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An Atomic Decomposition of the Predual of BMO(ρ)

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Abstract

We study the Orlicz type spaces H_ω, defined as a generalization of the Hardy spaces H^p for $p \leqslant 1$. We obtain an atomic decomposition of H_ω, which is used to provide another proof of the known fact that BMO(ρ) is the dual space of H_ω (see S. Janson, 1980, [J]).

Introduction

The purpose of this work is to study the spaces H_ω, obtained as a generalization of the Hardy spaces H^p taking $\omega(t) = t^p$. For more general ω, the space H_ω was considered before by Janson in [J]. There, the author proves that BMO(ρ) is the dual space of H_ω, with ρ and ω related by $t^n \rho(t) \omega^{-1}(1/t^n) = 1$. The main result of this paper is an atomic decomposition of H_ω.

The atomic decomposition of H^p spaces starts with the work of Hertz in the martingale setting ([H]). Since then many authors have been studying the problem in different situations: R. R. Coifman [CO], Latter [L], Latter and Uchiyama [LU], Calderón [C], Macías and Segovia [MS], etc. Since most of the work in the atomic decomposition relies on Calderón-Zygmund type lemmas, we accomplish the problem in the setting of spaces of homogeneous type for which the Calderón-Zygmund method has been worked out by Macías [M] and Macías and Segovia [MS].

As corollary of the atomic decomposition we obtain another proof of the fact that BMO(ρ) is the dual space of H_ω. As a by-product we get the equivalence of BMO(ρ) and BMO(ρ, q) ($1 < q < \infty$) without using John-Niremberg type Lemmas (see [A] for a proof of John-Niremberg lemma in this context). We would like to point out that the atomic space is given by an Orlicz type norm which in the case of $\rho(t) = t^{1/p-1}$ provides the usual atomic H^p spaces.

In the first section we give the notation and definitions that we shall use in the sequel. We introduce the atomic spaces $H^{\rho,q}$, $1 < q \leqslant \infty$, the maximal spaces H_ω and the spaces BMO(ρ, q).

In Section 2 we state the main results: atomic decomposition, Theorem 2.1, and duality, Theorem 2.2.

In Section 3 we prove the basic properties of the growth functions ω and ρ, in particular Lemma 3.1 provides the tool for further work with this type of functions.

In Section 4 we prove Theorem 2.1. The key for this proof is the Calderón-Zygmund type Lemma 4.9. Other important tools in the proof of Theorem 2.1 are interesting by themselves: the maximal space H_ϕ is continuouly included in the Orlicz space L_ϕ for ϕ a Young function (Theorem 4.15). Theorem 2.1 also provides an important consequence namely, the spaces $H^{\rho,\infty}$ and $H^{\rho,q}$ ($1 < q < \infty$) are equivalent.

Finally, Theorem 2.2 is proved in Section 5.

1. Notation and Definitions

Let X be a set. A function $d: X \times X \to \mathbb{R}^+ \cup \{0\}$ shall be called a quasi-distance on X if there exists a finite constant K such that

(1.1) $$d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

(1.2) $$d(x, y) = d(y, x),$$

and

(1.3) $$d(x, y) \leqslant K[d(x, z) + d(z, y)]$$

for every x, y, and z in X.

In a set X, endowed with a quasi-distance $d(x, y)$, the balls

$$B(x, r) = \{y: d(x, y) < r\}, \qquad r > 0,$$

form a basis of neighbourhoods of x for the topology induced by the uniform structure on X.

We shall say that a set X, with a quasi-distance $d(x, y)$ and a non-negative measure μ defined on a σ-algebra of subsets of X which contains the balls $B(x, r)$, is a normal space of homogeneous type if there exist four positive and finite constans A_1, A_2, K_1 and $K_2 \leqslant 1 \leqslant K_1$, such that

$$A_1 r \leqslant \mu(B(x, r)) \quad \text{if} \quad r \leqslant K_1 \mu(X) \tag{1.4}$$

$$B(x, r) = X \quad \text{if} \quad r > K_1 \mu(X) \tag{1.5}$$

$$A_2 r \geqslant \mu(B(x, r)) \quad \text{if} \quad r \geqslant K_2 \mu(\{x\}) \tag{1.6}$$

$$B(x, r) = \{x\} \quad \text{if} \quad r < K_2 \mu(\{x\}). \tag{1.7}$$

We note that, under these conditions, there exist two finite constants, $a > 1$ and A, such that

$$0 < \mu(B(x, ar)) \leqslant A\mu(B(x, r)) \tag{1.8}$$

holds for every x in X and $r > 0$.

