

Fundamental Solutions and Asymptotic Behaviour for the p -Laplacian Equation

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Abstract

We establish the uniqueness of fundamental solutions to the p -Laplacian equation

$$(PLE) \quad u_t = \operatorname{div}(|Du|^{p-2}Du), \quad p > 2,$$

defined for $x \in \mathbb{R}^N$, $0 < t < T$. We derive from this result the asymptotic behaviour of nonnegative solutions with finite mass, *i.e.* such that $u(\cdot, t) \in L^1(\mathbb{R}^N)$. Our methods also apply to the porous medium equation

$$(PME) \quad u_t = \Delta(u^m), \quad m > 1,$$

giving new and simpler proofs of known results. We finally introduce yet another method of proving asymptotic results based on the idea of asymptotic radial symmetry. This method can be useful in dealing with more general equations.

1. Introduction

The p -Laplacian equation

$$(1.1) \quad u_t = \operatorname{div}(|Du|^{p-2}Du), \quad p > 1$$

admits a fundamental solution, *i.e.* a function $w \geq 0$ which solves (1.1) in a weak sense in $Q = \mathbb{R}^N \times (0, \infty)$ and takes on the initial data

$$(1.2) \quad u(x, 0) = M\delta(x), \quad M > 0.$$

It is given by the following selfsimilar expression (the «Barenblatt» solution [Ba])

$$(1.3) \quad w(x, t) = t^{-k} (C - q|\xi|^{p/(p-1)})_+^{(p-1)/(p-2)}$$

where

$$\xi = xt^{-k/N}, \quad k = \left(p - 2 + \frac{p}{N}\right)^{-1}, \quad q = \frac{p-2}{p} \left(\frac{k}{N}\right)^{1/(p-1)}$$

and C is related to the mass M by $C = cM^\alpha$, with $\alpha = p(p-2)k/N(p-1)$ and $c = c(p, N)$ determined from the condition $\int w(x, t) dx = M$.

In this paper we prove that $w = w_M$ is the unique solution of (1.1), (1.2) which is nonnegative (and satisfies the natural growth conditions as $|x| \rightarrow \infty$). Moreover, these are the only nonnegative solutions of (1.1) which take on the initial value $u(x, 0) = 0$ for $x \neq 0$, apart from the trivial solution $u = 0$ (*i.e.* they have a positive isolated singularity at $(0, 0)$, see Theorem 1, Section 3).

In the course of our proof we estimate the rate of expansion of supports of nonnegative solutions and give necessary and sufficient conditions for the existence of a waiting time.

We then use Theorem 1 to prove that w represents the asymptotic behaviour as $t \rightarrow \infty$ of any nonnegative solution with mass $\int u_0(x) dx = M$. (Theorem 2, Section 4.) This result was known in one space dimension. In fact, for $N = 1$ Esteban and Vázquez [EV] establish a detailed description of the asymptotic behaviour of solutions and interfaces for a class of equations which generalizes both (1.1) and the porous medium equation

$$(1.4) \quad u_t = \Delta(u^m), \quad m > 1,$$

under the assumptions: $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$, u_0 has compact support.

Our method uses in a strong way the similarity properties attached to the power-like nonlinearity of the equation.

Corresponding results are also known for the porous medium equation for $N \geq 1$. Uniqueness of solutions with a bounded measure as initial datum was proved by Pierre [P]. Asymptotic behaviour is due to Friedman and Kamin [FK]. Of course, the results are well-known for the heat equation, which corresponds to the case $p = 2$ or $m = 1$ above.

Our proofs of Theorem 1 and 2 apply without any essential modification to equation (1.4), thus establishing the uniqueness of fundamental solutions (which is a part of Pierre's result) and the asymptotic behaviour of [FK] by different methods. We will give detailed references to the background

material as the proof proceeds. The corresponding self-similar solutions w are explicitly stated in Section 5.

In the last section we give yet another proof of the asymptotic behaviour both for the porous medium and p -Laplacian equations which uses completely different methods. Since the proof is analogous in both cases we have chosen to present it only for the porous medium equation. It is based on the idea of asymptotic radial symmetry.

Finally, we recall that in the stationary case

$$(1.5) \quad -\operatorname{div}(|Du|^{p-2}Du) = 0, \quad x \neq 0$$

the description of solutions with a singularity at the origin is due to [FV] and [KV]. In this work Kichenassamy and Veron establish a classification of solutions with an isolated singularity at the origin, generalizing classical results by Serrin. For equation

$$(1.6) \quad -\operatorname{div}(|Du|^{p-2}Du) + |u|^{q-1}u = 0$$

a similar study is done in [FV].

