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Non-Negative Solutions to Fast Diffusions

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Dedicated to the memory of our friend José Luis Rubio de Francia

Introduction

The purpose of this work is to study the class of non-negative continuous weak solutions of the non-linear evolution equation

$$(1.1) \qquad \frac{\partial u}{\partial t} = \Delta \varphi(u), \qquad x \in \mathbb{R}^n, 0 < t < T \leqslant +\infty.$$

Here the non-linearity φ is assumed to be continuous, increasing, with $\varphi(0) = 0$. In the case when $\varphi(u)$ is super-linear when u tends to $+\infty$, the corresponding study was carried out in [7] (and in the pure power case in [2], [6] and [4]). Here, we treat the case when $\varphi(u)$ is sub-linear when u tends to $+\infty$ (but not *too* sub-linear, as we shall see below). In doing so, we complement and extend (and make extended use of) some results of M. A. Herrero and M. Pierre [12], who have studied the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad 0 < m < 1, \text{ in } \mathbb{R}^n \times (0, \infty),$$

with any initial data f in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. They prove that, given any such f, there exists a unique (in an appropriate class) solution u to the Cauchy problem. Moreover, in the case when $m > (n-2)/n$, they establish a regularizing effect from L^1_{loc} to L^∞_{loc}, for the solution, exploiting the well-known Aronson-Bénilan [1] differential inequality. It is known (see [5]) that for $0 < m \leqslant (n-2)/n$, such a regularizing effect cannot hold.

Note that existence of solutions for arbitrary $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in sharp contrast with the super-linear case ([2], [4], [6], [7]) where, even to get existence for finite time, growth conditions at infinity on the initial data are necessary.

The quantitative assumptions on our non-linearity φ are

$$(1.2)\quad \begin{cases} \text{(i)} & a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} \leqslant \dfrac{1}{a}, \qquad u > 0 \\ \text{(ii)} & \dfrac{n-2}{n} + a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} \leqslant 1 - a, \qquad u > u_0 \end{cases}$$

Assumption (i) is a global, scale invariant, polynomial growth for φ, while (ii) reflects sub-linearity for u large, but also reflects a power behavior larger than $(n-2)/n$ for u large.

Under (1.2), we prove (Theorem 3.11) that any non-negative continuous weak solution of (1.1) in $\mathbb{R}^n \times (0, T)$, $0 < T \leqslant +\infty$ has an initial trace μ, as $t \downarrow 0$, which is a locally finite Borel measure (in fact, only (1.2)(i) is needed for this). We also establish the L^1_{loc} into L^∞_{loc} regularizing effect (Corollary 3.20). Here, the full force of (1.2) is needed, since, without the left-hand side of (1.2)(i)), this fails. At this point, there is a crucial methodological difference with the work of Herrero and Pierre ([12]), since the Aronson-Bénilan inequality [1] is unavailable to us, given our assumptions on φ. We then prove (Theorem 3.22), that given any locally finite, positive Borel measure μ there exists a non-negative continuous weak solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$, with initial trace μ. Again, the left-hand inequality of (1.2)(ii) is needed here, because of the work of Brézis and Friedman [5], which shows that for $\varphi(u) = u^m$, $0 < m \leqslant (n-2)/n$, there is no solution with initial data the delta mass. Moreover, the right-hand inequality in (1.2)(ii) is also needed, since there are no growth restrictions on the measure μ at ∞. Finally, (Theorem 3.28), we show that non-negative, continuous weak solutions of (1.2) in $\mathbb{R}^n \times (0, T)$, $0 < T \leqslant +\infty$, are uniquely determined by their initial trace. To show this, we adopt techniques in [16], [12], [6], and [7]. This result may be somewhat surprising, since such solutions verify no growth restrictions at infinity. A corollary of these results (Theorem 3.33) is that any non-negative continuous weak solution of (1.1) in $\mathbb{R}^n \times (0, T)$, $0 < T \leqslant +\infty$, with φ verifying (1.2) has a unique extension to a solution in $\mathbb{R}^n \times (0, \infty)$. This again is in sharp contrast with the super-linear case, where blow-up in finite time can occur.

Several interesting questions along the lines of this work remain. For instance, one would like to be able to classify, if one does not assume the left-hand inequality in (1.2)(ii), the class of «admissible» initial traces. Also, it would be of interest to develop, assuming (1.2), the theory for the initial-Dirichlet problem in finite cylinders, along the lines of [8]. Another important question is whether, under (1.2), non-negative continuous weak solutions of (1.1), and weak solutions of (1.1) are the same, as in the super-linear case ([9], [10]). We hope to return to some of these questions at a later time.

All the solutions u considered in this work are non-negative. We will sometimes omit specific mention of this fact.

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2. Preliminary Results

In this section we collect some preliminary results that will be used throughout the paper. Most of the proofs are omitted, since they are essentially contained in [7]. We will work with continuous, strictly increasing φ on $0 \leqslant u < +\infty$, that are positive on $0 < u < +\infty$, with $\varphi(0) = 0$. In this section we only impose the growth condition

$$0 < a \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant \frac{1}{a}, \qquad 0 < u < \infty \tag{2.1}$$

and the normalization

$$\varphi(1) = 1. \tag{2.2}$$

When φ verifies (2.1) and (2.2), we say that $\varphi \in \mathfrak{A}_a$.

We say that u is a continuous non-negative weak solution of

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \quad \text{in} \quad \Omega \subset \mathbb{R}^{n+1} \tag{2.3}$$

if u is continuous in Ω, $u \geqslant 0$, and (2.3) holds in Ω in the distribution sense.

Set $B = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\tau_1 < \tau_2$, and let $Q = B \times (\tau_1, \tau_2)$. Denote by $\partial_p Q$ the parabolic boundary of Q, *i.e.*, $\partial_p Q = \bar{Q} \setminus (B \times \{\tau_2\})$.

