z_k^{(j)})|^2}{|I_{j,k}|}.$$

Consequently,

$$\int_{I_{j,k}} |G'(x)|\, dx \leqslant A(1 - |G(z_k^{(j)})|^2). \tag{4.4}$$

And, in particular, adding up (4.4) over all j, k and G we obtain that

$$\operatorname{length}(F^{-1}(\Omega \cap \mathbb{R})) \leqslant A \sum_{w \in F^{-1}(Z)} (1 - |w|^2).$$

But we are assuming that we have already shown that $F^{-1}(Z)$ is an interpolating sequence and so, in particular, that the measure

$$\mu = \sum_{w \in F^{-1}(Z)} (1 - |w|^2)\delta_w$$

is finite (as a matter of fact, that μ is a Carleson measure). The interpolation constants of $F^{-1}(Z)$ depend only on Ω and thus so does the mass of μ; this implies that (4.1) holds.

All that remains is to show that $F^{-1}(Z)$ is an interpolating sequence. But before doing so let us remark that the argument above (which appears in [GGJ]) is general. In fact, given Ω (not necessarily Denjoy), split the intersection with Ω of a given line L into disjoint intervals J_k so that in each interval J_k

$$\frac{1}{100} \leqslant \frac{\operatorname{dist}(z, \partial\Omega)}{\operatorname{length}(J)} \leqslant 100.$$

Let z_k be the center of J_k. Then if $F^{-1}(\{z_k\})$ is interpolating with contants depending on Ω alone one deduces that Ω is a Hayman-Wu domain. Conversely, if Ω is Hayman-Wu then using that Ω has strong barrier one may show that the inverse image of such a sequence is interpolating.

There is an argument introduced by Garnett-Gehring-Jones for checking whether $F^{-1}(Z)$ is interpolating or not by transfering the problem to a harmonic measure estimate on Ω itself. If we assume that Ω has an strong barrier then we have that $F^{-1}(Z)$ is interpolating if and only if there is $\epsilon < 1/4$ and $a > 0$ so that if for $z \in Z$ we define

$$H_\epsilon(z) = \sum_{z' \in Z \setminus \{z\}} \bar{\Delta}(z', \epsilon \operatorname{dist}(z', \partial\Omega)) \cap \mathbb{R}.$$

Then

$$\omega(z, \partial\Omega, \Omega \setminus H_\epsilon(z)) \geqslant a, \quad \text{for all} \quad z \in Z. \tag{4.5}$$

This appears in [Po2] and in [JM]. If $z' = \infty$ by $\bar{\Delta}(\infty, \epsilon \operatorname{dist}(\infty, \partial\Omega))$ we mean $\bar{\mathbb{R}} \setminus (p - (1/\epsilon) \operatorname{diam} \partial\Omega, q + (1/\epsilon) \operatorname{diam} \partial\Omega)$. It turns out that for Denjoy domains with homogeneous complement (4.5) can be easily checked. This could be done as follows: if $z \in Z \setminus \{\infty\}$, then $\Delta(z, (1/8) \operatorname{dist}(z, \partial\Omega)) \subset \Omega \setminus H_\epsilon(z)$; by Harnack's inequality it is enough to estimate

$$\omega(z + id, \partial\Omega, \Omega \setminus H_\epsilon(z))$$

from below, where

$$d = \frac{1}{16} \operatorname{dist}(z, \partial\Omega),$$

But

$$\omega(z + id, \partial\Omega, \Omega \setminus H_\epsilon(z)) \geqslant \omega(z + id, \partial\Omega, U),$$

(where U is the upper half plane).

Let $b \in \partial\Omega$ be such that $|z - b| = \operatorname{dist}(z, \partial\Omega)$, using again Harnack's inequality we see that we just need to estimate $\omega(b + id, E, U)$ form below. But from the explicit expression of the Poisson kernel of the upper half plane we readily see that

$$\omega(b + id, E, U) \geqslant C\frac{|(b - 10d, b + 10d)|}{d}$$

where C is an absolute constant. And this gives the desired result. (For $z = \infty$ one needs a minor variation of the argument.)

Necessity

Assume that Ω is a Hayman-Wu domain. We want to check that $\partial\Omega$ is homogeneous. Write $E = \partial\Omega$.

We already know that E satisfies the capacity condition (1.1).

We use the notation of the proof of the sufficiency.