2. Preliminaries

Since equation (1.1) is degenerate at the points where $Du = (u_{x_1}, \dots, u_{x_N})$ vanishes we need to introduce a suitable concept of solution. A measurable function $u = u(x, t)$ defined in $Q_T = \mathbb{R}^N \times (0, T)$, $T > 0$, is a weak solution of (1.1) if

$$u \in C((0, T]; L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^1(0, T; W^{1,p-1}_{\text{loc}}(\mathbb{R}^N))$$

and for every test function

$$\phi \in W^{1,\infty}(0, T; L^\infty(\mathbb{R}^N)) \cap L^\infty(0, T; W^{1,\infty}(\mathbb{R}^N))$$

having compact support, the following identity holds

$$(2.1) \quad \iint_{Q_T} (-u\phi_t + |Du|^{p-2}Du \cdot D\phi) \, dx \, dt = 0$$

($Du \cdot D\phi$ means scalar product). Di Benedetto and Herrero [dBH] study the existence and uniqueness of weak solutions. They define the norm

$$(2.2) \quad \|f\|_r = \sup_{R \geq r} R^{-\lambda} \int_{B_R(0)} |f| \, dx, \quad \lambda = N + \frac{p}{p-2}$$

for $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $r > 0$; $B_s(x) = \{y \in \mathbb{R}^N: |y - x| < s\}$. They prove the following results in particular.

Result 1. For every $u_0 \in L^1_{loc}(\mathbb{R}^N)$ with $\|u_0\|_r < \infty$, $r > 0$, there exists a time $T = T(u_0)$ and a weak solution $u(x, t)$ of (1.1) in Q_T which takes on the initial data u_0 : $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$ and $u(\cdot, t) \rightarrow u_0$ in $L^1_{loc}(\mathbb{R}^N)$. Moreover for $0 < t \leq T_r(u_0)$, where

$$(2.3) \quad T_r(u_0) = C_0 \|u_0\|_r^{-(p-2)},$$

we have

$$(2.4) \quad \|u(\cdot, t)\|_r \leq C_1 \|u_0\|_r$$

$$(2.5) \quad |u(x, t)| \leq C_2 t^{-k} R^{p(p-2)} \|u_0\|_r^{pk/N} \text{ if } |x| \leq R, \quad r \leq R$$

$$(2.6) \quad |Du(x, t)| \leq C_3 t^{-(N+1)k/N} R^{2/(p-2)} \|u_0\|_r^{2k/N} \text{ if } |x| < R, \quad r \leq R$$

where $C_i = C_i(N, p)$, $i = 0, \dots, 3$. Finally u and Du are Hölder continuous in $\mathbb{R}^N \times [\tau, T(u_0) - \tau]$ with Hölder constants and exponents depending upon N , p , C_i , τ and $\|u_0\|_r$.

Result 2. Two weak solutions u, v of (1.1) defined in Q_T , such that $\|u(\cdot, t)\|_r$ and $\|v(\cdot, t)\|_r$ are uniformly bounded for $0 \leq t \leq T$ and some $r > 0$, and such that

$$(2.7) \quad u(\cdot, t) - v(\cdot, t) \rightarrow 0 \text{ in } L^1_{loc}(\mathbb{R}^N)$$

as $t \downarrow 0$, are equal in Q_T , $u \equiv v$.

Result 3. (Maximum Principle) If under the above conditions $u_0 \leq v_0$ a.e. in \mathbb{R}^N , then $u \leq v$ in their common strip of definition Q_T .

Result 4. A nonnegative solution of (1.1) in a domain Q_T admits a unique initial trace u_0 which is a σ -finite Borel measure.

Result 5. (Harnack inequality) Let u be a nonnegative solution of (1.1) in Q_T . Then for every $R > 0$; $\tau, \theta > 0$ with $\tau + \theta \leq T$.

$$(2.8) \quad R^{-N} \int_{B_R} u(x, \tau) dx \leq C_4(p, N) \left\{ \left(\frac{R^p}{\theta} \right)^{\frac{1}{p-2}} + \left(\frac{\theta}{R^p} \right)^{\frac{N}{p}} u(0, \tau + \theta)^{1 + \frac{N(p-2)}{p}} \right\}$$

As a matter of fact, existence is proved in [dBH] for initial data which are σ -finite Borel measures under the restriction of finite $\|\cdot\|_r$ norm (suitably defined for measures). One of the open problems mentioned in the paper is

the uniqueness of such solutions. A partial answer to this question will be provided by our Theorem 2.

Corresponding results are well known for the porous medium equation. Analogues of Results 1, 2 and 3 were proved by B\u00e9nilan, Crandall and Pierre [BCP], while Results 4 and 5 are due to Aronson and Caffarelli [AC]. As mentioned above the uniqueness of solutions with bounded measures as initial data was settled by Pierre [P]. This result was later extended to measures with a growth rate as $|x| \rightarrow \infty$ by Dahlberg and Kenig [DK].