We shall say that a normal space of homogeneous type (X, d, μ), is of order α, $0 < \alpha < \infty$, if there exists a finite constant K_3 satisfying

$$|d(x, z) - d(y, z)| \leqslant K_3 r^{1-\alpha} d(x, y)^{\alpha}, \tag{1.9}$$

for every x, y and z in X, whenever $d(x, z) < r$ and $d(y, z) < r$ (see [MS]).

In this paper $X = (X, d, \mu)$ shall mean a normal space of homogeneous type of order $\alpha, 0 < \alpha \leqslant 1$, and we shall often refer to the constants that appear in (1.3) to (1.9) as the constants of the space.

Let ρ be a positive function defined on $\mathbb{R}^+$. We shall say that ρ is of upper type m (respectively, lower type m), if there exists a positive constants c such that

$$\rho(st) \leqslant ct^m \rho(s) \tag{1.10}$$

for every $t \geqslant 1$ (respectively, $0 < t \leqslant 1$). A non-decreasing function ρ of finite upper type such that $\lim_{t \to 0^+} \rho(t) = 0$ is called a growth function.

We shall say that a positive function ρ is quasi-increasing if there exists a constant c such that

$$\rho(s) \leqslant c\rho(t) \quad \text{for} \quad s \leqslant t.$$

Let ρ be a quasi-increasing function. We consider the function $\rho^{-1}(s)$ defined by

$$\rho^{-1}(s) = \sup \{t : \rho(t) \leqslant s\}$$

for those values of s at which the supremum is a real number. Clearly $\rho^{-1}(\rho(s)) \geqslant s$. It is easy to see that if the function $\rho(t)$ is continuous and strictly increasing then the function $\rho^{-1}(t)$ is the ordinary inverse function of $\rho(t)$.

We shall understand that two positive functions are equivalents if their quotient is bounded above and below by two positive constants.

Let $\psi(x)$ be an integrable function on bounded subsets of X. For any ball B, we denote

$$m_B(\psi) = \mu(B)^{-1} \int_B \psi(x)\, d\mu(x)$$

and, as is usual, the Hardy-Littlewood maximal function by

$$M(\psi)(x) = \sup m_B(|\psi|),$$

where the supremum is taken over all balls B containing x.

Definition 1.11. *Let $1 \leqslant q < \infty$ and ρ a growth function plus a non-negative constant or $\rho \equiv 1$. A function $f(x)$, integrable on bounded subsets, belongs to* BMO(ρ, q) *if there exists a constant c such that the inequality*

$$\left[\mu(B)^{-1} \int_B |f(x) - m_B(f)|^q\, d\mu(x)\right]^{1/q} \leqslant c\rho(\mu(B))$$

holds for every ball B. The least constant c satisfying the inequality above shall be denoted by $\|f\|_{\mathrm{BMO}(\rho,q)}$. When ρ is the constant function $\rho \equiv 1$ and $1 \leqslant q < \infty$, the space BMO$(1, q)$ *coincides with the space of functions of bounded mean oscillation* BMO. *The space* BMO$(\rho, 1)$ *shall be denoted by* BMO(ρ).

Let ρ be a growth function. We shall say that a function $\psi(x)$ belongs to Lip(ρ), *if there exists a finite constant c such that*

$$|\psi(x) - \psi(y)| \leqslant c\rho(d(x, y)) \tag{1.12}$$

for every x and y in X. The least constant c satisfying this condition shall be denoted by $\|\psi\|_{\mathrm{Lip}(\rho)}$. When $\rho(t)$ is the function t^β, $0 < \beta < \infty$, we shall say that $\psi(t)$ is in Lip(β) *and, in this case, $\|\psi\|_\beta$ indicates its norm.*

In [MS], Macías and Segovia introduce the space of distributions $(E^\alpha)'$ as the dual of the space E^α consisting of all functions with bounded support, belonging to Lipschitz β, $0 < \beta < \alpha$.

