Revista Matemática Iberoamericana
Vol. 4, n.° 1, 1988

L^p Estimates for Degenerate Elliptic Equations

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Dedicated to the memory of J. L. Rubio de Francia

Introduction

In this note we are going to address the question of when is a second order differential operator «controlled» by a subelliptic second order differential operator.

By a second order subelliptic operator on a compact manifold M we mean an operator L that in local coordinates is of the form

$$L = \Sigma a^{ij}(x)\partial x_i \, \partial x_j + \Sigma b^j(x) \, \partial x_j + c(x),$$

with a^{ij}, b^j, c all real and C^∞, $(a^{ij})^T = (a^{ij})$ positive semidefinite, and L satisfies a subelliptic estimate: for some $\epsilon > 0$

$$\|u\|_{H^\epsilon} \leqslant c(\|Lu\|_2 + \|u\|_2), \qquad u \in C^\infty(M)$$

where H^s denote the classical Sobolev spaces and $\| \ \|_p$ stands for the norm in $L^p = L^p(M, d\mu)$ for some smooth positive measure μ on M (fixed from now on).

Let now P be a second order differential operator on M. We want to know under which conditions an a priori inequality

$$(*) \qquad \|Pu\|_p \leqslant C_p(\|Lu\|_p + \|u\|_p), \qquad u \in C^\infty(M), \qquad 1 < p < \infty$$

holds.

If $P = \Sigma b^{ij}(x)\,\partial x_i\,\partial x_j + \cdots$ in local coordinates, then testing with $u(x) = e^{itx\cdot\xi}\phi(x)$ with $\phi \in C_0^\infty$ and letting $t \to +\infty$, we see that if the inequality (*) holds for some p, $1 < p < \infty$, then

$$|\Sigma b^{ij}(x)\xi_i\xi_j| \leqslant C\,\Sigma a^{ij}(x)\xi_i\xi_j,$$

i.e. the principal symbol for P is bounded by a constant times the principal symbol of L.

We will show in this article that if P is selfadjoint (with respect to μ), then this condition is also sufficient.

We will also state a more technical necessary and sufficient condition for first order operators, much on the flavor of the results of Fefferman an Phong [1], where some sufficient conditions for the L^2 estimates are given.

For prior results see [3, 4].

2. Background

We are going to review some facts about subelliptic operators that will be needed, specially the results of [3] on which this paper relies.

Assume first that L is self-adjoint, $L = \Sigma a^{ij}(x)\,\partial x_i\,\partial x_j + \cdots$ in local coordinates. We say a tangent vector at x, $\Sigma\alpha^i\,\partial x_i$, is subunit if $(\Sigma\alpha^i\xi_i)^2 \leqslant \Sigma a^{ij}(x)\xi_i\xi_j$ for all ξ.

With this, one can associate a distance d to L whose corresponding balls are given by [2]:

$$B_L(x;\lambda) = \{y \in M\colon \text{there exists } \phi\colon [0,\lambda] \to M \text{ Lipschitz with } \phi(0) = x,$$
$$\phi(\lambda) = y \text{ and } \phi'(t) \text{ subunit at } \phi(t) \text{ for a.e. } t\}.$$

Observe that $B_L(x;\lambda) = B_{\lambda^2 L}(x;1)$.

In the case of a general L, write $L = L^{sa} + X_0$, with L^{sa} selfadjoint and X_0 a vector field we set

$$B_L(x;\lambda) = B_{\lambda^2 L^{sa} + \lambda^4 X_0^* X_0}(x;1) \qquad (= B_{\lambda^2 L}(x;1)).$$

(*a*) *Standard balls.* We are going to recall the construction of the «standard» balls $B_\lambda(x) = B_\lambda^L(x)$ as in [1], [3]. These balls are equivalent to the metric balls in the sense that

$$B_L(x;c\lambda) \subseteq B_\lambda^L(x) \subseteq B_L(x;C\lambda).$$

Assume for simplicity that $x = 0$ and L is selfadjoint. Let $L = \Sigma a^{ij}(x)\,\partial x_i\,\partial x_j + \cdots$ in local coordinates and consider

$$\lambda^2 L + \lambda^{2N}\Delta = \lambda^2\Sigma a_\lambda^{ij}(x)\,\partial x_i\,\partial x_j + \cdots,$$

where Δ is the Laplacian $\partial^2_{x_1} + \cdots + \partial^2_{x_n}$ and $N >> 1/\epsilon$ (as in [2] the $\lambda^{2N}\Delta$ term is introduced for technical reasons, mainly to assume we are dealing with polynomials when convenient).