We will need in our study some precise information about how supports propagate. It is well-known that for $p > 2$ equation (1.1) enjoys the *finite propagation property*, by which we mean that the support of a solution travels a finite distance in a finite time; in particular a compact support at time $t = 0$ stays compact for $t > 0$, cf. [DH], [DV]. We will deal from this moment on with nonnegative solutions, $u \geq 0$.

Let $\Omega(t) = \{x \in \mathbb{R}^N : u(x, t) > 0\}$ the positivity set of u at time t , $0 < t \leq T$. We know that the family $\{\Omega(t)\}_{t>0}$ is expanding in time (essentially a consequence of the Maximum Principle and the property of expanding supports for the solutions w_M). We begin by characterizing the points at which u stays zero for a certain time.

Proposition 2.1. *Given $x \in \mathbb{R}^N$ we have $u(x, t) > 0$ for every $t > 0$ if and only if*

$$(2.9) \quad B(x) = \sup_{R>0} R^{-\lambda} \int_{B_R(x)} |u_0| = \infty.$$

Moreover if $B(x) < \infty$ then $u(x, t) = 0$ for

$$(2.10) \quad 0 < t < C_0(p, N)B(x)^{-(p-2)}.$$

PROOF. (2.10) is a simple consequence of (2.5) and (2.3) above taking x as the origin, so that $B(x) = \lim_{r \rightarrow 0} \| |u_0| \|_r$ and letting $R = r \rightarrow 0$.

For the positivity when $B(x) = \infty$ we use the Harnack inequality with $x = 0$, $\tau + \theta = t$, divide both members by $R^{p/(p-2)}$ and let $R, \tau \rightarrow 0$ to conclude that $u(0, t) > 0$ for every $t > 0$. Remark that $B(0) = \infty$ is equivalent to

$$\lim_{R \rightarrow 0} R^{-\lambda} \int_{B_R(0)} u_0(x) dx = \infty. \quad \square$$

In case u_0 does not vanish a.e. in any neighborhood of x , and $B(x) < \infty$, then for some positive time $u(x, t) = 0$ and x will belong to the free boundary separating the regions $[u > 0]$ and $[u = 0]$ in Q_T . Therefore we have a stationary interface and

$$(2.11) \quad t^*(x) = \sup \{t \geq 0 : u(x, t) = 0\}$$

is called the *waiting time* at x . Proposition 2.1 gives a necessary and sufficient condition for a waiting time to occur.

On the contrary, if u_0 vanishes in a neighborhood of x , then we can estimate the time it takes for the solution to reach x . This estimate can be precised if $u_0 \in L^1(\mathbb{R}^N)$ as follows. Let

$$(2.12) \quad d(x) = \sup \{R: u_0(y) = 0 \text{ a.e. in } B_R(x)\}$$

be the distance from x to the support of u_0 and let us define for a set $S \subset \mathbb{R}^N$ the ρ -neighborhood as $N_\rho(S) = \{y \in \mathbb{R}^N: d(y, S) \leq \rho\}$.

Proposition 2.2. *Let $x \in \mathbb{R}^N$ with $d(x) > 0$ and let $u_0 \in L^1(\mathbb{R}^N)$, then $u(x, t) = 0$ for*

$$(2.13) \quad 0 \leq t \leq C_0(p, N)d^{\lambda(p-2)}\|u_0\|_1^{-(p-2)}$$

therefore for $T > t_2 > t_1 \geq 0$, $\Omega(t_2)$ is contained in the ρ -neighborhood of $\Omega(t_1)$, with

$$(2.14) \quad \rho = C_5(p, N)(\|u_0\|_1^{p-2}(t_2 - t_1))^{k/N}.$$

PROOF. (2.13) follows from the observation that under our assumptions the supremum in the definition of $B(x)$ is taken for $R \geq d(x)$ and choosing the worst possibility. (2.14) is a consequence of (2.13) changing the origin of time to $t = t_1$. \square

It will be of great importance in the next section that, but for the constant C_5 , formula (2.14) is the exact rate of propagation of the explicit solution w .

The above development follows closely the similar study by Caffarelli, Vázquez and Wolanski [CVW] for the porous medium equation.

