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Global Models of Riemannian Metrics

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Introduction

In this paper we give certain Riemannian metrics on the manifolds $S^{n-1} \times S^1$ and S^n ($n \geqslant 2$), which have the property to determine these manifolds, up to diffeomorphisms.

The global expressions used for Riemannian metrics are based on the global expression for exterior forms studied in [4]. In [3] one finds certain metrics using global expressions that differ from the type we propose.

To some extent, Theorem 3 is a «generalization for metrics» in an arbitrary dimension, of a theorem proved in [2] for certain volume forms on surfaces.

1. Examples and Theorems on Surfaces

The following example illustrates the context in which our statements are made.

Let us consider in $\mathbb{R}^3 - 0$ the quadratic form:

$$m = (x_1\,dx_2 - x_2\,dx_1)^2 + (x_1\,dx_3 - x_3\,dx_1)^2 + (x_2\,dx_3 - x_3\,dx_2)^2.$$

A simple calculation proves that a vector v is isotropic if and only if

$$v = \lambda\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right), \qquad \lambda \in \mathbb{R}. \qquad (*)$$

Hence, m is a Riemannian metric over all surfaces in $\mathbb{R}^3$ whose tangent plane is transverse to the position vector field. In particular, if $i: S^2 \to \mathbb{R}^3 - 0$

is the ordinary inclusion, the metric $i^*(m)$ on S^2 admits the global expression

$$i^*(m) = (f_1\, df_2 - f_2\, df_1)^2 + (f_1\, df_3 - f_3\, df_1)^2 + (f_2\, df_3 - f_3\, df_2)^2$$

where $f_j\colon S^2 \to \mathbb{R}$, $j = 1, 2, 3$ are global functions given by $f_j = x_j \cdot i$.

As a consequence of $(*)$ the following theorem is easily proved

Theorem 1. *Let M be a compact connected surface having a Riemannian metric m that admit the global expression:*

$$m = (f_1\, df_2 - f_2\, df_1)^2 + (f_1\, df_3 - f_3\, df_1)^2 + (f_2\, df_3 - f_3\, df_2)^2$$

where $f_i\colon M_2 \to \mathbb{R}$, $i = 1, 2, 3$ are C^∞-global functions. Then M_2 is diffeomorphic to the sphere S^2.

PROOF OF THEOREM 1. Given the metric m on M_2, let us consider the map $\varphi\colon M_2 \to \mathbb{R}^3 - 0$ expressed by

$$\varphi(p) = (f_1(p), f_2(p), f_3(p)).$$

Lemma 1. *The following statements are equivalent*

(*a*) *m is a Riemannian metric on M_2.*

(*b*) *φ is an immersion transverse to the vector field*

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \quad on \quad \mathbb{R}^3.$$

The proof of this lemma is basically the remark made in $(*)$.

Let $\Pi\colon \mathbb{R}^3 - \{0\} \to S^2$ be given by $\Pi(x) = x/|x|$. Lemma 1 proves that if m is a Riemannian metric, $\pi \cdot \varphi\colon M_2 \to S^2$ is a covering map, hence $\pi \cdot \varphi$ is a diffeomorphism. □

Let us now consider the metric

$$m_1 = (1 + 4 \sin^2 \theta_2 \cos^2 \theta_2)\, d\theta_1^2 + d\theta_2^2$$

on the torus $T^2 = S^1 \times S^1$ and

$$m_2 = (x_1\, dx_2 - x_2\, dx_1)^2 + (x_1\, dx_3 - 2x_3\, dx_1)^2 + (x_2\, dx_3 - 2x_3\, dx_2)^2$$

on the sphere S^2.

An easy calculation proves that both metrics admit the global expression

$$m = (f_1\, df_2 - f_2\, df_1)^2 + [f_1(f\, dg - g\, df) - 2fg\, df_1]^2 + [f_2(f\, dg - g\, df) - 2fg\, df_2]^2$$

where f_1, f_2, f and g are global C^∞-functions.

Theorem 2. *Let M_2 be a compact connected surface having a Riemannian metric that admits the global expression*:

$$m = (f_1\,df_2 - f_2\,df_1)^2 + [f_1(f\,dg - g\,df) - 2fg\,df_1]^2 + [f_2(f\,dg - g\,df) - 2fg\,df_2]^2$$

and let

$$H = \{p \in M_2 \mid f_1(p) = f_2(p) = 0\}.$$

(*a*) *If $H \neq \varnothing$ then M_2 is diffeomorphic to the sphere S^2.*
(*b*) *If $H = \varnothing$ then M_2 is diffeomorphic to the torus T^2.*

PROOF OF THEOREM 2.

Lemma 2. $(f_1\,df_2 - f_2\,df_1)(p) = 0$ *if and only if $p \in H$. Moreover, H is finite.*

PROOF. There is an obvious implication. If $f_1(p)\,df_2(p) - f_2(p)\,df_1(p) = 0$ with $f_1(p) \neq 0$ then

$$df_2(p) = \frac{f_2(p)}{f_1(p)}\,df_1(p),$$

and substituting in the expression for the metric m at that point, we obtain

$$m(p) = \left(1 + \frac{f_2^2}{f_1^2}\right)[f_1(f\,dg - g\,df) - 2fg\,df_1]^2(p)$$

which admits isotropic vectors. Hence $f_1(p) = 0$ and likewise $f_2(p) = 0$.

Finally, if $p \in H$, then

$$m(p) = 4f^2g^2(df_1^2 + df_2^2)(p)$$

and therefore $(df_1 \wedge df_2)(p) \neq 0$, hence H is finite. □

We define

$$\omega_1 = f_1(f\,dg - g\,df) - 2fg\,df_1$$

and

$$\omega_2 = f_2(f\,dg - g\,df) - 2fg\,df_2.$$

Lemma 3. $\omega_1 \wedge \omega_2 = 0$ *if and only if* $fg = 0$.