We know that for some $\epsilon > 0$ and $a = a(\epsilon) > 0$

$$\omega(z, E, \Omega\backslash H_\epsilon(z)) \geqslant a, \quad \text{for every} \quad z \in A.$$

It is easy to check that $E \cup \hat{H}_\epsilon(z)$ is homogeneous with a constant depending only on ϵ (and not on E). Here $\hat{H}_\epsilon(z)$ is the part of $H(z)$ not lying in the component of ∞ of $\bar{\mathbb{R}}\backslash E$. Clearly

$$\omega(z, E, \Omega\backslash \hat{H}_\epsilon(z)) \geqslant a, \quad \text{for every} \quad z \in Z.$$

Let $V = [p, q]$ be the smallest interval containing E. We shall check that for an appropriate constant $M = M(\epsilon)$ we have for all $y \in V\backslash E$ that

$$|\Delta(y, M \operatorname{dist}(y, E)) \cap E| \geqslant C \operatorname{dist}(y, E) \tag{4.7}$$

where $C = C(\epsilon)$.

This will be enough as the following simple lemma shows.

Lemma. *Let $A \subset [0, 1]$ be a closed set and assume that there exist constants η, N such that if $y \in [0, 1]\backslash A$*

$$|(y - Nd(y), y + Nd(y)) \cap A| \geqslant \eta\, d(y),$$

where $d(y) = \operatorname{dist}(y, A)$ then

$$|A| \geqslant \eta/8N.$$

PROOF OF LEMMA. Let $J_y = (y - Nd(y), y + Nd(y))$.

Consider

$$B = \bigcup_{y \in [0,1] \setminus A} J_y .$$

We may choose points y_j so that

$$B = \bigcup_j J_{y_j}$$

and

$$\sum_j \chi_{y_j} \leqslant 2\chi_I$$

(*i.e.* no point of B is in more than two of the J_{y_j}). Then

$$|A \cap B| = \int_A \chi_B \geqslant \frac{1}{2} \sum_j |A \cap J_{y_j}| \geqslant \frac{\eta}{2} \sum_j d(y_j) \geqslant \frac{\eta}{4N} |B| .$$

Now, $A \cap B \subset A$, and $B \supset [0,1] \setminus A$ so that

$$|A| \geqslant \frac{\eta}{4N} (1 - |A|)$$

and so

$$|A| \geqslant \frac{\eta}{8N} .$$

It is clear that in order to check (4.7) for all $y \in V \setminus E$ it is enough to do so when y is one of the points $z_k^{(j)}$.

Since both the data and the desired conclusion are translation and scale invariant, we may assume that $z_k^{(j)} = 0$, $1 \in E$, and $\operatorname{dist}(z_k^{(j)}, E) = 1$.

Around $1/2$ there is an interval of length 2ϵ which lies in $\partial H_\epsilon(0)$. Then there exists $M = M(\epsilon)$ so that

$$\omega(0, \mathbb{R} \setminus (-M(\epsilon), M(\epsilon)), \Omega \setminus \hat{H}_\epsilon(z) \setminus [-M(\epsilon), M(\epsilon)]) \leqslant a/2 .$$

Therefore we see that

$$\omega(0, E \cap [-M(\epsilon), M(\epsilon)], \Omega \setminus \hat{H}_\epsilon(z)) \geqslant a/2 . \tag{4.8}$$

We define two sets $\tilde{E}, \tilde{K}$ as follows: we let $\tilde{E}$ be the set $E \cap [-M(\epsilon), M(\epsilon)]$ and $\tilde{K}$ be the set $E \cup ([-M(\epsilon), M(\epsilon)] \cap \hat{H}_\epsilon(0))$. Consider $\tilde{\Omega} = \hat{\mathbb{C}} \setminus \tilde{K}$. We know from (4.8) that

$$\omega(0, \tilde{E}, \tilde{\Omega}) \geqslant a/2 .$$

Again $\tilde{K}$ is homogeneous with a constant depending only on ϵ, and since $\tilde{K} \subset [-M(\epsilon), M(\epsilon)]$ then we know that $\omega(\infty, \bullet, \tilde{\Omega})$ is absolutely continuous with respect to length and in fact, that the Radon-Nikodym derivative h is in L^p, for some $p > 1$. More precisely.

$$\omega(\infty, \bullet, \tilde{\Omega}) = h\,dx$$

and for $p = p(\epsilon) > 1$ and $T = T(\epsilon)$ we have

$$\int_{\partial\tilde{\Omega}} |h(x)|^p\,dx \leqslant T(\epsilon).$$

This is the heart of the matter. It is due to Jones and Marshall ([JM]).