3. Uniqueness of the Fundamental Solution

Let u be a nonnegative weak solution of (1.1) defined in Q_T , $T > 0$ and such that it takes continuously the initial data

$$(3.1) \quad u(x, 0) = 0 \quad \text{for } x \neq 0.$$

This solution can exhibit a *singular* behaviour as $(x, t) \rightarrow (0, 0)$. In fact, by Result 4 there exists a σ -finite Borel measure μ such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t)\phi(x) dx = \int_{\mathbb{R}^N} \phi d\mu$$

for every ϕ continuous and compactly supported in \mathbb{R}^N . Because of (3.1) μ is supported in $\{0\}$, therefore $\mu = M\delta$ and we have

$$(3.2) \quad \lim_{t \rightarrow 0} \int_{B_R} u(x, t) \, dx = M$$

for every $R > 0$. We say that u is a *fundamental* solution if it is a nonnegative weak solution and satisfies (3.1), (3.2) for some $M > 0$. The functions w defined in (1.3) are fundamental solutions of (1.1). We prove the following result.

Theorem 1. *The only nonnegative solutions of (1.1) satisfying (3.1) are the fundamental solutions w_M and the trivial solution.*

The proof is divided into several steps. We begin by estimating the support of any such solution u .

Lemma 3.1. *$u(\cdot, t)$ has compact support for every $t > 0$. Moreover $\text{supp}(u(\cdot, t)) \subset B_{\rho(t)}(0)$ with $\rho(t) \rightarrow 0$ as $t \rightarrow 0$.*

PROOF. Let $\hat{u}(x, t) = w_{M_1}(x, t + \tau)$ with some M_1 and $\tau > 0$. For R and θ small enough $\hat{u}(x, t) \geq u(x, t)$ on $\Gamma_1 = \{(x, t): |x| = R, 0 \leq t \leq \theta\}$. The inequality is true if θ is small enough by (3.1). Consider the domain

$$(3.3) \quad D_R = \{(x, t): |x| > R, 0 < t < \theta\}.$$

Since $u = 0 \leq \hat{u}$ on the bottom: $|x| \geq R$ the Maximum Principle (a variation of Result 3 above) implies that

$$(3.4) \quad u \leq \hat{u}$$

in D_R . In particular $u(\cdot, t)$ has compact support, since \hat{u} does for $0 < t < \theta$. Since we have the finite propagation property, $u(\cdot, t)$ is compactly supported for all t .

By (3.4), for $0 < t < \theta$

$$(3.5) \quad \text{supp}(u(\cdot, t)) \subset B_{r(t)}(0)$$

with $r(t) = c(t + \tau)^{k/N}$, c depends on M_1 . Letting $\tau \rightarrow 0$ we conclude that the support of $u(\cdot, t)$ shrinks to $\{0\}$ as $t \rightarrow 0$. \square

A simple consequence of Lemma 3.1 is the fact that $u(\cdot, t) \in L^1(\mathbb{R}^N)$. By conservation of mass and (3.2) we have

Corollary 3.2. $\int u(x, t) \, dx = M$ for every $M > 0$.

We may now exclude the case $M = 0$, where necessarily $u = 0$. In the sequel we assume that $M > 0$. We also know that u exists for all times $t > 0$ since it has finite mass (cf. [dBH]).

We need also L^∞ bounds. Such bounds have been proved by [Ve], [B], [HV] and are known as «smoothing effects». We need here the *sharp* version that is contained in [V2]. See [V1] for the porous medium case.

Lemma 3.3. *For every $t > 0$ and every solution u with initial data u_0 we have*

$$(3.6) \quad |u(x, t)| \leq w_M(0, t) = c_*(p, N)M^{pk/N}t^{-k}$$

where $M = \|u_0\|_1$ and $k = (p - 2 + p/N)^{-1}$.

We shall use later the sharp constant c_* in an essential way.

Our next step consists in comparing u with a Barenblatt solution. In order to simplify the calculation we set $M = 1$, what implies no loss of generality using a rescaling of our solution (see (3.18) below).

Lemma 3.4. *There exists $M' \geq 1$ such that*

$$(3.7) \quad u(x, t) \leq w_{M'}(x, t) \text{ in } Q_T$$

PROOF. We first improve our control on the support of $u(x, t)$. Let $\tau > 0$ be small so that $\text{supp}(u(\cdot, \tau)) \subset B_\epsilon(0)$ with $\epsilon \approx 0$. Then by (2.14) the support of $u(\cdot, t)$, $t > \tau$, is contained in a ball of radius

$$C_5(p, N)(t - \tau)^{k/N} + \epsilon$$

(recall that we are assuming that $\|u(\cdot, \tau)\|_1 = 1$). Letting $\epsilon \rightarrow 0$, we obtain $C_5 t^{k/N}$ which corresponds to a Barenblatt solution with a certain mass M_1 .