PROOF. From the definition of ω_1 and ω_2, if $fg = 0$ it is obvious that $\omega_1 \wedge \omega_2 = 0$. Conversely, if $\omega_1 \wedge \omega_2 = 0$, from the relation $2fg(f_1\,df_2 - f_2\,df_1) = f_2\omega_1 - f_1\omega_2$ we obtain

$$2fg(f_1\,df_2 - f_2\,df_1)\wedge\omega_1 = 0$$
$$2fg(f_1\,df_2 - f_2\,df_1)\wedge\omega_2 = 0.$$

The fact that m is a metric implies that two of the three forms $f_1\,df_2 - f_2\,df_1$, ω_1, ω_2 must be independent at each point, then either $(f_1\,df_2 - f_2\,df_1)\wedge\omega_1 \neq 0$ or $(f_1\,df_2 - f_2\,df_1)\wedge\omega_2 \neq 0$, hence $fg = 0$. □

Remark 1. From the expression for m, it is deduced that f and df (g and dg, respectively) cannot have common zeros, and either the set $f = 0$ ($g = 0$ respectively) is empty or it is made up of a finite number of disjoint circles. Hence ω_1 and ω_2 are independent in a dense open set.

Let us now define $\omega = f_1\omega_1 + f_2\omega_2$. We have

Lemma 4. *$\omega(p) = 0$ if and only if $p \in H$.*

PROOF. If $p \in H$ obviously $\omega(p) = 0$. If $\omega(p) = 0$ and $f_1^2(p) + f_2^2(p) \neq 0$, Lemma 3 implies that $fg = 0$. Moreover,

$$\omega = f_1\omega_1 + f_2\omega_2 = (f_1^2 + f_2^2)(f\,dg - g\,df) - 2fg(f_1\,df_1 + f_2\,df_2)$$

which implies $0 = (f_1^2 + f_2^2)(f\,dg - g\,df)(p)$ and then $(f\,dg - g\,df)(p) = 0$ and the expression for the metric at this point would be $(f_1\,df_2 - f_2\,df_1)^2$, which is a contradiction. □

Remark 2. Since H is finite (Lemma 2), ω has a finite number of singularities.

Let us consider on M_2 the vector fields X, Y which are dual, with respect to the metric m, of the 1-forms $f_1\,df_2 - f_2\,df_1$ and ω. Lemma 2 and Remark 2 imply that X and Y have a finite number of singularities.

Lemma 5. *X and Y are orthogonal with respect to the metric m.*

PROOF. The vector fields are defined by the relations

$$m(X, \bullet) = f_1\,df_2 - f_2\,df_1$$
$$m(Y, \bullet) = \omega.$$

Lemma 5 is equivalent to proving that $\omega(X) = 0$.
From the expression for m, it is deduced that

$$m(X, \bullet) = (f_1\,df_2 - f_2\,df_1)(X)(f_1\,df_2 - f_2\,df_1) + \omega_1(X)\omega_1 + \omega_2(X)\omega_2$$

which implies

$$[1 - (f_1\,df_2 - f_2\,df_1)(X)](f_1\,df_2 - f_2\,df_1) = \omega_1(X)\omega_1 + \omega_2(X)\omega_2$$

and from the relation used in Lemma 3 we have

$$[1 - (f_1\,df_2 - f_2\,df_1)(X)]\frac{f_2\omega_1 - f_1\omega_2}{2fg} = \omega_1(X)\omega_1 + \omega_2(X)\omega_2.$$

From Lemma 3 it is deduced that in the dense open set $fg \neq 0$ the following relations are satisfied:

$$\lambda f_2 = \omega_1(X)$$
$$-\lambda f_1 = \omega_2(X)$$

where

$$\lambda = \frac{1}{2fg}[1 - (f_1\,df_2 - f_2\,df_1)(X)].$$

By multiplying the relations above by f_1 and f_2 respectively, and adding we obtain $(f_1\omega_1 + f_2\omega_2)(X) = 0$ and therefore $\omega(X) = 0$ if $fg \neq 0$.

Since the function $\omega(X)$ is defined on all of M_2 and it is zero in a dense open subset, we obtain $\omega(X) \equiv 0$ on M_2. □

Corollary 1. *M_2 is orientable.*

PROOF. From Lemma 5 it is concluded that $X \wedge Y$ is a 2-vector field on M_2 with a finite number of singularities, hence M_2 is orientable. □

2. Conclusion

(*a*) Let $H \neq \varnothing$ and $p \in H$. From the global expression for m it can be deduced that $df_1 \wedge df_2(p) \neq 0$ and hence f_1 and f_2 can be taken as coordinates in a neighbourhood of p. Therefore the vector field $f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}$ is well defined in a neighbourhood of each singularity of the vector field Y.

As the equality

$$(f_1\,df_2 - f_2\,df_1)\left(f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}\right) = f_1f_2 - f_2f_1 = 0$$

is satisfied in each of these neighbourhoods, the vector field $f_1 \frac{\partial}{\partial f_1} + f_2 \frac{\partial}{\partial f_2}$ is orthogonal to X and it follows from Lemma 5 that

$$Y = \mu\left(f_1 \frac{\partial}{\partial f_1} + f_2 \frac{\partial}{\partial f_2}\right)$$

in a neighbourhood of each singularity of Y.

Consequently, Y has a finite number of singularities (Remark 2), every singularity has an index of $+1$ (following from before), M_2 is orientable (Corollary 1) and hence M_2 is diffeomorphic to the sphere S^2.

(*b*) If $H \neq \emptyset$, according to Remark 2 Y is a vector field without singularities, hence the Euler characteristic is $\chi(M_2) = 0$. As M_2 is orientable, it is deduced that $M_2 = T^2$.