For x in X and $0 < \gamma < \alpha$ we consider the class $T_\gamma(x)$, of functions ψ belonging to E^α satisfying the following condition: there exists r such that $r \geqslant K_2\mu(\{x\})$, supp $\psi \subset B(x, r)$ and

$$r\|\psi\|_\infty \leqslant 1 \quad \text{and} \quad r^{1+\gamma}\|\psi\|_\gamma \leqslant 1. \tag{1.13}$$

Given γ, $0 < \gamma < \alpha$, we define the γ-maximal function $f_\gamma^*(x)$ of a distribution f on E^α by

$$f_\gamma^*(x) = \sup\{|f(\psi)| : \psi \in T_\gamma(x)\}. \tag{1.14}$$

Definition 1.15. *Let ρ be a growth function plus a non-negative constant or $\rho \equiv 1$. A (ρ, q) atom, $1 < q \leqslant \infty$, is a function $a(x)$ on X satisfying:*

(1.16) $$\int_X a(x)\, d\mu(x) = 0,$$

(1.17) *the support of $a(x)$ is contained in a ball B and*

(1.18) $$\left[\mu(B)^{-1} \int_B |a(x)|^q\, d\mu(x)\right]^{1/q} \leqslant [\mu(B)\rho(\mu(B))]^{-1}$$

if $q < \infty$ or

$$\|a\|_\infty \leqslant [\mu(B)\rho(\mu(B))]^{-1}, \quad \textit{if} \quad q = \infty.$$

If $\mu(X)$ is finite, we may assume that $\mu(X) = 1$. In this case, we also suppose that $\rho(1) = 1$ and we consider the characteristic function of X as a (ρ, q) atom. Clearly, when $\rho(t) = t^{1/p-1}$, $p \leqslant 1$, a (ρ, q) atom is a (p, q) atom in the sense of [M].

Definition 1.19. *Let $0 < \gamma < \alpha$. Assume that ω is a growth function of lower type l such that $l(1+\gamma) > 1$. We define*

$$H_\omega = H_\omega(X) = \left\{f \in (E^\alpha)' \colon \int \omega[f_\gamma^*(x)]\, d\mu(x) < \infty\right\}$$

and we denote

$$\|f\|_{H_\omega} = \inf\left\{\lambda > 0 \colon \int \omega\left(\frac{f_\gamma^*(x)}{\lambda^{1/l}}\right) d\mu(x) \leqslant 1\right\}.$$

It is easy to see that H_ω is a complete metrizable topological vector space with respect to the quasi-distance induced by $\| \quad \|_{H_\omega}$. Moreover, H_ω is continuously included in $(E^\alpha)'$.

Definition 1.20. *Let ω be a growth function of positive lower type l. If $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$, we define $H^{\rho,q}(X) = H^{\rho,q}$, $1 < q \leqslant \infty$, as the linear space of all distributions f on E^α which can be represented by*

(1.21) $$f(\psi) = \sum_i b_i(\psi),$$

for every ψ in E^α, where $\{b_i\}_i$ is a sequence of multiples of (ρ, q) atoms such that if supp $(b_i) \subset B_i$ *then*

(1.22) $$\sum_i \mu(B_i)\omega(\|b_i\|_q \mu(B_i)^{-1/q}) < \infty.$$

We introduce a quasi-distance in $H^{\rho,q}$. Given a sequence of multiples of (ρ, q)-atoms, $\{b_i\}_i$, we denote

$$\Lambda_q(\{b_i\}) = \inf\left\{\lambda: \sum_i \mu(B_i)\omega\left(\frac{\|b_i\|_q \mu(B_i)^{-1/q}}{\lambda^{1/l}}\right) \leqslant 1\right\} \tag{1.23}$$

and we define

$$\|f\|_{H^{\rho,q}} = \inf \Lambda_q(\{b_i\}), \tag{1.24}$$

where the infimum is taken over all possible representations of f of the form (1.21).

2. Statement of the results

The main theorems in this work are the following.

Theorem 2.1. *Let ω be a function of lower type l such that $l(1+\gamma) > 1$, $0 < \gamma < \alpha$. Assume that $\omega(s)/s$ is non-increasing. Let $\rho(t)$ be the function defined by $t\rho(t) = 1/\omega^{-1}(1/t)$. Then $H_\omega \equiv H^{\rho,q}$ for every $1 < q \leqslant \infty$.*

Theorem 2.2. *Let $\rho(t)$ and $\omega(t)$ as in Theorem* 2.1. *Then,* $(H_\omega)' \equiv \mathrm{BMO}(\rho)$.

We observe that from Proposition 3.10, it turns out that the function $\rho(t)$ defined in Theorem 2.1 satisfies the conditions of Definitions 1.11 and 1.15.