Let $\varphi \in \mathfrak{A}_a$, $0 < a < 1$, and let $g \in C(\partial_p Q)$ be a given non-negative function. Consider the boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta\varphi(v) & \text{in} \quad Q \\ v = g & \text{in} \quad \partial_p Q \end{cases} \tag{2.4}$$

A function $v(x, t)$ is said to be a weak solution of (2.4) in Q, if $v \in C([\tau_1, \tau_2]; L^1(B)) \cap L^\infty(Q)$, $v \geqslant 0$, and v satisfies the integral identity

$$\begin{aligned} &\int_Q \int \left[\varphi(v)\,\Delta\eta + v\frac{\partial\eta}{\partial t}\right] dx\,dt \\ &= \int_{\tau_1}^{\tau_2} \int_{\partial B} \varphi(g)\frac{\partial\eta}{\partial N}\,d\sigma\,dt + \int_B v(x, \tau_2)\eta(x, \tau_2)\,dx - \int_B g(x, \tau_1)\eta(x, \tau_1)\,dx \end{aligned} \tag{2.5}$$

for all smooth functions η on $\bar{Q}$, which vanish on $\partial B \times [\tau_1, \tau_2]$. Here $\partial/\partial N$ denotes the exterior normal derivative on ∂B, and σ denotes the surface measure on ∂B.

Lemma 2.6. *Let u be a continuous non-negative weak solution of* (2.3) *in $\Omega \subset \mathbb{R}^n \times \mathbb{R}$. Suppose that $\bar{Q} \subset \Omega$, and let $g = u|_{\partial_p Q}$. Then, u is a weak solution of* (2.4) *in Q.*

The proof given in [2], Theorem 3.1 applies without changes.

We next need some maximum principles and approximation results.

Lemma 2.7. *Let $g_1, g_2 \in C(\partial_p Q)$, and let v_1, v_2 be weak solutions of* (2.4) *with boundary values g_1 and g_2 respectively. If $0 \leqslant g_1 \leqslant g_2$, then $v_1 \leqslant v_2$ in Q.*

The proof given in [7], Lemma 2.3 applies verbatim.

Corollary 2.8. *Let $g \in C(\partial_p Q)$, $g \geqslant 0$, and assume that v is a weak solution of* (2.4), *and that v is continuous in $\bar{Q}$. Choose $G_k \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $g_k = G_k|_{\partial_p Q}$ are strictly positive, $g \leqslant g_{k+1} \leqslant g_k$, $g_k \to g$ uniformly. Let $\varphi_k \in C^\infty(0, \infty)$, $\varphi_k \in \mathfrak{A}_a$, $\varphi_k \to \varphi$ uniformly on compact subsets of* $[0, \infty)$. *Let v_k solve $\partial v_k/\partial t = \Delta\varphi_k(v_k)$ in Q, $v_k = g_k$ on $\partial_p Q$. Then, each $v_k \in C^\infty(\bar{Q})$, and $v_k \to v$ uniformly on compact subsets of Q.*

The proof is the same as the one of Corollary 2.1 of [7]. The next preliminary result that we need is an extension of M. Pierre's uniqueness theorem [16], to our class of non-linearities $\mathfrak{A}_a$. Again, the proof is given in [7], Lemma 2.12.

Lemma 2.9. *Suppose that $\varphi \in \mathfrak{A}_a$, and that u and v are continuous, non-negative weak solutions of* (2.3) *in $\mathbb{R}^n \times (0, \infty)$, $n \geqslant 3$. Suppose that*

$$\sup_{t>0} \int_{\mathbb{R}^n} [u(x, t) + v(x, t)]\, dx < \infty,$$

and that $u, v \in L^\infty(\mathbb{R}^n \times [\tau_1, \tau_2])$ for each $0 < \tau_1 < \tau_2 < +\infty$. Suppose also that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} [u(x, t) - v(x, t)]\eta(x)\, dx = 0$$

for each $\eta \in C_0^\infty(\mathbb{R}^n)$. Then, $u \equiv v$ in $\mathbb{R}^n \times (0, T)$.

We now need some *a priori* estimates. Let $1/2 < r < \rho < 2$, $S = B_r \times (-r^2, 0]$, $R = B_\rho \times (-\rho^2, 0]$, where $B_r = \{x \in \mathbb{R}^n \colon |x| < r\}$.

Lemma 2.10. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in R, where $\varphi \in \mathfrak{A}_a \cap C^\infty[0, \infty)$. Then,*

$$\|u\|_{L^\infty(S)} \leqslant C\left\{1 + \frac{1}{(\rho - r)^N}\int_R\int u^p\,dx\,dt\right\}^\theta, \tag{2.11}$$

where C, p, θ, N are positive constants which depend only on n and a.

The proof of Lemma 2.10 follows from a variant of the Moser ([4]) iteration technique, which is essentially given in [7], Lemma 3.1.

Corollary 2.12. *Let u be a continuous non-negative weak solution of* (2.3), *with φ in $\mathfrak{A}_a$, in R. Then,* (2.11) *holds.*

PROOF. Corollary 2.12 follows from 2.6, 2.8 and 2.10.

3. Nonnegative solutions

In this section we establish the main properties of nonnegative continuous weak solutions u of (2.3), in the strips $S_T = \mathbb{R}^n \times (0, T)$, $0 < T \leqslant +\infty$, under the assumptions (2.1) and (2.2) on the non-linearity φ, together with the additional assumption

$$\frac{n-2}{n} + a \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant a, \qquad u \geqslant u_0 \tag{3.1}$$

where $0 < a < 1$. If we also impose the normalization $u_0 = 1$, we say that $\varphi \in \mathfrak{B}_a$. It is easy to see that (2.1) and (3.1) imply the existence of numbers μ, ν, γ, $0 < \mu < 1$, $0 < \nu < +\infty, \gamma > (n-2)/n$ which depend only on a, so that, if $\varphi \in \mathfrak{B}_a$,

$$\text{(3.2)}\quad \begin{array}{lll} \text{(i)} & \varphi(u) \leqslant u^\mu, & u \geqslant 1 \\ \text{(ii)} & \varphi(u) \leqslant u^\nu, & 0 < u < 1 \\ \text{(iii)} & \varphi(u) \geqslant u^\gamma, & u \geqslant 1 \end{array}$$