For M' large enough $w_{M'}(x, t) \geq c_* t^{-k}$ for $\xi = xt^{-k/N} \leq C_5$. By (3.6) we know that $u(x, t) \leq c_* t^{-k}$ for every $x \in \mathbb{R}^N$. On the other hand $u(x, t) = 0$ if $\xi \geq C_5$, therefore $u(x, t) \leq w_{M'}(x, t)$. Notice that $\int u(x, t) dx = 1$ implies that $M' \geq 1$. \square

We now take the minimum of these bounds

$$(3.8) \quad M = \inf \{M': w_{M'}(x, t) \geq u(x, t)\}.$$

By Lemma 3.4 such M exists, $M \geq 1$ and $w_M \geq u$ in Q . If we prove that $M = 1$ the theorem is complete since $\int u = \int w$, hence $u = w$. Therefore we are left with the task of excluding the possibility $M > 1$. This we do next.

Lemma 3.5. $M = 1$.

PROOF. We assume that M is larger than 1 and arrive at a contradiction by showing that it can be decreased further.

Step 1. Take $t = 1$. We know that $u(x, 1)$ is bounded above by $w_M(x, 1)$ and also by c_* , therefore if we consider the solution U which at time $t = 1$ takes on the data

$$(3.9) \quad U(x, 1) = \min \{c_*, w_M(x, 1)\}$$

it will be clear that for $t \geq 1$

$$(3.10) \quad u(x, t) \leq U(x, t) \leq w_M(x, t)$$

In fact, since $U(0, 1) \leq c_*$ and c_* is strictly less than $w_M(0, 1) = c_*M^{pk/N}$, by continuity, there exist $t_1 > 1$ and $r > 0$ such that

$$(3.11) \quad U(x, t_1) < w_M(x, t_1) \quad \text{for } |x| < r.$$

We claim that $U(\cdot, t_1)$ and $w_M(\cdot, t_1)$ do not touch at any point inside the support of $w_M(\cdot, t_1)$, i.e. for

$$(3.12) \quad |x| < a(p, N)(M^{p/N}t_1)^k = \rho_M(t_1).$$

This is so because of the Strong Maximum Principle: since for $0 < |x| < \rho$, w_M is a classical solution of (1.1) with $|Dw_M| > 0$, if they were to touch at a point (x_0, t_1) , then $Du(x_0, t_1) = Dw_M(x_0, t_1) \neq 0$. By the continuity of Du , in a neighborhood of (x_0, t_1) , $u - w_M$ is also a solution of a linear strictly parabolic equation with continuous coefficients, so the Strong Maximum Principle applies and forces $u \equiv w_M$, a contradiction.

The solutions U and w_M could touch at the free boundary $|x| = \rho(t_1)$. To tackle this situation we slightly change w_M by putting in a small delay and consider $w_M(x, t_1 + \tau)$ instead of $w_M(x, t_1)$ for a small $\tau > 0$. We claim that $U(\cdot, t_1)$ is strictly below $w_M(x, t_1 + \tau)$ inside the support of the latter function, $\{x: |x| \leq \rho_M(t_1 + \tau)\}$.

For this we need a small technical diversion.

Lemma 3.5. $w_M(x, t) > w_M(x, t + \tau)$ in a region $|x| \leq c(\tau, p, N)\rho_M(t)$ and $w_M(x, t + \tau) > w_M(x, t)$ for $c(\tau, p, N)\rho_M(t) < |x| < \rho_M(t + \tau)$. Moreover as $\tau \rightarrow 0$

$$(3.13) \quad c(\tau, p, N) \rightarrow c_\# = ((p - 2)k)^{(\rho - 1)/p} < 1.$$

PROOF. It is reduced to knowing the sign of $\partial w_M(x, t)/\partial t$. After a typically cumbersome computation we find that

$$\frac{\partial}{\partial t} w_M(x, t) = 0 \quad \text{for } |x| = \rho_M(t)c_\#,$$

being negative for smaller $|x|$ and positive otherwise. \square

Now it is clear how to prove the pending claim. Take a small $\delta > 0$. If $\tau > 0$ is small enough we have

$$(3.14) \quad U(x, t_1) < w_M(x, t_1 + \tau)$$

for $|x| \leq c_{\#} \rho_M(t_1) + \delta$ because U is strictly separated in this compact region from $w_M(x, t_1)$ and $\tau \approx 0$. On the other hand, for $c_{\#} \rho_M(t_1) + \delta \leq |x| \leq \rho_M(t_1 + \tau)$

$$(3.15) \quad U(x, t_1) \leq w_M(x, t_1) < w_M(x, t_1 + \tau).$$

Now we come to the crucial part. Since $U(\cdot, t_1)$ is strictly separated from $w_M(\cdot, t_1 + \tau)$ inside their supports, which are also strictly separated, we may slightly reduce the mass of w and have still the relationship

$$(3.16) \quad U(x, t_1) \leq w_{M-\epsilon}(x, t_1 + \tau), \quad x \in \mathbb{R}^N.$$