From Harnack's inequality (and a bit of Poincaré geometry), we have

$$\omega(\infty, \tilde{E}, \tilde{\Omega}) \geqslant a'.$$

$(a' = a'(a, \epsilon) = a'(\epsilon))$. Therefore

$$a' \leqslant \int_E |h(x)|\,dx \leqslant T(\epsilon)^{1/p}|\tilde{E}|^{1-1/p}.$$

And so

$$|E \cap [-M(\epsilon), M(\epsilon)]| \geqslant c = c(\epsilon)$$

and we are done.

5. Fully Accessible Domains

This is a notion that has been introduced and studied by Patterson, [Pa1], [Pa2], Pommerenke [Po3], [Po4], [Po5], and Sullivan [S2]. A Fuchsian group Γ is called *fully accessible* if the action of Γ on $\partial\Delta$ is fully dissipative *i.e.* if there is a measurable set $B \subset \partial\Delta$ so that if $\gamma \in \Gamma \setminus \{id\}$, $|\gamma(B) \cap B| = 0$ and $|\partial\Delta \setminus \bigcup_{\gamma\in\Gamma} \gamma(B)| = 0$, or in other terms that the action of Γ on $\partial\Delta$ has a measurable fundamental set.

A domain is called *fully accessible* if its covering group is fully accessible.

Patterson showed in [Pa1] that if $\delta(\Gamma) < 1/2$ then Ω is fully accesible. On the other hand fully accesible domains may have $\delta(\Omega) = 1$. One such example is provided by $\Omega = \Delta^* \setminus \{a_n\}$, where $a_n \to 0$. It is easy to see that Ω is fully accessible (see Theorem 3 or Example 1 in [Po4]) but $\delta(\Omega) = 1$. See Remark 1.

It is reasonable to expect that Hayman-Wu domains must be fully accesible. We can only show this for Denjoy domains. In that case a Hayman-Wu domain satisfies that if F is the symmetric universal covering map with $F(0) = \infty$, Γ its covering group, and D_0 the associated Dirichlet region at 0 then

$$\sum_{\gamma\in\Gamma} \text{length}\,(\partial(\gamma(D_0))) < \infty$$

(see [FH]).

This clearly implies that

$$\left|\partial\Delta \setminus \bigcup_{\gamma} \gamma(\partial D_0 \cap \partial\Delta)\right| = 0,$$

which gives that Ω is fully accessible. Another argument to show this is provided by two characterizations. Assume that Ω is a Denjoy domain. We have seen that Ω is Hayman-Wu if and only if Ω is homogeneous; on the other hand it has been shown by D. Hamilton and the author that Ω is fully-accesible if and only if harmonic measure in $\partial\Omega$ is absolutely continuous with respect to arc length (see Remark 2). But Carleson, [C], showed that for homogeneous sets harmonic measure is in fact absolutely continuous.

Remark 1. Let a_n be a sequence of numbers converging to zero. Let

$$\Omega = \Delta^* \setminus \{a_k\}_{k=1}^{\infty}.$$

Now

$$\delta(\Omega) \geqslant \delta(\Delta \setminus \{0, a_n\}).$$

This follows from the results about signatures in [Pa2], but in [F] it is shown that $\delta(\Delta \setminus \{0, a_n\}) \to 1$ as $n \to \infty$ therefore $\delta(\Omega) = 1$.

Remark 2. We simply sketch the argument. It is based on the special form of the Dirichlet's, D_0, and Green's, G_0, fundamental region associated to the covering map F with takes 0 to ∞ and is symmetric under complex conjugation. The Dirichlet region is mapped under F on $\mathbb{C} \setminus [p, q]$ where $[p, q]$ is the smallest closed interval which contains $\partial\Omega$. Since ∂D_0 is rectifiable it follows that if Γ is fully accessible then $\omega(\infty, \cdot, \partial\Omega)$ is absolutely continuous with respect to length. Conversely, since the Green's region is mapped onto $\mathbb{C} \setminus [p, q]$ one sees that if $\omega(\infty, \cdot, \partial\Omega)$ is absolutely continuous with respect to length then the Green's measure is absolutely continuous with respect to $d\theta$, and this is equivalent to full accessibility; (see [Po3] for definitions and this last result).

6. Proof of Theorem A

We start with

Lemma. *Let G be a Fuchsian group and denote by $D_0(G)$ the Dirichlet region of G at 0. Then*

$$\sum_{g\in G} \text{length}\,(\partial g(D_0(G)) \leqslant \pi^2 \sum_{g\in G} (1 - |g(0)|^2)^{1/2}.$$

PROOF. The domain $g(D_0(G))$ is contained in

$$\{z: \rho(z, g(0)) \leqslant \rho(z, 0)\} = H(g(0)).$$

By a result of B. Brown, [B], we have that

$$\text{length}\,(\partial g(D_0(G))) \leqslant \frac{\pi^2}{2}\,\text{diam}\,(g(D_0(G))).$$

But

$$\text{diam}\,(H(g(0))) = 2(1 - |g(0)|^2)^{1/2},$$

and so the result follows.