We will also need the following elementary estimate

$$\int_b^a \frac{du}{Au^\beta + Bu^\mu} \approx \begin{cases} \dfrac{1}{B}[a^{1-\mu} - b^{1-\mu}], & Ca > 2 \\ \dfrac{1}{A}[a^{1-\beta} - b^{1-\beta}], & Ca \leqslant 2, \end{cases} \tag{3.3}$$

where $0 < \beta < \mu < 1$, $A > 0$, $B > 0$, $C = (B/A)^{1/(\mu-\beta)}$, and as usual $\simeq$ means that the ratio is bounded above and below. (3.3) remains valid if $\beta = \mu$, provided $A = B$. Let now μ, v be as in (3.2), and define $\beta = \min\{\mu, \nu\}$. Then, $0 < \beta \leqslant \mu < 1$, and,

$$\varphi(u) \leqslant u^{\beta}, \qquad 0 < u < 1. \tag{3.4}$$

Following Herrero and Pierre ([12]), for any number θ, $0 < \theta < 1$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \geqslant 0$, we set

$$C_\theta(\psi) = \left[\int_{\mathbb{R}^n} |\Delta\psi|^{1/(1-\theta)} \psi^{-\theta/(1-\theta)}\right]^{1-\theta}. \tag{3.5}$$

Clearly $C_\theta(\psi)$ could be infinite, but, if $C_\theta(\psi) < \infty$, the same is true for $C_\theta(\psi_{x_0,R})$, where, for $x_0 \in \mathbb{R}^n$, $R > 0$,

$$\psi_{x_0,R}(x) = \psi\left(\frac{x - x_0}{R}\right) \Big/ R^n. \tag{3.6}$$

Moreover, once β, μ are fixed as above, we can find $0 \leqslant \bar{\psi} \leqslant 1$,

$$\operatorname{supp} \bar{\psi} \subset \{|x| < 2\},$$

$\bar{\psi} \equiv 1$ on $\{|x| < 1\}$, so that $C_\beta(\bar{\psi}) < +\infty$, $C_\mu(\bar{\psi}) < +\infty$. (See the remarks after (3.6) in [12].) We will fix such a $\bar{\psi}$ for the remainder of the paper.

Our first lemma is merely a version of Lemma 3.2 in [12], in our context.

Lemma 3.6. *Let u be a continuous non-negative weak solution of* (2.3) *in* $\mathbb{R}^n \times (0, T)$, *with* $\varphi \in \mathcal{B}_a$. *Let* β, μ *be as in* (3.3), (3.2) *respectively, and let* $\psi \in C_0^\infty(\mathbb{R}^n)$ *be such that* $A = C_\beta(\psi) < +\infty$, $B = C_\mu(\psi) < +\infty$. *Let*

$$G(b) = \int_0^b \frac{du}{Au^\beta + Bu^\mu}.$$

Let

$$f(t) = \int_{\mathbb{R}^n} u(x, t)\psi(x)\, dx.$$

Then

$$|G(f(t)) - G(f(s))| \leqslant |t - s| \quad \textit{for} \quad 0 < s, t < T. \tag{3.7}$$

PROOF. By Corollary 2.8, the following formal calculation is justified.

$$f'(t) = \int_{\mathbb{R}^n} \varphi(u(x, t))\, \Delta\psi(x)\, dx,$$

and hence,

$$\begin{aligned}|f'(t)| &\leqslant \int_{u\le 1}\varphi(u(x,t))|\Delta\psi|\,dx+\int_{u>1}\varphi(u(x,t))|\Delta\psi|\,dx\\ &\leqslant \int_{u\le 1}u^{\beta}(x,t)|\Delta\psi|\,dx+\int_{u>1}u^{\mu}(x,t)|\Delta\psi|\,dx\\ &\leqslant C_\beta(\psi)f(t)^\beta+C_\mu(\psi)f(t)^\mu,\end{aligned}$$

by Hölder's inequality. The lemma now follows by integrating the differential inequality.

Corollary 3.7. *Let u be as in Lemma 3.6, $\bar\psi$ as above. Then, there exists $C_1>0$, $C_2>0$, which depend only on $\beta,\mu,\bar\psi$ and n such that, if $0<s,t<T$, and*

$$\max\left\{\int_{\mathbb{R}^n}u(x,t)\bar\psi_{x_0,R}(x)\,dx,\ \int_{\mathbb{R}^n}u(x,s)\bar\psi_{x_0,R}(x)\,dx\right\}\leqslant C_1, \tag{3.8}$$

then

$$\left|\left[\int_{\mathbb{R}^n}u(x,t)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta}-\left[\int_{\mathbb{R}^n}u(x,s)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta}\right| \leqslant C_2|t-s|/R^2, \tag{3.9}$$

while if the max *in* (3.8) *is bigger than C_1,*

$$\left|\left[\int_{\mathbb{R}^n}u(x,t)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu}-\left[\int_{\mathbb{R}^n}u(x,s)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu}\right| \leqslant C_2|t-s|/R^2. \tag{3.10}$$

PROOF. Note that $C_\theta(\bar\psi_{x_0,R})=R^{-2}C_\theta(\bar\psi)$. The corollary now follows from (3.17) and (3.3).

We are now ready to establish the existence of a trace as $t\downarrow 0$

Theorem 3.11. *Let u be a continuous non-negative weak solution of* (2.3) *in $\mathbb{R}^n\times(0,T)$, with $\varphi\in\mathcal{B}_a$. Then there exists a unique locally finite Borel measure μ on $\mathbb{R}^n$, such that*

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^n}u(x,t)\psi(x)\,dx=\int_{\mathbb{R}^n}\psi(x)\,d\mu(x),$$

for all $\psi\in C_0^\infty(\mathbb{R}^n)$.