By the Maximum Principle the same holds for $t \geq t_1$. Moreover, since $u \leq U$ for $t \geq 1$ we get

$$(3.17) \quad u(x, t) \leq w_{M-\epsilon}(x, t + \tau) \quad \text{for } x \in \mathbb{R}^N, \quad t \geq t_1.$$

Step 2. We apply to U the rescaling operator

$$(3.18) \quad T_\theta U = \frac{1}{\theta^k} U\left(\frac{x}{\theta^{k/N}}, \frac{t}{\theta}\right).$$

Let us call $U_\theta = T_\theta U$. U_θ is a solution of (1.1) for $t \geq \theta$ which takes on the initial data

$$(3.19) \quad U_\theta(x, \theta) = \min \{c_* \theta^{-k}, w_M(x, \theta)\},$$

since w_M is invariant under T_θ . Moreover (3.16) translates into

$$(3.20) \quad U_\theta(x, t) \leq w_{M-\epsilon}(x, t + \theta\tau) \quad \text{for } t \geq \theta t_1, x \in \mathbb{R}^N.$$

The initial conditions (3.19) and our estimates on u imply again that $U_\theta \geq u$ for $t \geq \theta t_1$. Hence

$$(3.21) \quad u(x, t) \leq w_{M-\epsilon}(x, t + \theta\tau) \quad \text{for } t \geq \theta t_1.$$

At last we see the end. We let $\theta \rightarrow 0$ and obtain

$$(3.22) \quad u(x, t) \leq w_{M-\epsilon}(x, t) \quad \text{for } x \in \mathbb{R}^N \quad \text{and } t > 0.$$

We are done with Lemma 3.4 and with Theorem 1. \square

Remark. When we apply the above method to the porous medium equation, everything is similar. Even an important step is simpler. In our proof of Lemma 3.5 we have to be careful to eliminate the possibility that U touches w at $x = 0$, since the equation degenerates ($Dw_M = DU = 0$) and the Strong Maximum Principle does not apply. This difficulty does not occur for equation (1.4).

4. Asymptotic Behaviour

We discuss in this section the asymptotic behaviour of nonnegative solutions with finite mass. We prove the following result.

Theorem 2. *Let u be a nonnegative solution of (1.1) with $u_0 \in L^1(\mathbb{R}^N)$. Then*

$$(4.1) \quad \lim_{t \rightarrow \infty} t^k |u(x, t) - w_M(x, t)| = 0$$

uniformly in $x \in \mathbb{R}^N$. Here $M = \int u_0(x) dx$.

PROOF. (i). To begin with, we assume that u_0 has compact support. We consider the family $\{u_\lambda = T_\lambda u\}$ of rescaled solutions, where T_λ is defined by (3.18). Since T_λ is mass preserving, $\int u_{\lambda,0}(x) dx = M$ for every $\lambda > 0$, therefore the family $\{u_\lambda\}$ is uniformly bounded in $\mathbb{R}^N \times (\tau, \infty)$ for any $\tau > 0$ (by estimate (3.6)). Also u_λ and Du_λ are uniformly equicontinuous on compact subsets of Q , (by Result 1, Section 2).

Therefore for every sequence $\lambda_n \rightarrow 0$ there exists a subsequence (which we also denote by λ_n) such that u_λ converges uniformly to a continuous function \bar{u} and $Du_\lambda \rightarrow D\bar{u}$, also uniformly. It is clear that \bar{u} will be a weak solution of (1.1).

We claim that $\bar{u} = w_M$. In order to prove this we recall that by (2.14) the support of $u(\cdot, t)$ is contained in a ball of radius $R(t)$ with

$$(4.2) \quad R(t) \leq c\rho_M(t) + R_0,$$

where $c = c(p, N) \geq 1$ and R_0 is radius of a ball $B_{R_0}(0) \supset \text{supp}(u_0)$. This implies that for any fixed $t > 0$ the supports of the family $\{\bar{u}_\lambda(\cdot, t)\}_{\lambda > 1}$ are uniformly bounded by $R(t)$. Therefore $u_\lambda(\cdot, t)$ converges in $L^1(\mathbb{R}^N)$ to $\bar{u}(\cdot, t)$ so that

$$(4.3) \quad \int \bar{u}(\cdot, t) dx = M.$$

Every solution has an initial trace, μ . Since the support of $\bar{u}(\cdot, t)$ shrinks as $t \rightarrow 0$ its initial trace is a Dirac mass. Hence \bar{u} is a fundamental solution and, by Theorem 1, $\bar{u} = w_M$.