3. Basic Lemmas on Growth Functions

Proposition 3.1. *Let ω be a function of positive lower type l such that $\omega(s)/s$ is non increasing. Then the following properties hold*

$$l \leqslant 1, \tag{3.2}$$

$$\frac{\omega(s)}{s^p} \tag{3.3}$$

is quasi-increasing for $p \leqslant l$,

$$\tilde{\omega}(t) = \int_0^t \frac{\omega(s)}{s}\, ds \tag{3.4}$$

is a continuous function of positive lower type $l \leqslant 1$ equivalent to ω, (3.5) $$\lim_{t \to 0^+} \tilde{\omega}(t) = 0,$$

(3.6) $\tilde{\omega}$ *is strictly increasing.*
(3.7) $\tilde{\omega}$ *is subaditive,*

(3.8) $$\frac{\tilde{\omega}(s)}{s}$$

is non-increasing and

(3.9) $$\frac{\tilde{\omega}(s)}{s^p}$$

is quasi-increasing for $p \leqslant l$.

PROOF. Since $\omega(s)/s$ is non-increasing and $\omega(s)$ is of lower type l, we have

$$\omega(1) \leqslant \frac{\omega(s)}{s} \leqslant cs^{l-1}\omega(1)$$

for $s < 1$, and (3.2) holds. To prove (3.3) take $s \leqslant t$ and $p \leqslant l$. From the lower type property of ω, we obtain

$$\frac{\omega(s)}{s^p} \leqslant c\left(\frac{s}{t}\right)^l \frac{\omega(t)}{s^p} \leqslant c\frac{\omega(t)}{t^p}.$$

Let us prove (3.4). Clearly $\omega(t) \leqslant \tilde{\omega}(t)$. On the other hand using the lower type l of ω, we get

$$\tilde{\omega}(t) = \int_0^1 \frac{\omega(st)}{s} ds \leqslant c\omega(t) \int_0^1 s^{l-1} ds = cl^{-1}\omega(t).$$

Properties (3.5) and (3.6) follow immediately from the definition fo $\tilde{\omega}$. In order to prove (3.7), let us observe that

$$\begin{aligned} \tilde{\omega}(a+b) &= \int_0^a \frac{\omega(s)}{s} ds + \int_a^{a+b} \frac{\omega(s)}{s} ds \\ &\leqslant \tilde{\omega}(a) + \int_0^b \frac{\omega(s)}{s} ds \\ &= \tilde{\omega}(a) + \tilde{\omega}(b), \end{aligned}$$

where the inequality follows from the fact that $\omega(s)/s$ is non-increasing. Let $t_1 \geqslant t_2$, then

$$\frac{\tilde{\omega}(t_1)}{t_1} = \frac{1}{t_1}\int_0^{t_1} \frac{\omega(s)}{s}\,ds = \frac{1}{t_2}\int_0^{t_2} \frac{\omega(st_1/t_2)}{st_1/t_2}\,ds \leqslant \frac{1}{t_2}\int_0^{t_2} \frac{\omega(s)}{s}\,ds = \frac{\tilde{\omega}(t_2)}{t_2},$$

which proves (3.4). Finally, (3.9) can be proved as (3.3) using (3.8). □

Let us observe that the results stated in Section 2 are invariant under change of equivalent growth functions. So that, there is no loss of generality in assuming that ω satisfies all properties of $\tilde{\omega}$ in Proposition 3.1, that is, we shall suppose that ω verifies (3.4) to (3.9).

Proposition 3.10. *Given $\omega(t)$, let $\rho(t)$ be the function defined on $\mathbb{R}^+$ by*

$$\rho(t) = \frac{t^{-1}}{\omega^{-1}(t^{-1})}.$$

Then $\rho(t)$ is a positive non decreasing function of upper type $l^{-1} - 1 \geqslant 0$.

Proof. In order to prove that ρ is non-decreasing, let us take $t_1 \geqslant t_2 > 0$. Since $\omega^{-1}(t_1^{-1}) \leqslant \omega^{-1}(t_2^{-1})$ and $\omega(s)/s$ is non-increasing, we have

$$\rho(t_1) = \frac{\omega(\omega^{-1}(t_1^{-1}))}{\omega^{-1}(t_1^{-1})} \geqslant \frac{\omega(\omega^{-1}(t_2^{-1}))}{\omega^{-1}(t_2^{-1})} = \rho(t_2).$$