PROOF. Let $s=T/2$, and consider $0<t<T/2$. The continuity of u in $\mathbb{R}^n\times(0,T)$, together with Corollary 3.7, shows that $\{u(x,t)\}$ has locally uniformly bounded mass. Hence, given $\{t_j\}\to 0$, we can find a subsequence $\{t_{r_j}\}\to 0$, and a locally finite Borel measure μ such that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} u(x, t_{r_j})\psi(x)\,dx = \int_{\mathbb{R}^n}\psi(x)\,d\mu(x)$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

Suppose that $\{s_j\} \to 0$, and let us consider the corresponding $\{s_{\theta_j}\}$ and $\tilde{\mu}$. We want to show that $\mu = \tilde{\mu}$. However, once again, Corollary 3.7, applied to $t = t_{r_j}$, $s = s_{\theta_j}$, shows that, for all x_0, R,

$$\int \bar{\psi}_{x_0, R}\,d\mu = \int \bar{\psi}_{x_0, R}\,d\tilde{\mu}.$$

The non-trivality of $\bar{\psi}$ now implies that $d\mu = d\tilde{\mu}$, and the theorem follows.

We now turn our attention to local L^∞ bounds for continuous weak solutions. This is the heart of the matter, and where we are forced to depart from the techniques in [12], due to the general class of non-linearities considered, since the Aronson-Bénilan inequality ([1]) is not available. Also, this is the only point where the lower bound in (3.1) enters (and of course, it necessarily enters here (see [5] and [12])).

Lemma 3.12. *Let u be a continuous, non-negative weak solution of* (2.3) *in* $Q^* = \{(x, t): |x| < 2, -4 < t < 0\}$, *with* $\varphi \in \mathcal{B}_a$. *Let*

$$Q = \{(x, t): |x| < 1, -1 < t < 0\},$$

and define $H_\varphi(s)$ to be 1 *for* $0 < s < 1$, *and* $s[\varphi(s)/s]^{n/2}$ *for* $s \geqslant 1$. *Then, there is a constant* $C = C(0, n) > 0$ *such that*

$$\|H_\varphi(u)\|_{L^\infty(Q)} \leqslant C\{\|u\|_{L^1(Q^*)} + 1\}.$$

PROOF. First note that the lower bound in (3.1) implies that H_φ is increasing for $s > 1$. Next, let p be as in Lemma 2.10 (and we can also assume that $p \geqslant 1$). For such a fixed p, we define now $H_{p,\varphi}(s)$ to be 1 for $0 < s \leqslant 1$, and $s^p[\varphi(s)/s]^{n/2}$ for $s \geqslant 1$. Again, $H_{p,\varphi}$ is increasing for $s \geqslant 1$. Let $F_{p,\varphi}(s)$ be the inverse function to $H_{p,\varphi}$, defined again for $s \geqslant 1$. We first claim that

$$u(0, 0) \leqslant CF_{p,\varphi}\left\{\iint_{Q^*} u^p(x, t)\,dx\,dt + 1\right\} \tag{3.13}$$

In order to establish (3.13) we first note that if u is a continuous weak solution of $\partial u/\partial t = \Delta\varphi(u)$ in Ω, and $\alpha > 0$, $\beta > 0$, $\gamma > 0$ are given, and we define $v(x, t) = u(\alpha x, \beta t)/\gamma$, then v is a continuous weak solution of $\partial v/\partial t = \Delta\psi(v)$ in the appropriate Ω', where $\psi(s) = \beta\varphi(\gamma s)\alpha^{-2}\gamma^{-1}$. Suppose now that $\beta/\alpha^2 \geqslant 1$, and $\varphi \in \mathcal{B}_a$. Let $G_\varphi(\gamma) = \gamma/\varphi(\gamma)$, which by the right-hand side inequality in (3.1), is increasing for $\gamma \geqslant 1$. Let $\Gamma_\varphi(s)$ be its inverse function defined for $s \geqslant 1$, and choose $\gamma = \Gamma_\varphi(\beta/\alpha^2)$. Then, $\psi(1) = \beta\varphi(\gamma)\alpha^2\gamma^{-1} = 1$, and it is easy to see that $\psi \in \mathcal{B}_a$.

Assume now that (3.13) fails. We can then find $\varphi_k \in \mathcal{B}_a$, u_k continuous weak solutions such that

$$u_k(0,0) \geqslant kF_{p,\varphi_k}\left\{\iint_{Q^*} u_k^p(x,t)\,dx\,dt + 1\right\}.$$

First note that $F_{p,\varphi_k}(1) = 1$, and that $F_{p,\varphi_k}(s)$ is increasing for $s \geqslant 1$, so that $u_k(0,0) \geqslant k$. Because of (2.11), this forces that

$$I_k = \iint_{Q^*} u_k^p(x,t)\,dx\,dt \to +\infty \quad \text{as} \quad k \to \infty.$$

For α_k small (to be chosen momentarily), let

$$v_k(x,t) = \frac{u_k(\alpha_k x, t)}{\gamma_k}, \qquad \gamma_k = \Gamma_{\varphi_k}\left(\frac{1}{\alpha_k^2}\right).$$

The v_k are continuous weak solutions of $\partial v/\partial t = \Delta\psi_k(v)$, $\psi_k \in \mathcal{B}_a$ in Q^*, by our previous discussion.

$$\int_{|x|<2/\alpha_k}\int_{-4}^{0} v_k^p(x,t)\,dx\,dt = \frac{1}{\Gamma_{\varphi_k}(1/\alpha_k^2)^p}\,\alpha_k^{-n} I_k.$$

Choose α_k so that

$$\frac{1}{\Gamma_{\varphi_k}(1/a_k^2)^p}\,\alpha_k^{-n} I_k = 1,$$

or equivalently

$$I_k = \frac{\Gamma_{\varphi_k}(1/\alpha_k^2)^p}{1/\alpha_k^n}.$$

This is possible because $\Gamma_{\varphi_k}^p(s)/s^{n/2}$ is increasing for $s \geqslant 1$, by the left-hand side inequality in (3.1), and $I_k \to +\infty$. Moreover, $\alpha_k \to 0$ as $k \to \infty$. But then,

$$\begin{aligned}
v_k(0,0) &= \frac{1}{\varphi_{\varphi_k}(1/\alpha_k^2)}\,u_k(0,0)\\
&\geqslant \frac{k}{\Gamma_{\varphi_k}(1/\alpha_k^2)}\,F_{p,\varphi_k}\{I_k + 1\}\\
&\geqslant \frac{k}{\Gamma_{\varphi_k}(1/\alpha_k)}\,F_{p,\varphi_k}(I_k)\\
&= kF_{p,\varphi_k}\left\{\frac{\Gamma_{\varphi_k}(1/\alpha_k^2)^p}{\alpha_k^n}\right\}\frac{1}{\Gamma_{\varphi_k}(1/\alpha_k^2)}\\
&= k,
\end{aligned}$$

by the definition of F_{p,φ_k}. However, this contradicts (2.11), by our choice of α_k, and thus (3.13) is established.