By the uniqueness of the limit the whole family converges as $\lambda \rightarrow \infty$. Now observe that (4.1) is equivalent to $u_\lambda(x, 1) \rightarrow w_M(x, 1)$ uniformly in x as $\lambda \rightarrow \infty$.

(ii) For general u_0 we consider an approximating sequence $\{u_{0n}\}$ such that u_{0n} has compact support and $0 \leq u_{0n} \leq u_{0, n+1} \leq u_0$.

We use the L^1 -dependence of solutions on initial data (contraction property in $L^1(\mathbb{R}^N)$, cf. [Be]): if u, \hat{u} are two solutions with initial data $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$, then for every $t > 0$

$$(4.1) \quad \|u(\cdot, t) - \hat{u}(\cdot, t)\|_1 \leq \|u_0 - \hat{u}_0\|_1.$$

Now assume that u_0 has mass M and u_{0n} has M_n with $M_n \uparrow M$. We repeat the rescaling above both on u and u_n to obtain families $\{T_\lambda u\}, \{T_\lambda u_n\}$. Along a subsequence $\lambda_n \rightarrow \infty$ $T_{\lambda_n} u \rightarrow \bar{u}$ while $T_{\lambda_n} u_n \rightarrow w_{M_n}$ by step (i). But since

$$\|T_{\lambda_n} u(\cdot, t) - T_{\lambda_n} u_n(\cdot, t)\|_1 = \|u(\cdot, \lambda_n t) - u_n(\cdot, \lambda_n t)\|_1 \leq \|u_0 - u_{0n}\|_1$$

in the limit we will get

$$\|\bar{u} - w_{M_n}\|_1 \leq \|u_0 - u_{0n}\|_1$$

which as $n \rightarrow \infty$ gives $\bar{u} = w_M$. This ends the proof. \square

Remark. Our proof applies literally to the porous medium equation, thus giving a new proof of the asymptotic behaviour of [FK], easily derived from the uniqueness result for fundamental solutions.

5. Another Approach to Asymptotic Behaviour

We give another method of proof of Theorem 2 which does not rely on the uniqueness of fundamental solutions. It uses instead the idea that solutions with compact support become asymptotically radially symmetric and the principle of concentration comparison introduced by Vázquez in [V2] for the porous medium equation, and valid also for the p -Laplacian equation. Since this is of interest also for the porous medium equation we give the proof for this equation.

We consider a solution $u \geq 0$ of the problem

$$(5.1) \quad u_t = \Delta(u^m) \quad \text{for } (x, t) \in Q, \quad m > 1$$

$$(5.2) \quad u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N$$

with $u_0 \in L^1(\mathbb{R}^N)$, $\int u_0(x) dx = M$. We want to prove

Theorem 3. *As $t \rightarrow \infty$*

$$t^k |u(x, t) - w_M(x, t)| \rightarrow 0$$

uniformly in $x \in \mathbb{R}^N$, where now $k = (m - 1 + 2/N)^{-1}$ and w_M is Barenblatt's solution

$$(5.3) \quad w_M(x, t) = t^{-k} \left(C - \frac{(m-1)k}{2mN} |\xi|^2 \right)_+^{1/(m-1)}$$

with $\xi = xt^{-k/N}$ and C is determined in terms of M, m and N .

PROOF. (i) Using the L^1 -dependence of solutions on initial data which is also true for (5.1), (5.2) and arguing as in (ii) of Theorem 2 we may assume that u_0 has compact support,

$$(5.4) \quad \text{supp}(u_0) \subset B_{R_0}(0).$$

(ii) The asymptotic symmetry of the solution will be a consequence of the following Lemma, which is a variant of Proposition 2.1 of [AC] (see also [CVW], [BA]).

Lemma 5.1. For any $|x_1| \geq R_0$ and any $t > 0$ we have

$$(5.5) \quad u(x_1, t) \geq u(x_2, t)$$

if $|x_2| \geq |x_1| + 2R_0$.

PROOF. We draw the hyperplane Π which bisects the segment x_1x_2 . Assume that this hyperplane leaves the support of u_0 in the same half-space Ω_1 as x . We consider the solutions

$$\begin{aligned} u_1(x, t) &= u(x, t), \\ u_2(x, t) &= u(\pi(x), t), \end{aligned}$$

where $\pi(x)$ is the symmetric of $x \in S = \Omega_1 \times (0, \infty)$ with respect to Π . Since $u_1 = u_2$ on the lateral boundary $\Pi \times [0, \infty)$ and $u_1 \geq u_2 = 0$ at the bottom $\bar{\Omega}$ we conclude that $u_1 \geq u_2$ in S , i.e. $u(x, t) \geq u(\pi(x), t)$. Since $x_1 \in \Omega$ and $x_2 = \pi(x_1)$, $u(x_1, t) \geq u(x_2, t)$ for $t > 0$.