The proof of the Theorem is complete. □

3. Examples of Metrics in Arbitrary Dimension

The generalization of Theorem 2 to an arbitrary dimension is motivated by the following examples

(*a*) Let us consider the mapping

$$h_r: S^{n-1} \times S^1 \to \mathbb{R}^n \times \mathbb{R}^2,$$

given by

$$h_r(p, \theta) = (p, \cos r\theta, \sin r\theta); \qquad r = 1, 2, 3, \ldots$$

and the quadratic form in $\mathbb{R}^n \times \mathbb{R}^2$:

$$m = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} \omega_{ij}^2 + \sum_{k=1}^{n} [x_k(y_1\,dy_2 - y_2\,dy_1) - 2y_1y_2\,dx_k]^2$$

where $(x_i)_{i=1,\ldots,n}$, $(y_i)_{i=1,2}$ are the coordinates in $\mathbb{R}^n$ and $\mathbb{R}^2$ respectively, and $\omega_{ij} = (x_i\,dx_j - x_j\,dx_i)$ for $i < j$.

A simple calculation proves that $h_r^*(m)$ is a Riemannian metric in $S^{n-1} \times S^1$.

For $r = 1$ and $n = 2$, the metric $h_1^*(m)$ in $S^1 \times S^1$ is the metric m_1 that appears in Section 1.

(*b*) Let us consider in $\mathbb{R}^{n+1}$ the quadratic form

$$m = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} \omega_{ij}^2 + \sum_{k=1}^{n} [x_k(y_1\,dy_2 - y_2\,dy_1) - 2y_1y_2\,dx_k]^2,$$

with $y_1 = 1$, $y_2 = x_{n+1}$. If $i: S^n \to \mathbb{R}^{n+1}$ is the ordinary inclusion, it can easily be proved that $i^*(m)$ is a metric in S^n.

In the case where $n = 2$, we have the metric m_2 of S^2 from Section 1.

(*c*) Let us consider in $\mathbb{R}^{n+1}$ and for each $r = 0, 1, 2, \ldots$, the quadratic form

$$m_r = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} \omega_{ij}^2 + \sum_{k=1}^{n} \tau_k^2$$

where

$$\omega_{ij} = x_i\, dx_j - x_j\, dx_i \quad \text{for} \quad i<j$$

$$\tau_k = [x_k(\cos f(x_{n+1}) d \sin f(x_{n+1}) - \sin f(x_{n+1}) d \cos f(x_{n+1})) - 2 \sin f(x_{n+1}) \cos f(x_{n+1})\, dx_k].$$

$$f(x_{n+1}) = \frac{\pi}{4} + \left(\frac{\pi}{2} + r\pi\right)\left(\frac{x_{n+1}+1}{2}\right) + \frac{\pi}{2}.$$

In the following remark we show that for every $r \in \mathbb{N}$, $i^*(m_r)$ is a Riemannian metric on S^n.

Remark 3. If we define

$$\alpha = \frac{df}{dx_{n+1}} = \frac{1}{2}\left(\frac{\pi}{2} + r\pi\right), \qquad \beta = \sin 2f(x_{n+1}),$$

then

$$\tau_k = x_k \alpha\, dx_{n+1} - \beta\, dx_k, \qquad k = 1, \ldots, n.$$

To see that $i^*(m_r)$ is a metric on S^n it is enough to check that at every point of S^n one can choose n independent 1-forms among the ω_{ij}'s and τ_k's. To do this we shall calculate the external product of certain n 1-forms by the form

$$\Omega = x_1\, dx_1 + x_2\, dx_2 + \cdots + x_n\, dx_n + x_{n+1}\, dx_{n+1}.$$

(1) If $\beta \neq 0$.

$$\Omega \wedge \tau_1 \wedge \cdots \wedge \tau_n = \beta^{n-1}[\alpha(1 - x_{n+1}^2) + \beta x_{n+1}]\, dx_1 \cdots dx_{n+1}$$

and the expressions for α and β imply that

$$\alpha(1 - x_{n+1}^2) + \beta x_{n+1} = 0$$

has no solution for $r = 0, 1, 2, \ldots$ with $-1 \leqslant x_{n+1} \leqslant 1$ [1].

(2) It $\beta = 0$, then $\tau_k = x_k \alpha\, dx_{n+1}$ and we calculate the following exterior products

$$\Omega \wedge \omega_{12} \wedge \omega_{13} \wedge \cdots \wedge \omega_{1n} \wedge \tau_1 = x_1^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$
$$\Omega \wedge \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{2n} \wedge \tau_2 = x_2^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Omega \wedge \omega_{1n} \wedge \omega_{2n} \wedge \cdots \wedge \omega_{n-1n} \wedge \tau_n = x_n^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$

Consequently, since $\alpha \neq 0$ and $x_{n+1}^2 \neq 1$ (because $x_{n+1} = \pm 1$ implies that $\beta \neq 0$), if $x_j \neq 0$ then $\omega_{1,j}; \omega_{2,j}; \ldots; \omega_{j-1,j}; \omega_{j,j+1}; \ldots; \omega_{j,n}$ are independent. On the other hand, if all the x_j's are zero then $x_{n+1} = \pm 1$ and therefore $\beta \neq 0$.