We next note that by using translation and dilations $(x,t) \to (\alpha x, \alpha^2 t)$, (3.13) has, as a consequence,

$$\|u\|_{L^\infty(B_{r/2}\times(-r^2/4,0])} \leqslant CF_{p,\varphi}\left\{\frac{1}{r^{n+2}}\iint_{B_r\times(-r^2,0)} u^p(x,t)\,dx\,dt + 1\right\} \tag{3.14}$$

for $0 < r < 2$. Finally, using (3.14), translations, the dilations $(x,t) \to (\alpha x, \alpha^2 t)$, and a simple covering argument, one can show

$$\|u\|_{L^\infty(S)} \leqslant CF_{p,\varphi}\left\{\frac{1}{(\rho - r)^n + (\rho - r)^2}\iint_R u^p(x,t)\,dx\,dt + 1\right\}, \tag{3.15}$$

where $S = B_r \times (-r^2, 0]$, $R = B_\rho \times (-\rho^2, 0]$, $1/2 < r < \rho < 2$.

To conclude the proof of Lemma 3.12, we use an argument which originates in the work of Hardy and Littlewood (see [11]), and which was first used in the context of the porous medium equation in [6].

We will first show that

$$\|u\|_{L^\infty(Q)} \leqslant C\{\|u\|_{L^1(Q^*)} + 1\}^\sigma, \tag{3.16}$$

where $\sigma = \sigma(a, n)$. Once (3.16) is established, the lemma will follow by repeating the argument that we used to establish (3.15), with $p = 1$, and using (3.16) instead of (2.11).

In order to establish (3.16) we need to point out to two properties of $F_{p,\varphi}$. First, $F_{p,\varphi}(As) \leqslant CA^\sigma F_{p,\varphi}(s)$, for $A \geqslant 1$, $s \geqslant 1$. This is an easy consequence of (3.1). Another easy consequence of (3.1) is that $F_{p,\varphi}(s^{p-1}) \leqslant Cs^{1-\epsilon}$, where $\epsilon = \epsilon(a, n)$.

Now, for $1/2 < r < 1$, let

$$\begin{aligned} S_r &= B_{2r} \times (-4r^2, 0], \\ m(r) &= \|u\|_{L^\infty(S_r)}, \\ I &= \iint_{Q^*} u(x,t)\,dx\,dt, \\ J &= \max\{I, 1\}. \end{aligned}$$

We want to show that

$$m(1/2) \leqslant C\{I + 1\}^\sigma. \tag{3.17}$$

If there exists r, $1/2 < r < 1$ such that $m(r) \leqslant 1$, we are done, and hence, we can assume that $m(r) > 1$ for all r, $1/2 < r < 1$. Pick now $1/2 < r < \rho < 1$. Then, (3.15) implies that

$$
\begin{aligned}
m(r) &\leqslant CF_{p,\varphi}\left\{\frac{1}{(r-\rho)^{n+2}}\iint_{S_\rho} u^p(x,t)\,dx\,dt+1\right\}\\
&\leqslant CF_{p,\varphi}\left\{\frac{4J}{(r-\rho)^{n+2}}m(\rho)^{p-1}\right\}\\
&\leqslant C\left\{\frac{J}{(r-\rho)^{n+2}}\right\}^{\sigma}F_{p,\varphi}(m(\rho)^{p-1})\\
&\leqslant C\left\{\frac{J}{(r-\rho)^{n+2}}\right\}^{\sigma}m(\rho)^{1-\epsilon}.
\end{aligned}
$$

Choose now γ, $0<\gamma<1$ so that $\theta=(1-\epsilon)/\gamma<1$, and let $\rho=r^\gamma$. Taking logarithms, we see that

$$
\log m(r)\leqslant C\log J+C\log\frac{1}{(r-r^\gamma)}+(1-\epsilon)\log m(r^\gamma).
$$

Integrating with respect to the measure dr/r, between $1/2$ and 1, we obtain

$$
\begin{aligned}
\int_{1/2}^{1}\log m(r)\frac{dr}{r}&\leqslant C\log J+C+\theta\int_{1/2^\gamma}^{1}\log m(r)\frac{dr}{r}\\
&\leqslant C\log J+C+\theta\int_{1/2}^{1}\log m(r)\frac{dr}{r}
\end{aligned}
$$

(3.17) immediately follows from this, and our lemma is established.

Remark 3.18. In the case when $\varphi(u)=u^m$, $0<m<1$, the technique of proof of Lemma 3.12 allows one to show

$$
\|u\|_{L^\infty(Q)}\leqslant C_q\left\{\left(\iint_{Q^*}u^q\right)^{\frac{1}{q-(1-m)n/2}}+1\right\}, \tag{3.19}
$$

for each q such that $q-(1-m)n/2>0$. Note that $q \doteq 1$ is allowed precisely when $m>(n-2)/n$, giving another explanation to the result in [5]. Moreover, at least in the range $m>(n-2)/n$, an inequality of the form

$$
\|u\|_{L^\infty(Q)}\leqslant C\left\{\iint_{Q^*}u^q+1\right\}^{\sigma}
$$

can only hold if $q-(1-m)n/2>0$, as can be seen by considering the Barenblatt solutions ([3], [1], [12], [5])

$$
U_\alpha(x,t)=t^{-\beta}\{\alpha+2m\gamma^{-1}|x|^2t^{-2\beta/n}\}^{-s},
$$

where $\alpha>0$, $s=\dfrac{1}{1-m}$, $\gamma=\dfrac{2}{1-m}-n$, $\beta^{-1}=m-1+\dfrac{2}{n}\cdot$