From all the cubes Q_δ with center 0 and side $\delta = 2^{-k}$ take the biggest one for which

$$\max_i \max_{Q_\delta} \lambda^2 a^{ii}_\lambda(x) \geqslant K\delta^2$$

where K is a constant much bigger than a bound on the a^{ij}'s and their derivatives up to order 2. It is no restriction to assume the maximum is reached for $i = 1$.

Since $Q_{2\delta}$ was not chosen, $\lambda^2 a^{ii}_\lambda(x) \leqslant 4K\delta^2$ for $x \in Q_{2\delta}$, *i.e.* $a^{ii}_\lambda(x) \leqslant 4K(\delta/\lambda)^2$ in $Q_{2\delta}$. Since $|\partial^\alpha a^{ij}_\lambda| << K$ if $|\alpha| \leqslant 2$ and $a^{11}_\lambda \geqslant 0$ then

$$\begin{aligned} a^{11}_\lambda(x) &\geqslant c(\delta/\lambda)^2 \quad \text{in} \quad Q_{\delta/\lambda} \\ a^{ii}_\lambda(x) &\leqslant 10K(\delta/\lambda)^2 \quad \text{in} \quad Q_{\delta/\lambda} \\ |a^{ij}_\lambda(x)| \leqslant [a^{jj}_\lambda(x) a^{ii}_\lambda(x)]^{1/2} &\leqslant 10K(\delta/\lambda)^2 \quad \text{in} \quad Q_{\delta/\lambda}. \end{aligned}$$

Also $|\partial^\alpha a^{ij}_\lambda(x)| \leqslant C_\alpha(\delta/\lambda)^{2-|\alpha|}$ in $Q_{\delta/\lambda}$ since that is clearly true for $|\alpha| = 0$, $|\alpha| \geqslant 2$ and so for $|\alpha| = 1$ by interpolation. After scaling by δ/λ and a change of variables (with bounds independent of δ/λ) we can assume

$$\lambda^2 L + \lambda^{2N}\Delta \approx \lambda^2\Big(\partial^2_{u_1} + \sum_{i,j\geq 2} r^{ij}(u_1, \bar{u})\,\partial u_i \,\partial u_j + \cdots\Big),$$

$\bar{u} = (u_2, \ldots, u_n)$. (Here $\Sigma c^{ij}\partial_i\partial_j + \cdots \approx \Sigma d^{ij}\partial_i\partial_j + \cdots$ means $s(c^{ij}) \leqslant (d^{ij}) \leqslant (1/s)(c^{ij})$ as matrices, s independent of the parameters). Also one has bounds for r^{ij} and its derivatives independent of δ, λ and $\sum_{i,j\geq 2} r^{ij}(u_1, \bar{u})\eta_i\eta_j \geqslant \lambda^{2N'}|\bar{\eta}|^2$ so a Taylor expansion around $u_1 = 0$ allows us to assume r^{ij} is a polynomial in u_1 (if $|u_1| \leqslant C\lambda$, λ small).

In these coordinates

$$B^L_\lambda(0) = (-\lambda, \lambda) \times B^{\bar{L}}_\lambda(0)$$

where

$$\bar{L} = \sum_{i,j\geq 2} \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} r^{ij}(u_1, \bar{u})\,du_1\,\partial u_i\,\partial u_j + \cdots$$

The process is completed by using induction on the dimension.

Composing all the changes of variables and scaling to the unit cube gives a map $\Phi: Q_1 \to B_\lambda(0)$. This map is of the form $\Phi = \Phi_1 \circ \cdots \circ \Phi_n$, where

$$\Phi_j(u) = \Big(u_1, \ldots, u_{j-1}, \frac{\delta_i}{\lambda}\phi_j(\lambda u_j, u_{j+1}, \ldots, u_n)\Big)$$

with ϕ_j and its inverse ψ_j having bounds for its derivatives independent of λ. As a consequence $|\partial^\alpha \Phi(u)| \leqslant C_\alpha \lambda^{|\alpha|}$ and the Jacobian $|\Phi'|$ satisfies $|\Phi'(z)| = \mu(B_\lambda(0))g(z)$, with $c \leqslant g(z) \leqslant C$, $|\partial^\alpha g(z)| \leqslant C_\alpha \lambda^{|\alpha|}$. Also the scaling Φ is defined in a much larger cube and the estimates above hold in $Q_s = \{z: |z_i| \leqslant s\}$ with constants depending on s too, of course (see [3] for more properties).