4. The Theorem in Arbitrary Dimension

Theorem 3. *Let M_n be a compact connected n-dimensional Hausdorff manifold, having a Riemannian metric m that admits the global expression*

$$m = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} (f_i\, df_j - f_j\, df_i)^2 + \sum_{k=1}^{n} [f_k(f\, dg - g\, df) - 2fg\, df_k]^2$$

where $f_1, f_2, \ldots, f_n, f, g$ are global C^∞-functions.
Let $H = \{p \in M_n \mid f_i(p) = 0 \quad i = 1, \ldots, n\}$.
Then

(*a*) *$H = \varnothing$ implies that M_n is diffeomorphic to $S^{n-1} \times S^1$.*
(*b*) *$H \neq \varnothing$ implies that M_n is diffeomorphic to the sphere S^n.*

Remark 4. Examples (*a*), (*b*) and (*c*) of Section 3 prove that on $S^{n-1} \times S^1$ and S^n there are metrics that admit the expression above.

Before beginning the proof of the Theorem, we include some comments on the quadratic form of example (*a*) which are essential for the proof.

Let us now consider in $\mathbb{R}^n \times \mathbb{R}^2$, with coordinates $(x_1, \ldots, x_n, y_1, y_2)$ the quadratic form

$$m_0 = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} (x_i\, dx_j - x_j\, dx_i)^2 + \sum_{k=1}^{n} [x_k(y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2\, dx_k]^2$$

and the vector fields

$$X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$$

$$Y = -2y_1 \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

By a simple calculation it can be proved that X and Y are isotropic and independent in the open set U of $\mathbb{R}^n \times \mathbb{R}^2$ where the following inequality is satisfied

$$y_1^2 y_2^2 + (y_1^2 + y_2^2)\left(\sum_{i=1}^{n} x_i^2\right) \neq 0.$$

If $p \notin U$ the quadratic form m_0 reduces to

$$m_0 = \sum_{i=1, j=2}^{n-1, n} (x_i\, dx_j - x_j\, dx_i)^2,$$

its rank being less than n because

$$X \rfloor m_0 = 0 \quad \text{if} \quad X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\cdot$$

Lemma 6. *Let $p \in U$ and let v_p be a vector where $v_p \rfloor m_0 = 0$ (i.e. $m_0(v_p v_p) = 0$ as m_0 is semidefined positive). Then $v_p = \lambda Y + \mu X$.*

PROOF. Let $v_p = v_1 + v_2$ be a vector where $p \in U$ and

$$v_1 = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i},$$

$$v_2 = \mu_1 \frac{\partial}{\partial y_1} + \mu_2 \frac{\partial}{\partial y_2}$$

$$m_0(v_p, v_p) = 0 \Leftrightarrow \begin{cases} (1)\ v_1 \rfloor \left(\displaystyle\sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} (x_i\, dx_j - x_j\, dx_i)^2 \right) = 0 \\ \quad \Leftrightarrow v_1 = \lambda \displaystyle\sum_{i=1}^{n} x_i \frac{\delta}{\partial x_i} \Leftrightarrow \lambda_k = \lambda x_k \\ (2)\ x_k [v_2 \rfloor (y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2] = 0;\ k = 1, 2, \ldots, n. \end{cases}$$

If $x_k(p) = 0$ for all values of k, then $y_1 y_2(p) \neq 0$ and (1) implies that $v_1 = 0$. Hence

$$v_p = \mu_1 \frac{\partial}{\partial y_1} + \mu_2 \frac{\partial}{\partial y_2} \quad \text{with} \quad X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \qquad Y = -2y_1 \frac{\partial}{\partial y_1},$$

and therefore v_p satisfies Lemma 6.

If

$$\sum_{k=1}^{n} x_k^2(p) \neq 0,$$

we would have from (2) that

$$v_2 \lrcorner (y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2 \lambda = 0,$$

whence

$$v_2 = -2\lambda y_1 \frac{\partial}{\partial y_1} + \mu\left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right).$$

Since

$$v_1 = \lambda \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

we have

$$v_p = \lambda\left(-2y_1 \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) + \mu\left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right). \quad \square$$

Corollary 2. *The quadratic form m_0 has constant rank n in U. If $p \notin U$ the rank of m_0 is less than n.*

The vector fields X, Y define (Lemma 6) a completely integrable 2-dimension ditribution $\mathfrak{F}$ in the open set U. Given also that $[X, Y] = 0$, the leaves of $\mathfrak{F}$ are the orbits of the Abelian group action $\mathbb{R}^2$ on the open set U, defined $\mathbb{R}^2 \times U \to U$ by $(s, t, p) \to \psi_s \cdot \varphi_t(p)$ where ψ_s and φ_t are the one-parameter group generated by Y and X respectively.

The relation between the manifold M_n, the metric m and the space $\mathfrak{F}$ is expressed by the Lemma below, which follows directly from Lemma 6.

Lemma 7. *Let M_n be a compact connected Hausdorff manifold, having a quadratic form that admits the global expression*:

$$m = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} (f_i\, df_j - f_j\, df_i)^2 + \sum_{k=1}^{n} [f_k(f\, dg - g\, df) - 2fg\, df_k]^2$$

and let $\varphi: M_n \to \mathbb{R}^n \times \mathbb{R}^2$ be given by $x_i = f_i$, $y_1 = f$, $y_2 = g$, $i = 1, \ldots, n$.

The following properties are equivalent

(*a*) *m is a Riemannian metric.*

(*b*) *φ is a transverse, immersion with respect to the distribution $\mathfrak{F}$.*

Remark 5. As a result, if m is a Riemannian metric on M_n expressed as above, and $\Pi: U \to \bar{U}$ is the quotient mapping to the leaf space of $\mathfrak{F}$, Lemma 7 proves that if $\bar{U}$ admits a quotient manifold structure, $\pi \cdot \varphi: M_n \to \bar{U}$ is then a local diffeormorphism.