On the other hand, it is easy to check that ω is of lower type l if and only if ω^{-1} is of upper type l^{-1}. So that, ω^{-1} satisfies

$$\omega^{-1}(s\tau) \geqslant c\tau^{1/l}\omega^{-1}(s)$$

for every $s \in \mathbb{R}^+$ and $\tau \leqslant 1$. Therefore, for $s \in \mathbb{R}^+$ and $t \geqslant 1$, we obtain

$$\rho(st) = \frac{s^{-1}t^{-1}}{\omega^{-1}(s^{-1}t^{-1})} \leqslant \frac{s^{-1}t^{-1}}{ct^{-1/l}\omega^{-1}(s^{-1})} = ct^{1/l-1}\rho(s),$$

which proves that $\rho(t)$ is of upper type $l^{-1} - 1$. □

4. Atomic Decomposition

In [MS], Macías and Segovia obtain the atomic decomposition of H_ω, with $\omega(t) = t^p$, on spaces of homogeneous type. In this section we shall adapt to our situation their scheme of proof. Therefore we shall prove only the technical lemmas which require suitable modifications, the results that we state without proof can be found in that paper.

In this and the following section we shall assume that ω satisfies (3.4) to (3.9) with $l(1 + \gamma) > 1$ for some γ, $0 < \gamma < \alpha$. The first two lemmas deal with geometric properties of the space.

Lemma 4.1. *Let $r > 0$, $x_0 \in X$ and $p > 1$. Then*

$$\int_{B(x_0, r)^c} [r/d(x, x_0)]^p \, d\mu(x) \leqslant c\mu(B(x_0, r)),$$

where c depends only on p and the constants of the space.

Lemma 4.2. *Let $0 < \beta$, $1 < q(1 + \beta)$ and M a positive integer. There exists a finite constant $c_{q,\beta,M}$ such that given any sequence of points $\{x_n\}$, and any sequence of positive numbers $\{r_n\}$, satisfying the condition that no point in X belongs to more than M balls $B(x_n, r_n)$, then*

$$\int \left\{ \sum_n \left[\frac{\mu(B(x_n, r_n))}{\mu(B(x_n, r_n)) + d(x, x_n)} \right]^{1+\beta} \right\}^q d\mu(x) \leqslant c_{q,\beta,M}\, \mu\left(\bigcup_n B(x_n, r_n) \right).$$

The following lemma and its corollary show that (ρ, q) atoms are in H_ω. Lemma 4.7 is a technical result to be used in Theorem 2.2.

Lemma 4.3. *Let $b(x)$ be a function in $L^q(X, d\mu)$, $1 < q \leqslant \infty$, with support contained in $B = B(x_0, R)$ and $\int b(x)\, d\mu(x) = 0$. Then, there exists a constant c, independent of $b(x)$, such that*

$$\int \omega[b_\gamma^*(x)]\, d\mu(x) \leqslant c\mu(B)\omega(\|b\|_q \mu(B)^{-1/q}).$$

PROOF. Let $\psi(x) \in T_\gamma(y)$ such that $\operatorname{supp}(\psi) \subset B(y, r)$. Then, from (1.6) and (1.13) it follows that

$$\left| \int b(x)\psi(x)\, d\mu(x) \right| \leqslant A_2 M(b)(y).$$

Therefore, $b_\gamma^*(y) \leqslant A_2 M(b)(y)$. Let $m = \|b\|_q \mu(B)^{-1/q}$. Thus, using (3.6) and (3.8), we have

$$\begin{aligned} \omega[b_\gamma^*(y)] &\leqslant c\omega[M(b)(y)] \leqslant c\omega[M(b)(y) + m] \\ &\leqslant c\left(\frac{M(b)(y)}{m} + 1 \right)\omega(m). \end{aligned}$$

Integrating on $B(x_0, 2KR)$, by (1.8) and the L^q-boundedness of M, we get

$$\begin{aligned} (4.4) \qquad \int_{B(x_0, 2KR)} \omega[b_\gamma^*(y)]\, d\mu(y) &\leqslant c\omega(m) \left[\frac{1}{m} \|M(b)\|_q \mu(B)^{1-1/q} + \mu(B) \right] \\ &\leqslant c\omega(m)\mu(B). \end{aligned}$$

On the other hand, let $y \notin B(x_0, 2KR)$ and $\psi \in T_\gamma(y)$ as before. We can assume that $B(x_0, R) \cap B(y, r) = \varnothing$. Consequently, $R < r$ and $d(y, x_0) < 2Kr$. Then,

$$\left|\int b(x)\psi(x)\,d\mu(x)\right| = \left|\int_B b(x)[\psi(x) - \psi(x_0)]\,d\mu(x)\right|$$
$$\leqslant \|b\|_q\left(\int_B |\psi(x) - \psi(x_0)|^{q'}\,d\mu(x)\right)^{1/q'}$$
$$\leqslant \|b\|_q\|\psi\|_\gamma R^\gamma \mu(B)^{1/q'}$$
$$\leqslant \|b\|_q\left[\frac{2KR}{d(y,x_0)}\right]^{1+\gamma} R^{-1}\mu(B)^{1/q'}$$
$$= m\left[\frac{2KR}{d(y,x_0)}\right]^{1+\gamma} R^{-1}\mu(B).$$