Corollary 3.20. *Let u be a continuous weak solution of* (2.3) *in* $B_{4R} \times (0, T)$, *continuous in* $\bar{B}_{4R} \times [0, T]$, *with* $\varphi \in \mathcal{B}_a$. *Suppose that* $T/R^2 > 1$. *Then,*

(3.21)
$$\sup_{|x|<R} H_\varphi(u(x, T)) \leqslant C\left\{\frac{1}{T^{n/2}} \int_{|x|<4R} u(x, 0)\, dx + \Gamma_\varphi(T/R^2) \cdot (T/R^2)^{n/2}\right\}.$$

PROOF. Apply Lemma 3.12 to

$$v(x, t) = \frac{u(\alpha x, \beta t)}{\gamma},$$

where $\alpha = R$, $\beta = T$, and $\gamma = \Gamma_\varphi(\beta/\alpha^2)$, and then apply Corollary 3.7 to the same v.

Note that in the case when $\varphi(u) = u^m$, $\frac{n-2}{n} < m < 1$, Theorems 3.20 is exactly Theorem 2.2 in [12].

We now turn our attention to the existence of solutions in $\mathbb{R}^n \times (0, \infty)$.

Theorem 3.22. *Let* μ *be a locally finite Borel measure on* $\mathbb{R}^n$. *Then, for any* $\varphi \in \mathcal{B}_a$, *there exists a continuous weak solution u to* (2.3) *in* $\mathbb{R}^n \times (0, \infty)$, *such that*

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} u(x, t)\eta(x)\, dx = \int_{\mathbb{R}^n} \eta(x)\, d\mu(x), \tag{3.23}$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

PROOF. Our starting point is the following classical result (see [15]): for any $f \in C_0^\infty(\mathbb{R}^n)$, there exists $u \in C([0, \infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$, which is a weak solution of (2.3) in $\mathbb{R}^n \times (0, \infty)$, and such tht $u(x, 0) = f(x)$. Moreover, the results in [17] show that, if φ verifies (2.1), then u is continuous in $\mathbb{R}^n \times (0, \infty)$.

Our next claim is that if $R \geqslant 1$, $\int_{B_{4R}} f\, dx \leqslant M$, then the corresponding u are equicontinuous in compact subsets of $B_R \times (0, \infty)$. To establish this, note that Corollary 3.20 establishes the local uniform boundedness in $B_R \times [R^2, +\infty)$, and hence [17] gives the equicontinuity there. If $t < R^2$, choose $r^2 = t < R^2$, and apply the same argument to $B_r \times [r^2, +\infty)$ and its translates, to obtain the desired equicontinuity.

Finally, note that if $\eta \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \eta \subset B_R$, $\int_{B_{4R}} f\, dx \leqslant M$, and u is as above, then

$$\left| \int \eta(x)[u(x, t) - f(x)]\, dx \right| \leqslant t C_{M,\eta}. \tag{3.24}$$

In fact,

$$\int \eta(x)[u(x,t) - f(x)] = \int_0^t \int_{\mathbb{R}^n} \Delta\eta\,(x)\varphi(u(x,s))\,dx\,ds,$$

and so, by (3.2)(i), the left-hand side of (3.24) is bounded by

$$\int_0^t \int_{\mathbb{R}^n} |\Delta\eta\,(x)|\,dx\,ds + \int_0^t \int_{\mathbb{R}^n} |\Delta\eta\,(x)|u^{\mu}(x,s)\,dx\,ds, \qquad 0 < \mu < 1.$$

An application of Hölder's inequality and Corollary 3.7 establishes (3.24).
Fix now the measure μ, and pick $f_k \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)\eta(x)\,dx = \int \eta(x)\,d\mu(x).$$

We can choose f_k so that, in addition

$$\int_{B_R} f_k(x)\,dx \leqslant M_R,$$

where M_R is independent of k. Let u_R be the corresponding solutions, constructed at the beginning of the proof. By equicontinuity (after possibly passing to a subsequence, which we still denote $\{u_k\}$), there exists a continuous weak solution u of (2.3), such that the u_k converge to u, uniformly on compact subsets of $\mathbb{R}^n \times (0,\infty)$. (3.24) now establishes (3.23).

Remark 3.25. The results in [5] show that, unless the left-hand inequality of (3.1) is verified (in the pure power case $\varphi(u) = u^m$), Theorem 3.22 fails when μ is the delta mass at the origin. As our proof shows, this is because of the lack of an $L^\infty - L^1$ «regularity» effect, as Lemma 3.12. This clarifies Remark 3.18.

Finally, we turn to the uniqueness of the solution constructed in Theorem 3.22. The general strategy is the one developed in [6]. As in [6] and [7], we start out with a version of the maximum principle. Its proof follows closely that of Theorem 2.3 in [12].

Lemma 3.26. *Let u, v be continuous weak solutions of* (2.3) *in*

$$S_T = \mathbb{R}^n \times (0,T),$$

with $\varphi \in \mathcal{B}_a$. Assume that

$$\lim_{t\downarrow 0} \int_{|x|<R} [v(x,t) - u(x,t)]^+\,dx = 0 \tag{3.27}$$

for all $R > 0$, where $A^+ = \max\{A, 0\}$. Then, $v \leqslant u$ in S_T.