5. The Leaf Space $\bar{U}$

The orbit passing through a point $p = (x_i, y_1, y_2)$ $i = 1, \ldots, n$, is expressed by $\psi_s\varphi_t(x_i, y_1, y_2) = (Ax_i, A^{-2}By_1, By_2)$ where $A = e^s$ and $B = e^t$. In order to calculate a model of the quotient of U by the action $\psi_s\varphi_t$, we consider the open sets

$$U_1 = \left\{ p \in U \,\middle|\, \sum_{i=1}^{n} x_i^2 \neq 0, y_1^2 + y_2^2 \neq 0 \right\}$$

and

$$U_1' = \{ p \in U \mid y_1 y_2 \neq 0 \}.$$

It is obvious that $U_1 \cup U_1' = U$, $U_1 \cap U_1' \neq \emptyset$ and both U_1 and U_1' are stable due to the action of $\mathbb{R}^2$.

In U_1, let us consider the submanifold

$$S^{n-1} \times S^1 = \left\{ p \in U_1 \,\middle|\, \sum x_i^2 = 1, y_1^2 + y_2^2 = 1 \right\}.$$

Lemma 8. *The leaf passing through $p \in U_1$ cuts $S^{n-1} \times S^1$ transversely at a single point, hence the mapping α: $U_1 \to S^{n-1} \times S^1$,*

$$\alpha(p) = \psi_s \cdot \psi_t(p) \cap (S^{n-1} \times S^1)$$

is a submersion.

Direct calculation proves that for

$$p = (x_i, y_1, y_2), \qquad \alpha(p) = (Ax_i, A^{-2}By_1, By_2)$$

where

$$A = \frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}} \quad \text{and} \quad B = \frac{1}{\sqrt{A^{-4}y_1^2 + y_2^2}}$$

The fact that the intersection is transversal follows immediately since these relations

$$\left(\sum_{i=1}^{n} x_i \, dx_i \right)(\lambda x + \mu Y) = 0 \qquad (y_1 \, dy_1 + y_2 \, dy_2)(\lambda x + \mu Y) = 0$$

imply that $\lambda = \mu = 0$. $\square$

Remark 6. The mapping $p \to (\alpha(p), A, B)$ of $U_1 \to (S^{n-1} \times S^1) \times \mathbb{R}_+^2$ is a diffeomorphism.

The open set U_1' has four connected components:

$$V_1 = \{p \in U_1' \mid y_1 > 0, y_2 > 0\}$$
$$V_2 = \{p \in U_1' \mid y_1 > 0, y_2 < 0\}$$
$$V_3 = \{p \in U_1' \mid y_1 < 0, y_2 > 0\}$$
$$V_4 = \{p \in U_1' \mid y_1 < 0, y_2 < 0\};$$

in each of which the following manifolds are considered

$$\Pi(1,1) = \{p \in V_1 \mid y_1 = 1, y_2 = 1\}$$
$$\Pi(1,-1) = \{p \in V_2 \mid y_1 = 1, y_2 = -1\}$$
$$\Pi(-1,1) = \{p \in V_3 \mid y_1 = -1, y_2 = 1\}$$
$$\Pi(-1,-1) = \{p \in V_4 \mid y_1 = -1, y_2 = -1\}.$$

Lemma 9. *The leaf passing through* $p \in V_i$, $(i = 1,2,3,4)$ *cuts* $\Pi_{(k,l)} \subset V_i$ $(k = 1, -1; l = 1, -1)$ *transversely at a single point. Consequently, the mapping* $\beta: U_1' \to \bigcup_{k,l=1,-1} \Pi_{(k,l)}$, defined by

$$\beta(p) = \psi_s \varphi_t(p) \cap \Pi_{(k,l)}, \quad \text{for} \quad p \in V_i \supset \Pi_{(k,l)}$$

is a submersion.

PROOF. The same calculation mentioned in Lemma 8 proves that

$$\psi_s \varphi_t(p) \cap \Pi_{(k,l)} = (Ax_i, A^{-2}By_1, By_2)$$

where

$$A = \left| \frac{y_1}{y_2} \right|, \qquad B = \frac{1}{|y_2|},$$

and $p \in V_i \supset \Pi_{(k,l)}$ $(k, l = 1, -1)$.

The transversality is now a consequence of the fact that the relations

$$dy_1(\lambda x + \mu Y) = 0$$
$$dy_2(\lambda x + \mu Y) = 0$$

imply that $\lambda = \mu = 0$ when $|y_1| = 1$ and $|y_2| = 1$. □

Due to the fact that $U_1 \cap U_1' \neq \varnothing$ the quotient $\bar{U}$ is obtained by identifying

$$(U_1 \cap U_1') \cap \Big(\bigcup_{k,l=1,-1,-1} \Pi_{(k,l)} \Big) \quad \text{with} \quad (U_1 \cap U_1') \cap (S^{n-1} \times S^1)$$

by means of the diffeomorphism which associates to each point of $U_1 \cap \Pi_{(k,l)}$, the intersection of the leaf passing through the point and $V_j \cap (S^{n-1} \times S^1)$, $\Pi_{(k,l)} \subset V_j$, *i.e.*

$$(**) \qquad U_1 \cap \Pi_{(k,l)} \to V_j \cap (S^{n-1} \times S^1)$$

given by

$$(x_i, k, l) \to \left(\frac{x_i}{\sqrt{\sum_{i=1}^{n} x_1^2}}, \frac{k \sum x_i^2}{\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)^2 + 1}}, \frac{l}{\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)^2 + 1}} \right)$$

with $\Pi_{(k,l)} \subset V_j$ for $j = 1, 2, 3, 4$ and $k, l = 1, -1$.

The fact that $\bar{U}$ is obtained by identifying open sets of the manifolds $S^{n-1} \times S^1$ and $\bigcup_{k,l=1,-1} \Pi_{(k,l)}$ by means of a diffeomorphism proves that $\bar{U}$ is a manifold. Moreover, the canonic applications

$$\alpha^1: S^{n-1} \times S^1 \to \bar{U}, \qquad \beta^1: \bigcup_{k,l=1,-1} \Pi_{(k,l)} \to \bar{U}$$

are diffeomorphic to their image.