We can suppose that $R \geqslant K_2\mu(\{x_0\})$, since otherwise $b \equiv 0$. Thus, by (1.6), we have

$$b_\gamma^*(y) \leqslant A_2 m\left[\frac{2KR}{d(y,x_0)}\right]^{1+\gamma}.$$

Applying ω to both sides, since ω is of lower type l and $\omega(s)/s$ is decreasing, we obtain

$$\omega[b_\gamma^*(y)] \leqslant c\omega(m)\left[\frac{2KR}{d(y,x_0)}\right]^{(1+\gamma)l} \tag{4.5}$$

for $y \notin B(x_0, 2KR)$. Integrating (4.5) on the complementary set of $B(x_0, 2KR)$, since $l(1+\gamma) > 1$, by Lemma 4.1, we get

$$\int_{B(x_0,2KR)^c} \omega[b_\gamma^*(y)]\,d\mu(y) \leqslant c\mu(B)\omega(m),$$

which together with (4.4) completes the proof of the lemma. □

Corollary 4.6. *Let* $\rho(t)$ *be the function defined in* (3.10). *If* $a(x)$ *is a* (ρ, q) *atom*, $1 < q \leqslant \infty$, *then there exists a constant c, independent of* $a(x)$, *such that*

$$\|a\|_{H_\omega} \leqslant c.$$

Lemma 4.7. *Set* $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$. *Let* $\{b_i\}_i$ *be a sequence of multiples of* (ρ, q) *atoms*, $1 < q \leqslant \infty$, *such that* $\Lambda_q(\{b_i\}) < \infty$ *and* $\alpha_i = \|b_i\|_q\mu(B_i)^{-1/q}/\omega^{-1}(\mu(B_i)^{-1})$, *where* $B_i \supset \operatorname{supp}(b_i)$. *Then there exists a contant c independent of the sequence* $\{b_i\}$ *such that* $\sum_i \alpha_i \leqslant c(\Lambda_q(\{b_i\}) + 1)^{1/l^2}$.

PROOF. By definition of α_i, we get

$$\mu(B_i)^{-1} = \omega(\|b_i\|_q\mu(B_i)^{-1/q}\alpha_i^{-1}). \tag{4.8}$$

We first note that $\{\alpha_i^l\}$ is bounded. In fact, for those i such that $\alpha_i > 1$, since ω is of lower type l with constant c_0, by (4.8), we have

$$\begin{aligned}\alpha_i^l &= \mu(B_i)\alpha_i^l\omega(\|b_i\|_q\mu(B_i)^{-1/q}\alpha_i^{-1})\\ &\leqslant c_0\mu(B_i)\omega(\|b_i\|_q\mu(B_i)^{-1/q})\\ &\leqslant c_0\sum_j \mu(B_j)\omega\left(\|b_j\|_q\mu(B_j)^{-1/q}\frac{(\Lambda_q(\{b_i\})+1)^{1/l}}{(\Lambda_q(\{b_i\})+1)^{1/l}}\right)\\ &\leqslant c_0(\Lambda_q(\{b_i\})+1)^{1/l} = a.\end{aligned}$$

Applying again (4.8) we obtain

$$\sum_i \alpha_i = a^{1/l}\sum_i \mu(B_i)\frac{a_i}{a^{1/l}}\,\omega\left(\frac{\|b_i\|_q\mu(B_i)^{-1/q}}{\alpha_i}\right).$$

Using that $\omega(s)/s$ is non-increasing, this is bounded by

$$a^{1/l}\sum_i \mu(B_i)\omega\left(\frac{\|b_i\|_q\mu(B_i)^{-1/q}}{a^{1/l}}\right) \leqslant a^{1/l}$$

since ω is increasing and $a^l \geqslant \Lambda_q(\{b_j\})$. □

One of the main tools in the proof of atomic decomposition of Hardy spaces is provided by a Calderón-Zigmund type lemma which allows us to split a given function into «good» and «bad» parts. In order to do this, let us take f belonging to H_ω. Consider $\omega(t) > \int \omega[f_\gamma^*(x)]\,d\mu(x)/\mu(X)$ and $\Omega = \{x: f_\gamma^*(x) > t\}$. By a Whitney's type lemma applied to the open set Ω, following [MS], we get a sequence of balls $B_n = B(x_n, r_n)$ and a partition of the unity $\{\phi_n\}$ associated to it. For each n, the expression

$$S_n(\psi)(x) = \phi_n(x)\left[\int \phi_n(z)\,d\mu(z)\right]^{-1}\int [\psi(x)-\psi(x)]\phi_n(z)\,d\mu(z)$$

defines a continuous operator from E^α into itself.