PROOF. Let $w = v - u$, and let q denote the characteristic function of the set where $u < v$. If u, v and φ are smooth, Kato's inequality ([13]) shows that

$$\Delta[\varphi(v) - \varphi(u)]^+ \geqslant q\Delta[\varphi(v) - \varphi(u)],$$

where

$$\frac{\partial w^+}{\partial t} = q\frac{\partial w}{\partial t}.$$

By (3.1), (3.2) and (3.4) we see that

$$[\varphi(v) - \varphi(u)]^+ \leqslant C\{([v-u]^+)^\mu + ([v-u]^+)^\beta\}.$$

Hence, (still under the assumption that u, v, φ are smooth), we have, for $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \geqslant 0$.

$$\begin{aligned}\frac{\partial}{\partial t}\int \psi(x)w^+(x,t)\,dx &\leqslant \int \psi(x)\,\Delta[\varphi(v) - \varphi(u)]^+\,dx \\ &\leqslant \int |\Delta\psi(x)|[\varphi(v) - \varphi(u)]^+\,dx \\ &\leqslant C\int |\Delta\psi(x)|([v-u]^+)^\mu + C\int |\Delta\psi(x)|([v-u]^+)^\beta \\ &\leqslant CC_\mu(\psi)\left(\int \psi(x)w^+(x,t)\,dx\right)^\mu \\ &\quad + CC_\beta(\psi)\left(\int \psi(x)w^+(x,t)\,dx\right)^\beta.\end{aligned}$$

Integrating this (one-sided) differential inequality in a manner similar to the proof of Lemma 3.6, using Corollary 2.8 to justify the formal argument above, and using (3.27), we see that, if $C_\mu(\psi) < \infty$, $C_\beta(\psi) < \infty$, there are constants C_1, C_2 such that, if

$$\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+\,dx \leqslant C_1,$$

then

$$\left(\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+\,dx\right)^{1-\beta} \leqslant C_2tR^{-2},$$

while if the above quantity exceeds C_1, then,

$$\left(\frac{1}{R^n}\int_{|x-\xi|<R} [v(x,t) - u(x,t)]^+\,dx\right)^{1-\mu} \leqslant C_2tR^{-2},$$

for any $R > 0$, $\xi \in \mathbb{R}^n$.

Let now $h(x, t) = \int_0^t [\varphi(v) - \varphi(u)]^+ \, dx$. It is easy to see that, for each $t > 0$, h is a subharmonic function of x. Hence, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
h(\xi, t) &\leqslant \frac{1}{\omega_n R^n} \int_{B_R(\xi)} h(x, t)\, dx \\
&\leqslant \frac{1}{\omega_n R^n} \int_0^t \int_{B_R(\xi)} [\varphi(v) - \varphi(u)]^+ \\
&\leqslant \frac{C}{R^n} \int_0^t \int_{B_R(\xi)} \{([v-u]^+)^\mu + ([v-u]^+)^\beta\} \\
&\leqslant \frac{C}{R^n} \int_0^t R^{n(1-\mu)}\left(\int_{B_R(\xi)} [v-u]^+\right)^\mu + R^{n(1-\beta)}\left(\int_{B_R(\xi)} [v-u]^+\right)^\beta \\
&\leqslant \frac{C}{R^n} \int_0^t R^{n(1-\mu)}\{s^{\mu/(1-\mu)} R^{-2\mu/(1-\mu)} + s^{\mu/(1-\beta)} R^{-2\mu/(1-\beta)}\} \\
&\quad + \frac{C}{R^n} \int_0^t R^{n(1-\beta)}\{s^{\beta/(1-\beta)} R^{-2\beta/(1-\beta)} + s^{\beta/(1-\mu)} R^{-2\beta/(1-\mu)}\}
\end{aligned}$$

For fixed t, this tends to 0 as $R \to \infty$, and hence $h(x, t) \equiv 0$, which establishes the lemma.

Our general uniqueness result now follows from the approximation procedure in [6], using Lemma 2.9 and Lemma 3.26.

Theorem 3.28. *Let u, v be continuous weak solutions of* (2.3) *in S_T, with $\varphi \in \mathcal{B}_a$. Assume that*

$$\lim_{t \downarrow 0} \int u(x, t)\eta(x)\, dx = \lim_{t \downarrow 0} \int v(x, t)\eta(x)\, dx$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$. Then $u \equiv v$ in S_T.

PROOF. Let μ be the locally finite Borel measure on $\mathbb{R}^n$, attached to u by Theorem 3.11. Pick $h \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant h \leqslant 1$, and let $w(x, t, h)$ be a solution in $\mathbb{R}^n \times (0, \infty)$, with initial trace $h\mu$. By Theorem 3.22, at least one such w exists. Moreover, Corollary 3.20 shows that any such w belongs to $L^\infty(\mathbb{R}^n \times [\tau_1, \tau_2])$, for each $0 < \tau_1 < \tau_2 < +\infty$. We now want to show that, for any such w,

$$\sup_{t>0} \int_{\mathbb{R}^n} w(x, t, h)\, dx < +\infty. \tag{3.29}$$

In fact, Corollary 3.7 easily implies that for each $t > 0$, $R > 0$,

$$(3.30)\quad \int_{R/2<|x|<R} w(x,t,h)\frac{dx}{|x|^{n-2}}$$
$$\leqslant \int_{R/2<|x|<2R} h(x)\frac{d\mu(x)}{|x|^{n-2}} + Ct^{1/(1-\beta)}R^{2-[2/(1-\beta)]} + Ct^{1/(1-\mu)}R^{2-[2/(1-\mu)]},$$

and hence

$$(3.31)\qquad \int_{|x|>1} w(x,t,h)\frac{dx}{|x|^{n-2}} < +\infty.$$

Pick now $\psi(x) \in C^\infty(\mathbb{R}^n)$, $\psi(x) > 0$, $\psi(0) = 1$, ψ bounded, $\Delta\psi \leqslant 0$, $\psi(x) \leqslant C/|x|^n$ for $|x| > 1$, $|\nabla\psi(x)| \leqslant C/|x|^{n-1}$ for $|x| > 1$, and let $\psi_R(x) = \psi(x/R)$. Let $\theta_N(x)$ be a $C_0^\infty(\mathbb{R}^n)$ function, $0 \leqslant \theta_N \leqslant 1$, $\theta_N \equiv 1$ for $|x| < N$, $\operatorname{supp}\theta_N \subset \{|x| < 2N\}$, $|\nabla\theta_N| \leqslant C/N$, $|\Delta\theta_N| \leqslant C/N^2$. Then, for $0 < s < t < +\infty$, (with $w(x,t) = w(x,t,h)$),

$$\begin{aligned}
\int w(x,t)\psi_R(x)\,dx - \int w(x,s)\psi_R(x)\,dx &= \lim_{N\to\infty}\int [w(x,t) - w(x,s)]\theta_N(x)\psi_R(x)\,dx \\
&= \lim_{N\to\infty}\int_s^t\int_{\mathbb{R}^n} \varphi(w(x,r))\Delta[\theta_N(x)\psi_R(x)]\,dx \\
&\leqslant \overline{\lim_{N\to\infty}}\int_s^t\int_{\mathbb{R}^n} \varphi(w(x,r))[\Delta\theta_N(x)\psi_R(x) \\
&\qquad + 2\nabla\theta_N(x)\nabla\psi_R(x)]\,dx\,dr.
\end{aligned}$$