(*b*) *Fundamental solution for L.* In [3] an approximate solution for L was constructed, that is, an operator K such that $LKu = u + Eu + Su$ with S smoothing and E a singular integral with respect to $d(x,y)$ whose L^p operator norm can be made arbitrarily small (for p fixed). In particular, it is not difficult to see that to prove an estimate of the form

$$\|Pu\|_p \leqslant C(\|Lu\|_p + \|u\|_p)$$

it suffices to show that PK is bounded in L^p.

The operator K is of the form ΣK_j with

$$K_j f(x) = \int K_j(x,y) f(y)\, d\mu(y);$$

here $K_j(x,y)$ is smooth and satisfies

(1) $\operatorname{supp} K_j \subseteq \{(x,y): d(x,y) \leqslant CR^{-j}\}$.

(2) $\int K_j(x,y)\, d\mu(y) = 0$.

(3) If $\Phi: Q_1 \to B_{R^{-j}}(x)$ is one of the scalings then

$$(3.1) \qquad |\partial_\omega^\alpha K_j(\Phi(w), y)| \leqslant C\alpha \frac{R^{-2j}}{\mu(B_L(x; R^{-j}))},$$

$$(3.2) \qquad |\partial_\omega^\alpha K_j(\Phi(w), y) - \partial_\omega^\alpha K_j(\Phi(w), y')| \leqslant C\alpha \frac{R^{-j}}{\mu(B_L(x; R^{-j}))} d(y,y')$$

if $d(y,y') \leqslant cR^{-j}$.

(3.3) Similarly for $K_j(y, \Phi(w))$.

(see [3] for these properties; (3) is not stated explicitely but it follows easily from the results there).

3. L^p Bounds for the Self-Adjoint Case

Assume P is a smooth, selfadjoint second order differential operator in M. Assume also that P has no zeroth order term. In local coordinates $d\mu = h(x)\,dx$, $P = (1/h)\Sigma\partial_i(hb^{ij}\partial_j)$ and $L = \Sigma a^{ij}(x)\partial_i\partial_j + \cdots$, where $b(x,\xi) = \Sigma b^{ij}(x)\xi_i\xi_j$ and $a(x,\xi) = \Sigma a^{ij}(x)\xi_i\xi_j$ are the principal symbols of P and L respectively.

The basic ingredient in the proof of the estimates in the following

Lemma. *Assume* $|b(x, \xi)| \leqslant Ca(x, \xi)$ *and that* $\Phi: Q_1 \to B_\lambda(y)$, λ *small, is one of the scalings described above. Then the pullback of* P *by* Φ satisfies

$$\Phi^*(\lambda^2 P) = \Sigma d^{ij}\partial_i\partial_j + \Sigma d^j \partial_j$$

where the d's and their derivatives have bounds independent of λ, Φ.

PROOF. Assume for simplicity of notation that $y = 0$. Under Φ, h is transformed into $h(\Phi(w))|\Phi'(w)|$, and this is of the form $\mu(B_\lambda(0))f(w)$, with $0 < c \leqslant f(w) \leqslant C$ and $|\partial_w^\alpha f| \leqslant C_\alpha$. Since the constant $\mu(B_\lambda(0))$ cancels out, we only need to check how the b^{ij}'s transform. We will do that by induction on the dimension, so assume the lemma holds in dimension $n - 1$ (the initial case of dimension 1 is done as the induction step).

Recall that in the construction of Φ we have first a change of variables $x = \delta\phi(u)/\lambda$ followed by a scalling by λ in u_1 (with bounds for ϕ, $\psi = \phi^{-1}$ and their derivatives independent of λ). The change

$$x = \frac{\delta}{\lambda}\phi(u)$$

sends $(\lambda^2 b^{ij}(x))$ to

$$(\lambda^2 \tilde{b}^{ij}(u)) = \lambda^2 \frac{\lambda^2}{\delta^2}\psi'(\phi(u))\left(b^{ij}\left(\frac{\delta}{\lambda}\phi(u)\right)\psi'^t(\phi(u))\right).$$

Claim. $|\partial^\alpha \tilde{b}^{ij}| \leqslant C_\alpha$.