Lemma 4.9. Calderón-Zygmund type. *Let f in H_ω and $b_n(\psi) = f(S_n(\psi))$ for $\psi \subset E^\alpha$. Then*

$$(4.10)\quad (b_n)_\gamma^*(x) \leqslant ct\left[\frac{r_n}{d(x,x_n)+r_n}\right]^{1+\gamma}\chi_{B(x_n,4Kr_n)^c}(x) + cf_\gamma^*(x)\chi_{B(x_n,4Kr_n)}(x)$$

and

$$(4.11)\quad \int \omega[(b_n)_\gamma^*(x)]\,d\mu(x) \leqslant c\int_{B(x_n,4Kr_n)} \omega[f_\gamma^*(x)]\,d\mu(x).$$

Moreover, the series $\sum_n b_n$ *converges in* $(E^\alpha)'$ *to a distribution* b *satisfying*

$$b_\gamma^*(x) \leqslant ct \sum_n \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{1+\gamma} + cf_\gamma^*(x)\chi_\Omega(x) \tag{4.12}$$

and

$$\int \omega[b_\gamma^*(x)]\, d\mu(x) \leqslant c \int_\Omega \omega[f_\gamma^*(x)]\, d\mu(x). \tag{4.13}$$

The distribution $g = f - b$ *satisfies*

$$g_\gamma^*(x) \leqslant ct \sum_n \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{1+\gamma} + cf_\gamma^*(x)\chi_{\Omega^c}(x). \tag{4.14}$$

PROOF. We shall only prove (4.11) and (4.13). To obtain (4.11), we first apply ω to inequality (4.10) and then we integrate on X. Thus, since ω is of lower type l, $l(1 + \gamma) > 1$, by Lemma 4.1 we get

$$\begin{aligned} \int_X \omega[(b_n)_\gamma^*(x)]\, d\mu(x) &\leqslant c\omega(t) \int_{B(x_n, 4Kr_n)^c} \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{(1+\gamma)l} d\mu(x) \\ &\quad + c \int_{B(x_n, 4Kr_n)} \omega[f_\gamma^*(x)]\, d\mu(x) \\ &\leqslant c \int_{B(x_n, 4Kr_n)} \omega[f_\gamma^*(x)]\, d\mu(x). \end{aligned}$$

Applying ω to (4.12), using the sub-additivity of ω and proceeding as above, (4.13) follows. □

The next result which shall be often used in the sequel, is also an statement of the inclusion $H_\phi \subset L_\phi$, where ϕ is a Young function, *i.e.* a convex, positive and increasing function on $\mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Its proof is similar to that of Theorem 3.25 in [MS].

Theorem 4.15. *Let* f *be a distribution on* E^α *and assume that* $f_\gamma^*(x)$ *belongs to* $L_\phi(X, d\mu)$. *Then, there exists a function* $f(x)$ *such that* $|f(x)| \leqslant cf_\gamma^*(x)$ *and*

$$f(\psi) = \int f(x)\psi(x)\, d\mu(x),$$

for every ψ *in* E^α.

Applying the preceding theorem with $\phi(t) = t^q$, $1 \leqslant q < \infty$, we obtain the density of L^q in H_ω.

Theorem 4.16. *Let f be a distribution on E^α belonging to H_ω. Then for $\epsilon > 0$ and $1 \leqslant q < \infty$ given, there exists a function $h(x)$ in $L^q(X, d\mu)$ such that*

$$\int \omega[(f - h)^*_\gamma(x)]\, d\mu(x) < \epsilon.$$

PROOF. There exists $\omega(t) > \mu(X)^{-1} \int \omega[f^*_\gamma(x)]\, d\mu(x)$ such that the inequality

$$\int_\Omega \omega[f^*_\gamma(x)]\, d\mu(x) < \epsilon \tag{4.17}$$

holds for $\Omega = \{x : f^*_\gamma(x) > t\}$. For this value of t, by lemma (4.9), we get the decomposition