If we now use the pointwise estimates for ψ_R, $\nabla\psi_R$, $\Delta\theta_N$, $\nabla\theta_N$, the support properties of the last two functions, (3.2), (3.30), and the boundedness of w in $\mathbb{R}^n \times [s,t]$, we see that the above $\overline{\lim}_{N\to\infty}$ is non-positive. Hence

$$\int w(x,t)\psi_R(x)\,dx \leqslant \int w(x,s)\psi_R(x)\,dx.$$

Moreover, (3.30), and our pointwise bounds on ψ show that

$$\lim_{s\to 0}\int w(x,s)\psi_R(x)\,dx = \int h(x)\psi_R(x)\,d\mu(x).$$

The last expression is bounded, independently of R, and so

$$\int w(x,t)\psi_R(x)\,dx \leqslant C$$

and Fatou's lemma implies (3.29), letting $R \to \infty$. Hence, Lemma 2.9 now shows there is only one such $w(x,t,h)$. We next claim that $w(x,t,h) \leqslant u(x,t)$.

Let $U_\epsilon(x, t)$ be a solution with initial data $h(x)u(x, \epsilon)$. Theorem 3.11 and the above argument show that there exists exactly one such U_ϵ, and that

$$\sup_{t>0} \int_{\mathbb{R}^n} U_\epsilon(x, t)\, dx \leqslant C \sup_{0<\epsilon<T/2} \int h(x)u(x, \epsilon)\, dx \leqslant C_h. \tag{3.32}$$

where the last inequality is a consequence of Corollary 3.7. Moreover, for any smooth η, we have that

$$\lim_{\epsilon\to 0} \int \eta(x)h(x)u(x, \epsilon)\, dx = \int \eta(x)h(x)\, d\mu(x).$$

Note now that $\lim_{\epsilon\to 0} U_\epsilon(x, t) = w(x, t)$. In fact, (3.32), Corollary 3.20 and the results in [17] show that $\{U_\epsilon(x, t)\}$ is equicontinuous on compact subsets of $\mathbb{R}^n \times (0, \infty)$. Let $\{U_{\epsilon_j}(x, t)\}$ be a subsequence, which converges uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$ to $\hat{w}(x, t)$. Clearly, $\hat{w}(x, t)$ is a continuous weak solution of (2.3) in $\mathbb{R}^n \times (0, \infty)$. We claim that for any $\eta \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{t\downarrow 0} \int \hat{w}(x, t)\eta(x)\, dx = \int h(x)\eta(x)\, d\mu(x).$$

In fact, the argument leading to (3.24), together with the second inequality in (3.32) show that

$$\left| \int \eta(x)[U_\epsilon(x, t+\epsilon) - h(x)u(x, \epsilon)]\, dx \right| \leqslant tC_h$$

which proves our claim. But the uniqueness of $w(x, t, h)$ then shows that $\hat{w}(x, t) = w(x, t, h)$, and our claim follows. The construction of $U_\epsilon(x, t)$ in [15] gives that $U_\epsilon(x, t) \in C([0, \infty); L^1(\mathbb{R}^n))$, so that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} |U_\epsilon(x, t) - h(x)u(x, \epsilon)|\, dx = 0.$$

By the continuity of $u(x, t)$,

$$\lim_{t\downarrow 0} \int_K |u(x, t+\epsilon) - u(x, \epsilon)| dx = 0,$$

for each $K \subset\subset \mathbb{R}^n$. Hence, by Lemma 3.26, $U_\epsilon(x, t) \leqslant u(x, t+\epsilon)$, and thus, $w(x, t, h) \leqslant u(x, t)$. Pick now $0 \leqslant h_j \leqslant h_{j+1} \leqslant 1$, $h_j \in C_0^\infty(\mathbb{R}^n)$, $\lim_{j\to\infty} h_j(x) \equiv 1$. By the construction of $w(x, t, h)$ given in Theorem 3.11,

$$w(x, t, h_j) \leqslant w(x, t, h_{j+1}).$$

Moreover, by the first part of the proof, $w(x, t, h_j) \leqslant u(x, t)$. Let $w_\infty(x, t)$ be a limit of some subsequence of $w(x, t, h_j)$, which exists and is a continuous weak solution, by equicontinuity. $w_\infty(x,t)$ has a trace, as $t\downarrow 0$ by Theorem 3.22. The trace is between $h_j\,d\mu$ and $d\mu$ for each j, and hence it equals $d\mu$. Since $w_\infty \leqslant u$,

$$\lim_{t\downarrow 0}\int_{|x|<R} |u(x,t) - w_\infty(x,t)| = 0$$

for all $R>0$. By Lemma 3.26, $u(x,t) = w(x,t)$. Similarly, $v(x,t) = w_\infty(x,t)$, and hence $u(x,t) = v(x,t)$.

Theorem 3.33. *Let u be a continuous weak solution of* (2.3) *in S_T, with $\varphi \in \mathcal{B}_a$. Then there exists a unique $\hat{u}$ in $S_\infty = \mathbb{R}^n \times (0,\infty)$, which is a continuous weak solution of* (2.3) *in S_∞, with $u = \hat{u}$ in S_T.*

Proof. Let μ be the trace of u given by Theorem 3.11, and let $\hat{u}$ be the corresponding solution in S_∞, constructed in Theorem 3.22. By Theorem 3.28, $u = \hat{u}$ in S_T.

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