If Γ satisfies that $U_{1/2}$ is bounded then for any group G conjugate to Γ we have

$$\sum_{g\in G} \text{length}\,(g(\partial D_0(G))) \leqslant \pi^2 \| U_{1/2}\|_\infty. \tag{6.1}$$

For $G = \omega^{-1}\Gamma\omega$, where $\omega \in \text{Möb}\,(\Delta)$, and then

$$\sum_{g\in G} (1 - |g(0)|^2)^{1/2} = \sum_{\gamma\in G} (1 - [\omega(0), \gamma(\omega(0))]^2)^{1/2} = U_{1/2}(\omega(0)).$$

Assume that a covering group Γ of Ω (and hence all) has $\| U_{1/2}\|_\infty < \infty$.
Let F be any universal covering map from Δ onto Ω. The group of deck transformations of Γ is a group G conjugate to Γ.
We want to estimate the length of the set $V = F^{-1}(\Omega \cap L)$ where L is an straight line. Since F is one-to-one on $g(D_0(G))$ we deduce form the Hayman-Wu theorem (see [GGJ]) that

$$\text{length}\,(V \cap g(D_0(G))) \leqslant C\,\text{length}\,(g(\partial D_0(G))) \tag{6.2}$$

where C is an absolute constant. But then using (6.1) and (6.2) we deduce that

$$\begin{aligned}\text{length}\,(V) &\leqslant \sum_{g\in G} \text{length}\,(\partial g(D_0(G))) + \sum_{g\in G} \text{length}\,(V \cap g(D_0(G)))\\ &\leqslant (1 + c) \sum_{g\in G} \text{length}\,(g(\partial D_0(G)))\\ &\leqslant (1 + c)\pi^2 \| U_{1/2}\|_\infty.\end{aligned}$$

7. An Example

We know that for a domain Ω, U_1 is bounded if and only if Ω posseses a strong barrier. Possesing a strong barrier means that Ω contains no doubly connected domains (separating $\partial\Omega$) of arbitrarily large modulus, or equivalently, in view of a theorem of Teichmüller ([Ah, p. 74]), that contains no ring (separating $\partial\Omega$) of arbitrarily large modulus (see [BP], [Po1]).

Let us define the modulus of a domain Ω as

$$M(\Omega) = \sup\{\text{mod}(R): R, \text{ring}, R \subset \Omega, R \text{ separating } \partial\Omega\}.$$

The constant $M(\Omega)$ and the reciprocal of the ϵ in the definition of strong barrier are bounded by functions of each other.

Since $\delta(\Omega) < 1/2$ guarantees that there are no isolated boundary points it is tempting to guess that $\delta(\Omega) < 1/2$ implies that Ω posseses an strong barrier. Theorem A also points in that direction. Unfortunately

Example. *Given $\delta_0 > 0$ there exist a domain Ω with $\delta(\Omega) \leqslant \delta_0$ but $M(\Omega) = \infty$.*

In order to show that the exponent of a domain is close to 1 one only has to provide an example of a function $\varphi \in C_0^\infty(\Omega)$ with small

$$\frac{\iint_\Omega |\nabla\varphi|_\Omega^2 \, dx\, dy}{\iint_\Omega |\varphi|^2 \omega_\Omega}.$$

But the Rayleigh quotient is of no help here since at most it can be used to show that $\delta(\Omega) \leqslant 1/2$. We do have to look into the geometry of Ω.

Given a sequence ϵ_j of positive numbers tending to zero consider the domain

$$\Omega = \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \bar{\Delta}(2n+1, \epsilon_{|n|}) \setminus \bigcup_{n\in\mathbb{Z}} T(2n, \eta_n).$$

where if $a \in \mathbb{R}$ and $\eta > 0$

$$T(a, \eta) = \{a + iy: |y| \geqslant \eta\} \cup \{x + iy: |x - a| \leqslant 1/2, |y| = \eta\}.$$

If δ_0 is given we can choose the numbers η_n converging to zero so fast that $\delta(\Omega) \leqslant \delta_0$. Of course, $M(\Omega) = \infty$.