Finally, the condition for Π to leave $B_{R_0}(0)$ on the same side as x_1 is just $|x_2| \geq |x_1| + 2R_0$. \square

Suppose now that we consider the rescaled solutions $T_\lambda u$ given again by (3.18) with our present value of k . It happens that (5.5) holds if

$$(5.6) \quad |x_2| \geq |x_1| + 2R_0\lambda^{k/N}, \quad |x_1| \geq R_0\lambda^{-k/N}.$$

As in Section 4 we pass to the limit $\lambda_n \rightarrow \infty$ thanks to the boundedness and equicontinuity of the solutions which are well-known (cf. [FK]) and we obtain

$T_{\lambda_n} u \rightarrow \bar{u}$ uniformly on compact subsets of Q . Moreover the supports are uniformly bounded (cf. [CVW]) hence $T_{\lambda_n} u \rightarrow \bar{u}$ uniformly in $x \in \mathbb{R}^N$, $0 < \tau < t < 1/\tau$. Moreover $\bar{u}(\cdot, t) \in L^1(\mathbb{R}^N)$ with $\|\bar{u}(\cdot, t)\|_1 = M$. Passing to the limit $\lambda \rightarrow \infty$ in (5.5), (5.6) we deduce that

$$(5.7) \quad \bar{u}(x_1, t) \geq \bar{u}(x_2, t) \quad \text{if } |x_2| \geq |x_1|, \quad t > 0,$$

i.e. \bar{u} is radially symmetric and decreasing.

(iii) We now recall the concept of concentration for nonnegative, radially symmetric functions in $L^1(\mathbb{R}^N)$ as defined in [V1]. We say that f is less concentrated than g , $f \prec g$ if for every $r > 0$.

$$(5.8) \quad \int_0^r f(\rho)\rho^{N-1} d\rho \leq \int_0^r g(\rho)\rho^{N-1} d\rho.$$

We will use the property that the relation \prec is hereditary for problem (5.1), (5.2).

Lemma 5.2 [V1]. *Let u_1, u_2 be radially symmetric and nonnegative solutions of (5.1), (5.2) with initial data u_{01}, u_{02} . Then if $u_{01} \prec u_{02}$, we have $u_1(\cdot, t) \prec u_2(\cdot, t)$ for all $t > 0$.*

Obviously the most concentrated initial data are $u_0 = M\delta(x)$ corresponding to a Dirac mass. Therefore we have for every $t > 0$ and every u with mass M ,

$$(5.9) \quad \bar{u}(\cdot, t) \prec w_M(\cdot, t).$$

It easily follows from (5.9) that $\|u(\cdot, t)\|_\infty \leq \|w(\cdot, t)\|_\infty$, *i.e.* the smoothing effect. (The above argument needs a justification since δ is a measure, not a function. This is however done easily by approximation; in particular replace $M\delta$ with $w_M(\cdot, \tau)$, $\tau \approx 0$ and approximate u_{0n} from below.)

(iv) We need now an estimate converse to (5.9). This is done by first remarking that arguing as in Sections 3 and 4 with the estimates for the support of [CVW] we conclude that

$$(5.10) \quad \text{supp}(\bar{u}(\cdot, t)) \subset B_{R(t)}(0)$$

with

$$(5.11) \quad R(t) \leq c\rho_M(t)$$

where $\rho_M(t)$ is the radius of the support of w_M . Take $\tau > 0$, τ small. Then $\text{supp} \bar{u}(\cdot, \tau)$ is contained in a very small ball. Since also \bar{u} is radially symmetric and decreasing, it is easy to see that for some θ depending on τ

$$(5.12) \quad w_M(\cdot, \tau + \theta) \prec \bar{u}(\cdot, \tau).$$

This property will continue to be valid for $t \geq \tau$. Letting $\tau \rightarrow 0$, hence $\theta \rightarrow 0$, we obtain

$$(5.13) \quad w_M(\cdot, t) \searrow \bar{u}(\cdot, t) \quad \text{for } t > 0.$$

(5.13) and (5.9) imply that

$$(5.14) \quad \int_0^r w_M(\rho, t) \rho^{N-1} d\rho = \int_0^r \bar{u}(\rho, t) \rho^{N-1} d\rho,$$

for every $r > 0$, hence $w_M \equiv \bar{u}$.

(v) Since the limit \bar{u} is unique, the whole family $T_\lambda u$ converges to \bar{u} . \square

Remark. This proof generalizes easily to the p -Laplacian equation with $p > 1$. It can be also applied to $u_t = \Delta\phi(u)$, with assumptions on the nondecreasing function ϕ which make it resemble a power, as the ones imposed in [dPV].

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