To see this, it suffices to prove that $|\partial_x^\alpha b^{ij}| \leqslant C_\alpha(\delta/\lambda)^{2-|\alpha|}$ in $Q_{\delta/\lambda}$. Since this is clearly true for $|\alpha| \geqslant 2$, it suffices to show $|b^{ij}| \leqslant C(\delta/\lambda)^2$ (the derivatives of order one follow by an easy interpolation argument). Now

$$|\Sigma b^{ij}(x)\xi_i\xi_j| \leqslant C\Sigma a^{ij}(x)\xi_i\xi_j,$$

so

$$|b^{ii}(x)| \leqslant Ca^{ii}(x) \quad \text{and} \quad |b^{ij}(x)| \leqslant C(a^{ii}(x) + |a^{ij}(x)| + a^{jj}(x)).$$

Since by construction $|a^{ij}(x)| \leqslant C(\delta/\lambda)^2$ in $Q_{\delta/\lambda}$, that proves the claim.

Now, in the u-coordinates

$$\left|\sum_{i,j\geq 2} \tilde{b}^{ij}(u_1, \bar{u})\eta_i\eta_j\right| \leqslant C\sum_{i,j\geq 2} r^{ij}(u_1, \bar{u})\eta_i\eta_j$$

$$\leqslant C\sum_{i,j\geq 2}\frac{1}{2\lambda}\int_{-\lambda}^{\lambda} r^{ij}(u_1, \bar{u})\,du_1\,\eta_i\eta_j$$

(the principal symbol of $\bar{L}$) since

$$\sum_{i,j\geq 2} r^{ij}(u_1,\bar{u})\eta_i\eta_j \geqslant 0$$

is a polynomial in u_1.

If we now change variables $u_1 = \lambda w_1$, $\bar{u} = \bar{\Phi}(\bar{w})$, where $\bar{\Phi}\colon \bar{Q}_1 \to B_\lambda^{\bar{L}}(0)$ is the map corresponding to $\bar{L}$ and we call $(\bar{b}^{ij}(w))$ the matrix that $(\lambda^2 \bar{\mathscr{b}}^{ij}(u))$ is transformed into, we want to show that $|\partial_w^\alpha b^{ij}| \leqslant C_\alpha$.

By the induction hypothesis $|\partial_{\bar{w}}^{\bar{\alpha}} \bar{b}^{ij}| \leqslant C_\alpha$ if $i,j \geqslant 2$, so $|\partial_w^\alpha \bar{b}^{ij}| \leqslant C_\alpha$ for $i,j \geqslant 2$. Also $\bar{b}^{11}(w) = \lambda^2 \bar{\mathscr{b}}^{11}(\lambda w_1, \bar{\Phi}(\bar{w}))$, so $|\partial_w^\alpha \bar{b}^{11}| \leqslant C_\alpha$. We are left with $\bar{b}^{1j}$, $2 \leqslant j \leqslant n$. To deal with them, recall that $\bar{\Phi}$ is a composition of maps of the form

$$u \to (u_1, \ldots, u_j, (\delta_i/\lambda)\phi_j(\lambda u_{j+1}, u_{j+2}, \ldots, u_n)).$$

It is not difficult then to see that $|\partial^\alpha \bar{b}^{1j}| \leqslant C_\alpha(\lambda^{|\alpha|+2}/(\delta_1 \cdots \delta_n)) \leqslant C_\alpha^*$ if $|\alpha|$ is large enough ($\delta_1 \cdots \delta_n \geqslant c\lambda^{n+1/\epsilon}$ as a consequence of [2]). Again it suffices then to get the estimates for $|\alpha| = 0$, *i.e.* to prove the

Claim. $|\bar{b}^{1j}| \leqslant C$.