Remark 7.

(*a*) $\bar{U}$ comprises $S^{n-1} \times S^1$ and four points

$$p_1 = \beta^1((0, \ldots, 0, 1, 1)),$$
$$p_2 = \beta^1((0, \ldots, 0, 1, -1)),$$
$$p_3 = \beta^1((0, \ldots, 0, -1, 1)),$$
$$p_4 = \beta^1((0, \ldots, 0, -1, -1)),$$

because

$$\Pi_{(k,l)} - (U_1 \cap \Pi_{(k,l)}) = (0, \ldots, 0, k, l), \quad \text{for} \quad k, l = 1, -1,$$

where $S^{n-1} \times S^1$ has the usual differentiable structure.

(*b*) A base of open neighbourhoods of

p_1 is $p_1 \cup (S^{n-1} \times W^1)$ where W^1 is an open interval of S^1 with extremes (0, 1) contained in $y_1 > 0$, $y_2 > 0$.

p_2 is $p_2 \cup (S^{n-1} \times W^2)$ where W^2 is an open interval of S^1 with extremes (0, -1) contained in $y_1 > 0$ and $y_2 < 0$.

p_3 is $p_3 \cup (S^{n-1} \times W^3)$ where W^3 is an open interval of S^1 with extremes (0, 1) contained in $y_1 < 0$ and $y_2 > 0$.

p_4 is $p_4 \cup (S^{n-1} \times W^4)$ where W^4 is an open interval of S^1 with extremes (0, -1) contained in $y_1 < 0$, $y_2 < 0$.

See Fig. 1.

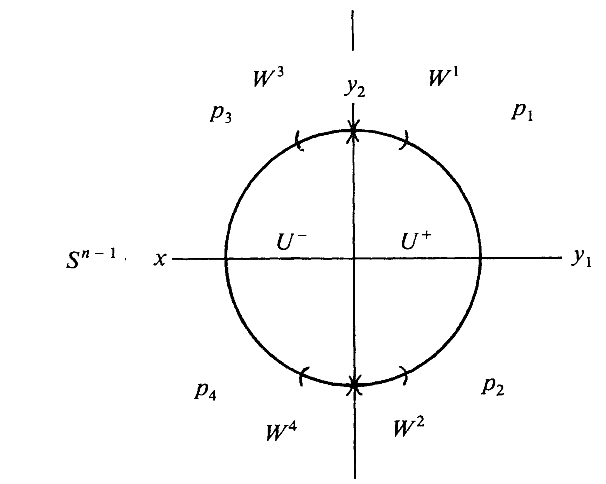

Fig. 1.

In fact, since β^1 is an open map it is sufficient to calculate

$$\beta^1[(B_\epsilon(0) - 0) \times (k, l)]$$

where $B_\epsilon(0)$ is the open ball of $\mathbb{R}^n$, centred at the origin with radius ϵ. This is obtained directly from the diffeomorphism (**) which transforms $(B(0) - 0) \times (k, l)$ into

$$S^{n-1} \times \left\{ \left(\frac{k \sum x_i^2}{\sqrt{(\sum x_i^2)^2 + 1}}, \frac{l}{\sqrt{(\sum x_i^2)^2 + 1}} \right) \middle| \sum x_i^2 < \epsilon \right\} = S^{n-1} \times W^i$$

where W^i is an interval in S^1 contained in the quadrant corresponding to the point $\beta^1(0, \ldots, 0, k, l)$, of extremes $(0, l)$.

(*c*) $\alpha'\alpha: U_1 \to \bar{U}$ and $\beta'\beta: U'_1 \to \bar{U}$ constitute the composition of the submersions α, β (Lemmas 8 and 9) with the diffeomorphisms α', β' respectively, $\alpha'\alpha$ and $\beta'\beta$ coincide on $U_1 \cap U'_1$ and define the quotient application $\Pi: U \to \bar{U}$, hence Π is a submersion.

(*d*) It should be noted that $\bar{U}$ is a compact, connected, non-Hausdorff manifold because p_1 and the points of $S^{n-1} \times (0, 1)$ do not have disjoint neighbourhoods. The same occurs with p_2 and $S^{n-1} \times (0, -1)$, p_3 and $S^{n-1} \times (0, 1)$ and p_4 and $S^{n-1} \times (0, -1)$.

(*e*) Note that $p \in M_n$ and $f_i(p) = 0$ for all $i = 1, \ldots, n$ is equivalent to $\pi \cdot \varphi(p)$ being equal to some p_i, $(i = 1, 2, 3, 4)$.

(*f*) Finally, $p \in M_n$ and $f(p) = 0$ is equivalent to $\pi \cdot \varphi(p) \in S^{n-1} \times (0, l)$, for $l = 1, -1$.

PROOF OF (*a*), THEOREM 3. From (*e*) of Remark 7, it is deduced that if $H = \emptyset$, $\pi\varphi$ is a local diffeomorphism from M_n onto $S^{n-1} \times S^1$ and consequently it is a covering map. From the classification of covering maps and since M_n is connected and compact, it is deduced that M_n is diffeomorphic to $S^{n-1} \times S^1$.

PROOF OF (*b*), THEOREM 3. For this proof it is necessary to assume that $n > 2$. Let

$$U^+ = S^{n-1} \times \left(\frac{3\pi}{2}, \frac{\pi}{2}\right) \quad \text{and} \quad U^- = S^{n-1} \times \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

(see Fig. 1) be open subsets of $\bar{U}$ and let $\pi\varphi: M_n \to \bar{U}$ be the local diffeomorphism obtained in Remark 5 of Lemma 7.

For the following we shall be using the next lemma, which is easy to prove.