We content ourselves with giving a proof of the following

Lemma. *Given $\delta_0 > 0$ and M_0 there exists a triply connected domain Ω with*

$$\delta(\Omega) \leqslant \delta_0 \quad \textit{and} \quad M(\Omega) \geqslant M_0.$$

Consider the domain

$$\Omega = \mathbb{C}\setminus\bar{\Delta}(1,\epsilon)\setminus\bar{\Delta}(-1,\epsilon)\setminus T(0,\eta).$$

The set $\{x+iy\colon |y|\leqslant \eta/2, |x|\leqslant 1/2\}$ will be called the tunnel. It is clear that if ϵ is small enough then $M(\Omega)\geqslant M_0$ (recall that $\eta\leqslant\epsilon$). We now fix ϵ and show that $\delta(\Omega)$ tends to zero as $\eta\to 0$.

Notice that Ω is symmetric under reflection on the imaginary axis ($(x+iy)^* = -x+iy$). Choose F so that $F(0)=1$ and $F(\bar{z})=F(z)^*$. We have to check that for s small (assuming η small) we have

$$\sum_{\gamma\in\Gamma} e^{-s\rho(0,\gamma(0))}<\infty \tag{7.1}$$

where Γ is the covering group of F. The group Γ is a free group in two generators. One generator, α, corresponds to the loop with base at 1, which surrounds $\bar{\Delta}(1,\epsilon)$ the other one, β, corresponds to the $*$-symmetric loop. We decompose the sum in (7.1) as follows

$$1+\sum_{k=1}^{\infty}\sum_{\gamma\in A_k} e^{-s\rho(0,\gamma(0))}, \tag{7.2}$$

where A_k denotes the collection of those elements of Γ of the form

$$\sigma = w_1^{p_1}w_2^{p_2}\cdots w_k^{p_k}$$

where w_i is α or β but $w_i\neq w_{i+1}$, $i=1,\dots,k-1$, and $p_i\in\mathbb{Z}\setminus\{0\}=\mathbb{Z}^*$. Consider $\sigma\in A_k$, we will estimate $\rho(0,\sigma(0))$ from below. Let h denote the length of the shortest geodesic in Ω which surrounds $\bar{\Delta}(1,\epsilon)$. This number h depends on ϵ and η but there exist $h_0=h_0(\epsilon)$ which depends only on ϵ so that $h\geqslant h_0$. (This could be seen by using the convergence results in [H]).

The segment from 0 to $\sigma(0)$ is mapped by F onto a curve $\hat{\sigma}$ which is locally a geodesic and

$$\rho(0,\sigma(0)) = l_\Omega(\hat{\sigma}) \qquad (=\text{the Poincaré length of } \hat{\sigma}).$$

With this information we may estimate $l_\Omega(\hat{\sigma})$ from below as follows:

$$l_\Omega(\hat{\sigma})\geqslant\left(\sum_{j=1}^{k}|p_j|-k\right)h_0+k\left(\frac{1}{\eta}\right).$$

For the length of a curve connecting the short sides of the tunnel is at least $1/\eta$ and $\hat{\sigma}$ «contains» k arcs connecting these short sides.

For a vector v in $(\mathbb{N}-\{0\})^k$ we write

$$\|v\| = \sum_{j=1}^{k}|v_j|.$$

Then we have that

$$\sum_{\gamma\in A_k} e^{-s\rho(0,\gamma(0))} \leqslant 2^k \sum_{v\in(\mathbb{N}-\{0\})^k} e^{-s(\|v\|h_0+k(1/\eta-h_0))}$$
$$= 2^k e^{-sk(1/\eta-h_0)} \sum_{v\in(\mathbb{N}-\{0\})^k} e^{-sh_0\|v\|}$$
$$= 2^k e^{-s(1/\eta-h_0)k}\left[\frac{e^{-sh_0}}{1-e^{-sh_0}}\right]^k$$

and given s if we choose η small enough we have that

$$e^{-s(1/\eta-h_0)} < \frac{1}{4}(e^{sh_0}-1)$$

and then the sum (7.2) is majorized by

$$1+\sum_{k=1}^{\infty}\frac{1}{2^k}=2.$$

Remark. The example shows that one can have δ small while U_1 is unbounded. On the other hand Theorem 3 shows that for plane domains if U_1 is bounded then $\delta < 1$. This last fact *does not hold for Riemann surfaces.* Consider a Z^3-cover R of a compact Riemann surface S. Now R has a Green's function (see, *e.g.*, [T, p. 484]) and since U_1 is invariant under the Z^3-action we have that U_1 of R is bounded. On the other hand it is easy to see that the infimum of the Rayleigh's quotient is zero, and so $\delta = 1$. This example was pointed out by A. Ancona.

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