To see this, observe that if $(\bar{r}^{ij}(w))$ is the matrix for

$$\sum_{i,j\leq 2} \lambda^2 r^{ij}(\lambda w_1, \bar{u})\,\partial u_i\,\partial u_j + \cdots$$

after the change of variables $\bar{u} = \bar{\Phi}(\bar{w})$ then we know that

$$\left|\sum_{i,j=1}^{n} \bar{b}^{ij}(w)\zeta_i\zeta_j\right| \leqslant C\Big(\zeta_1^2 + \sum_{i,j\leq 2} \bar{r}^{ij}(w)\zeta_i\zeta_j\Big)$$

so

$$|\bar{b}^{1j}(w)| \leqslant C(1 + \bar{r}^{jj}(w) + |\bar{b}^{11}(w)| + |\bar{b}^{jj}(w)|)$$

From those terms we only have to worry about checking that $\bar{r}^{jj}(w)$ is bounded. However that is a consequence of the induction hypothesis applied to $\sum_{i,j\geq 2} r^{ij}(\lambda w_1, \bar{u})\,\partial u_i\,\partial u_j + \cdots$, since as it was mentioned

$$\left|\sum_{i,j\geq 2} r^{ij}(\lambda w_1, \bar{u})\eta_i\eta_j\right| \leqslant C \sum_{i,j\geq 2} \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} r^{ij}(u_1, \bar{u})\,du_1\,\eta_i\eta_j.$$

Thus finishes the proof of the Lemma.

With the notation used above we can now state

Theorem. *Let L be a subelliptic second order operator, P a smooth self-adjoint second order differential operator and* $1 < p < \infty$. *Then the estimate*

$$\|Pu\|_p \leqslant C(\|Lu\|_p + \|u\|_p), \qquad u \in C^\infty(M)$$

holds if and only if $|b(x, \xi)| \leqslant Ca(x, \xi)$ *where* $b(x, \xi)$, $a(x, \xi)$ *are the principal symbols of P and L respectively.*

Proof. To show that $|b(x, \xi)| \leqslant Ca(x, \xi)$ is a necessary condition, take, in local coordinates, u of the form $u(x) = e^{itx \cdot \xi}\phi(x)$, $\phi \in C_0^\infty$, real. If the estimate $\|Pu\|_p \leqslant C(\|Lu\|_p + \|u\|_p)$ holds, we have

$$t^{2p} \int |\Sigma b^{ij}(x)\xi_i\xi_j|^p\phi(x)^p h(x)\, dx \leqslant C\Big(t^{2p} \int (\Sigma a^{ij}(x)\xi_i\xi_j)^p\phi(x)^p h(x)\, dx + O(t^p)\Big)$$

for all t, so

$$\int (|b(x, \xi)|^p - Ca(x, \xi)^p)\phi^p(x)h(x)\, dx \leqslant 0$$

for all $\phi \in C_0^\infty$ and hence $|b(x, \xi)|^p \leqslant Ca(x, \xi)^p$.

To prove the converse we can clearly assume that P has no zeroth order therm and, as it was mentioned, we only need to show that PK is bounded in L^p. Now $PK = \Sigma PK_j$, and PK_j is given by integration against a kernel $F_j(x, y) = P^x K_j(x, y)$.

The L^p estimate follows by classical arguments if we show that PK is a singular integral, *i.e.* if we prove that

(s.i.1) $F_j(x, y)$ is supported in $\{(x, y) : d(x, y) \leqslant cR^{-j}\}$ and

$$\int F_j(x, y)\, d\mu(x) = \int F_j(x, y)\, d\mu(y) = 0.$$

(s.i.2)
$$|F_j(x, y)| \leqslant c/\mu(B_L(x; R^{-j}))$$

and

$$|F_j(x, y) - F_j(x, y')| + |F_j(y, x) - F_j(y', x)| \leqslant C\frac{R^j}{\mu(B_L(x; R^{-j}))}d(y, y').$$

But (s.i.1) is a consequence of the properties (1), (2) of $K_j(x, y)$ and the fact that $P(1) = 0$. The estimates (s.i.2) follow from the property (3) of $K_j(x, y)$ and the lemma applied to the scaling $\Phi: Q_1 \to B_{R^{-j}}(x)$.