Lemma 10. *Let $\tau: M_n \to M'_n$ be a local diffeomorphism between two compact connected manifolds. Let U be an open set of M'_n satisfying the following conditions*

(*a*) $\tau^{-1}(U) \neq \emptyset$.
(*b*) *For all values of $x \in U$* and $y \in M'_n$ *there are disjoint open neighbourhoods of x and y, U^x, U^y.*

Then if C is a connected component of $\tau^{-1}(U)$, the mapping $\tau_{|C}: C \to U$ is a covering map with a finite number of folds.

Corollary 3. *If C^+ (C^- respectively) is a connected component of $(\pi\varphi)^{-1}(U^+)$ ($(\pi\varphi)^{-1}(U^-)$ respectively), then $\pi \cdot \varphi_{|C^+}: C^+ \to U^+$ ($\pi \cdot \varphi_{|C^-}: C^- \to U^-$ respectively) is a diffeomorphism.*

This follows directly from $n > 2$ and the fact that the open set U^+ (U^- respectively) of the manifold $\bar{U}$ fulfills the conditions of Lemma 10. □

Lemma 11. *Let $p \in M_n$ such that $\pi \cdot \varphi(p) = p_1$ or p_2. There exists a unique connected component C^+ of $(\pi\varphi)^{-1}(U^+)$ such that*

(1) *$p \cup C^+$ is open.*
(2) *$\pi \cdot \varphi_{|p \cup C^+}: p \cup C^+ \to p_1 \cup U^+$ ($p_2 \cup U^+$ respectively) is a diffeomorphism.*

(If $\pi \cdot \varphi(p) = p_3$ or p_4, an analogous statement is obtained by substituting C^+ by C^- and U^+ by U^-.)

PROOF. Let us assume that $\pi\varphi(P) = p_1$ and that W^p is an open connected neighbourhood of p such that $\pi\varphi_{|W^p}: W^p \to V^{p_1}$ is a local diffeomorphism, where

$$V^{p_1} = p_1 \cup \left(S^{n-1} \times \left(\frac{\pi}{2}, \frac{\pi}{2} - \epsilon \right) \right)$$

and $\epsilon > 0$ (see Fig. 1). Since $\pi \cdot \varphi(W^p - p) \subset U^+$ and $W^p - p$ is connected, there is a unique connected component C^+ of $(\pi \cdot \varphi)^{-1}(U^+)$ such that $W^p - p \subset C^+$. Hence, $p \cup C^+$ is open.

Part 2 of Lemma 11 results from Corollary 3. □

Let us suppose that the function f appearing in the global expression for the metric m, satisfies $f(p) \neq 0$ for every $p \in M_n$ (this is the case of the metric on S^n in (*b*) of Section 3 where $f \equiv 1$). If $f(p) > 0$ for every $p \in M_n$ as $\pi\varphi(M_n)$ is compact, it follows from Lemma 11 that there will be points $p, q \in M_n$ such that

(1) $p \cup C^+ \cup q$ is open in M_n.

(2) $\pi \cdot \varphi_{|p \cup C^+ \cup q}: p \cup C^+ \cup q \to p_1 \cup U^+ \cup p_2$ is a diffeomorphism.

Since $p_1 \cup U^+ \cup p_2$ is diffeomorphic to the sphere S^n (see Fig. 1) and M_n is compact and connected, we conclude that M_n and S^n are diffeomorphic. Likewise, if $f(p) < 0$ for every $p \in M_n$. This proves Theorem 3(*b*) in the case where $f(p) \neq 0$ for every $p \in M_n$.

The metrics constructed on S^n in (*c*) of Section 3 have the property $\{p \in S^n \mid f(p) = 0\} \neq \emptyset$.

From the global expression for the metric in Theorem 3, we conclude that $df(p) \neq 0$ in the case $f(p) = 0$, hence $f^{-1}(0)$ is the union of a finite number of $(n-1)$-dimension compact manifolds. Moeover,

$$f^{-1}(0) = (\pi \cdot \varphi)^{-1}\left(S^{n-1} \times \frac{\pi}{2} \cup S^{n-1} \times \frac{3\pi}{2} \right)$$

(see Fig. 1).

We will denote by $M_{n-1}^{\pi/2}$ a connected component of $(\pi\varphi)^{-1}\left(S^{n-1} \times \left\{ \frac{\pi}{2} \right\}\right)$ and by $M_{n-1}^{3\pi/2}$ a connected component of $(\Pi\varphi)^{-1}\left(S^{n-1} \times \left\{ \frac{3\pi}{2} \right\}\right)$.

Lemma 12.

$$\pi\varphi_{|M_{n-1}^{\pi/2}}: M_{n-1}^{\pi/2} \to S^{n-1} \times \frac{\pi}{2}$$

and

$$\pi\varphi_{|M^{3\pi/2}_{n-1}}: M^{3\pi/2}_{n-1} \to S^{n-1} \times \frac{3\pi\ \pi}{2}$$

are diffeomorphisms.

Moreover, for each $M^{\pi/2}_{n-1}$ *(*$M^{3\pi/2}_{n-1}$ *respectively) there are unique connected components* C^+ *and* C^- *of* $(\pi\cdot\varphi)^{-1}(U^+)$ *and* $(\pi\cdot\varphi)^{-1}(U^-)$*, such that*

$$\pi\cdot\varphi_{|C^-\cup M^{\pi/2}_{n-1}\cup C^+}: C^- \cup M^{\pi/2}_{n-1} \cup C^+ \to U^- \cup \left(S^{n-1}\times\frac{\pi}{2}\right) \cup U^+$$

$$\left(C^+ \cup M^{3\pi/2}_{n-1} \cup C^- \to U^+ \cup \left(S^{n-1}\times\frac{3\pi}{2}\right) \cup U^- \ \textit{respectively}\right)$$

is a diffeomorphism.