4. First Order Operators

Consider now the case of a smooth vector field Y on M. In local coordinates $Y = \Sigma y^j(x)\,\partial x_j$ and its symbol is $y(x, \xi) = i\Sigma y^j(x)\xi_j$. If $\Phi\colon Q_1 \to B_\lambda(m)$ is a scaling map then the symbol of $\Phi^*(Y)$, the pullback of Y by Φ, is given by $y(\Phi^c(z, \eta))$, where $\Phi^c(z, \eta) = (\Phi(z), \eta(\Phi'(z))^{-1})$ is the induced map on the cotangent space. We can now state

Theorem. *Let* $1 < p < \infty$. *The estimate*

$$\|Yu\|_p \leqslant C(\|Lu\|_p + \|u\|_p) \qquad u \in C^\infty(M)$$

holds if and only if there is a constant C_0 *such that*

$$\max_{\eta \in Q_1} \max_{z \in Q_1} |y(\Phi^c(z, \eta))| \leqslant C_0 \lambda^{-2}$$

for all scalings $\Phi\colon Q_1 \to B_\lambda(m)$, λ *small.*

PROOF. To show that the L^p estimate holds under the symbol condition we can argue as we did for P. We need to prove then that $\Phi^*(\lambda^2 Y) = \Sigma d^j(z)\,\partial z_j$ with $|\partial^\alpha d^j| \leqslant C_\alpha$, the C_α's independent of the scaling $\Phi\colon Q_1 \to B_\lambda(m)$. Assume $m = 0$. Recalling that Φ is a composition of maps

$$u \to (u_1, \ldots, u_{k-1}, (\delta_k/\lambda)\phi_k(\lambda u_k, \ldots, u_n))$$

it is not difficult to check that

$$|\partial_z^\alpha d^j(z)| \leqslant C_\alpha \frac{\lambda^{|\alpha|+2}}{\delta_1 \cdots \delta_n}$$

so if $|\alpha|$ is large $\partial^\alpha d^j$ is bounded. Since by assumption $d^j(z)$ is bounded, it follows that $\partial^\alpha d^j$ is bounded for all α.

To prove the converse, observe that applying the L^p inequalities to $v \circ \Phi^{-1}$ we get

$$\|\tilde{Y}v\|_p \leqslant C(\|\tilde{L}v\|_p + \lambda^2\|v\|_p), \qquad v \in C_0^\infty(Q_2)$$

where $\tilde{Y}, \tilde{L}$ are the pullbacks of $\lambda^2 Y$, $\lambda^2 L$ by the scaling of $B_\lambda(m)$ and $\|\ \|_p$ now denotes L^p norm with respect to Lebesgue measure (we can do that since the Jacobian $|\Phi'|$ is of the order of magnitude of $\mu(B_\lambda(m))$ in Q_2, so we can divide by it).

Taking now $v(z) = e^{z\cdot\eta}\phi(z)$ with $\eta \in Q_1$ and $\phi \in C_0^\infty(Q_2)$ a function with $\phi \equiv 1$ on Q_1 and using the fact that the coefficients of $\tilde{Y}$ have bounds independent of Φ and λ we get

$$\int_{Q_1} |\Sigma d^j(z)\eta_j|^p \, dz \leqslant C$$

where $\tilde{Y} = \Sigma d^j(z)\partial_{z_j}$. This, in turn, implies

$$|\lambda^2 y(\Phi^c(z,\eta))| = |\Sigma d^j(z)\eta_j| \leqslant C_0 \quad \text{for} \quad z \in Q_1.$$

In fact, using that $|\partial^\alpha(\Sigma d^j(z)\eta_j)| \leqslant C_\alpha$ if $|\alpha|$ is large and that

$$\max_{Q_1} |f(z)| \leqslant C\Big(\int_{Q_1} |f(z)|^p \, dz\Big)^{1/p}$$

for polynomials of some fixed degree, we get

$$\max_{Q_1} |\Sigma d^j(z)\eta_j| \leqslant C_1\Big(\int_{Q_1} |\Sigma d^j(z)\eta_j|^p\Big)^{1/p} + C_2 \leqslant C_0.$$

This finishes the proof.

5. Final Remarks

The same results hold if L^p is replaced by the Hölder spaces

$$\Gamma^\alpha(M) = \{f \text{ continuous in } M: |f(x) - f(y)| \leqslant Cd(x,y)^\alpha\}, \qquad 0 < \alpha < 1.$$

Also, by reducing it to the case of a compact manifold, one can get similar results for bounded open sets in $\mathbb{R}^n$.

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* Research supported in part by a NSF grant.