PROOF. The first statement of this lemma results from $\pi\varphi$ being a local diffeomorphism and from $n > 2$.

The second statement follows from Lemma 11 and the compactness of $M^{\pi/2}_{n-1}$ ($M^{3\pi/2}_{n-1}$ respectively). □

Remark 8. If $p \in C^+$ (C^- respectively) we have the relations

$$\sum_{i=1}^{n} f_i^2(p) \neq 0 \quad \text{and} \quad f(p) \neq 0,$$

(see Fig. 1). Hence the closure $\bar{C}^+$ ($\bar{C}^-$ respectively), according to Lemmas 11 and 12, is one and only one of the following sets:

$$p\cup C^+\cup q; \qquad p\cup C^+\cup M^{3\pi/2}_{n-1}; \qquad M^{\pi/2}_{n-1}\cup C^+\cup M^{3\pi/2}_{n-1}$$
$$(r\cup C^-\cup s;\ r\cup C^-\cup M^{\pi/2}_{n-1};\ M^{\pi/2}_{n-1}\cup C^-\cup M^{3\pi/2}_{n-1} \text{ respectively})$$

where $\pi\varphi(p) = p_1$, $\pi\varphi(q) = p_2$ ($\pi\varphi(r) = p_4$, $\pi\varphi(s) = p_3$ respectively).

The final part of the proof of *b*) in Theorem 3 now follows from previous lemmas.

Let us assume that $\pi\varphi(p) = p_1$ and let C^+ be the unique connected component of $(\pi\varphi)^{-1}(U^+)$ such that $p\cup C^+$ is open and $\pi\varphi_{|p\cup C^+}: p\cup C^+ \to p_1\cup U^+$ is a diffeomorphism (Lemma 9). In the closure of C^+ there can be either points or spheres (Remark 8). If there are only points in $\bar{C}^+$, then $\bar{C}^+ = p\cup C^+\cup q$. This case has already been considered and hence M_n is diffeomorphic to S^n.

Let us assume that $\bar{C}^+ = p\cup C^+\cup M^{3\pi/2}_{n-1}$ where

$$\pi\varphi_{|p\cup C^+\cup M^{3\pi/2}_{n-1}}: p\cup C^+\cup M^{3\pi/2}_{n-1} \to p_1\cup U^+\cup\left(S^{n-1}\times\frac{3\pi}{2}\right)$$

is a diffeomorphism and let C^- be the unique component of $(\pi\varphi)^{-1}(U^-)$ (Lemma 10) such that

(*a*) $C^+ \cup M_{n-1}^{3\pi/2} \cup C^-$ is open

(*b*) $\pi \cdot \varphi_{|C^+ \cup M_{n-1}^{3\pi/2} \cup C^-}: C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \to U^+ \cup \left(S^{n-1} + \frac{3\pi}{2}\right) \cup U^-$ is a diffeomorphism.

If

$$\bar{C}^- = M_{n-1}^{3\pi/2} \cup C^- \cup q,$$

all previous diffeomorphisms produce a single diffeomorphism

$$\pi\varphi: p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \cup q \to p_1 \cup U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right) \cup U^- \cup p_3$$

(see Fig. 1) where the second member is a manifold diffeomorphic to the sphere S^n. As M_n is connected it is diffeomorphic to S^n.

If

$$\bar{C}^- = M_{n-1}^{3\pi/2} \cup C^- \cup M_{n-1}^{\pi/2},$$

Lemma 12 proves that here is a component C^{+1} of $(\pi\varphi)^{-1}(U^+)$ such that $C^- \cup M_{n-1}^{\pi/2} \cup C^{+1}$ is open and

$$\pi\varphi_{|C^- \cup M_{n-1}^{\pi/2} \cup C^{+1}}: C^- \cup M_{n-1}^{\pi/2} \cup C^{+1} \to U^- \cup \left(S^{n-1} \times \frac{\pi}{2}\right) \cup U^{+1}$$

is a diffeomorphism ($U^{+1} = U^+$ but it is now diffeomorphic to C^{+1}).

Let us assume that $\bar{C}^{+1} = M_{n-1}^{\pi/2} \cup C^{+1} \cup q$ where $\pi\varphi(q) = p_2$. We have the diffeomorphisms

$$p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \xrightarrow{\pi\varphi} p_1 \cup U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right) \cup U^- = U_1.$$

$$C^- \cup M_{n-1}^{\pi/2} \cup C^{+1} \cup q \xrightarrow{\pi\varphi} U^- \cup \left(S^{n-1} \times \frac{\pi}{2}\right) \cup U^{+1} \cup p_2 = U_2.$$

By pasting the manifolds U_1 and U_2 through the identification of $U^- \subset U_1$ with $U^- \subset U_2$, the sphere S^n is obtained and consequently $\pi\varphi$ is a diffeomorphism

$$\pi\varphi \quad p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \cup M_{n-1}^{\pi/2} C^{+1} \cup q \to S^n.$$

Because the number of C^+ and C^- components is finite, the above process would use up these components and by Lemmas 9 and 10 would end in a single point. As M_n is connected, it would be diffeomorphic to S^n.

The proof of the theorem is complete. □

References

[1] Gonzalo, J. Classification results for contact forms, *Indag. Math.* **48**(1986), 289-312.

[2] Gonzalo, J., Varela, F. Some surfaces that are distinguishable by the global expression of their volume elements, *Math. Ann.* **267** (1984), 199-212.

[3] Gromov, M. Partial Differential Relations, Springer Verlag, Heildelberg, 1986.

[4] Varela, F. Formes de Pfaff, classe et perturbations, *Annales Inst. Fourier*, XXVI, **4**(1976), 239-271.

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