$$f = g + b.$$

Moreover, from (4.14), (4.2) and the fact that $\omega(s)/s$ is non-increasing, we have

$$\begin{aligned} \int g^*_\gamma(x)^q\, d\mu(x) &\leqslant ct^q \mu(\Omega) + c \int_{\Omega^c} f^*_\gamma(x)^q\, d\mu(x) \\ &\leqslant c \frac{t^q}{\omega(t)} \int \omega[f^*_\gamma(x)]\, d\mu(x) + ct^{q-1} \int_{\Omega^c} f^*_\gamma(x)\, d\mu(x) \\ &\leqslant c \frac{t^q}{\omega(t)} \int \omega[f^*_\gamma(x)]\, d\mu(x). \end{aligned}$$

Consequently, $g^*_\gamma(x)$ belongs to $L^q(X, d\mu)$. Hence, Theorem 4.15 with $\phi(t) = t^q$ implies that there exists a function $h(x)$ such that $|h(x)| < cg^*_\gamma(x)$ and the distribution on E^α induced by $h(x)$ coincides with g. Therefore $h(x) \in L^q(X, d\mu)$. On the other hand, from (4.13) and (4.17) it follows that

$$\begin{aligned} \int \omega[(f - h)^*_\gamma(x)]\, d\mu(x) &= \int \omega[(f - g)^*_\gamma(x)]\, d\mu(x) = \int \omega[b^*_\gamma(x)]\, d\mu(x) \\ &\leqslant c \int_\Omega \omega[f^*_\gamma(x)]\, d\mu(x) \leqslant c\epsilon. \quad \square \end{aligned}$$

The analogue to Lemma 4.9 in the case that f is a function is contained in the following Lemma.

Lemma 4.18. *Let $f(x) \in L^q(X, d\mu)$, $1 \leqslant q < \infty$. Assume that the distribution f on E^α induced by $f(x)$ belongs to H_ω and $|f(x)| \leqslant cf^*_\gamma(x)$ almost everywhere on X. With the same notation used in Lemma 4.9, let*

$$m_n = \left(\int \phi_n(z)\, d\mu(z) \right)^{-1} \int f(y) \phi_n(y)\, d\mu(y).$$

We have

$$|m_n| \leqslant ct. \tag{4.19}$$

(4.20) *The distribution on* E^α *induced by the function*

$$b_n(x) = (f(x) - m_n)\phi_n(x),$$

coincides with b_n.

(4.21) *The series* $\sum b_n(x)$ *converges for* $x \in X$ *and in* $L^q(X, d\mu)$. *Its sum induces a distribution on* E^α *which coincides with* b *and shall be denoted by* $b(x)$.

(4.22) *The function* $g(x) = f(x) - b(x)$ *satisfies*

$$g(x) = f(x)\chi_{\Omega^c}(x) + \sum m_n\phi_n(x)$$

and

$$|g(x)| \leqslant ct.$$

Moreover, $g(x)$ *induces a distribution on* E^α *which coincides with* g.

We shall need the following lemma, which is a consequence of Lemma 4.7.

Lemma 4.23. *Let* $\rho(t)$ *and* $\{b_i\}_i$ *as in Lemma* 4.7. *Then the series* $\sum_i b_i$ *converges in* $(E^\alpha)'$.

PROOF. Let us first assume that $l < 1$. Let D be a bounded subset of E^α, therefore there exists a ball $B(x_0, R)$ and a constant c such that $R > K_2\mu(\{x_0\})$ and for every $\psi \in D$ we have $\operatorname{supp}\psi \subset B(x_0, R)$, $\|\psi\|_\infty \leqslant c$ and $\|\psi\|_{l^{-1}-1} \leqslant c$. Observe that $\operatorname{Lip}(l^{-1} - 1) \cap D \subset \operatorname{Lip}(\rho)$ and

$$\|\psi\|_{\operatorname{Lip}(\rho)} \leqslant c(D)\|\psi\|_{l^{-1}-1} \leqslant \tilde{c}(D), \tag{4.24}$$

because $\rho(t)$ is of upper type $l^{-1} - 1$. From (4.24) and the definition of $\rho(t)$ we get

$$\begin{aligned}
\sup_{\psi \in D}\left|\sum_m^n b_i, (\psi)\right| &\leqslant \sup_{\psi \in D}\sum_m^n \int_{B_i} |b_i(x)|\,|\psi(x) - \psi(x_i)|\,d\mu(x) \\
&\leqslant c\sup_{\psi \in D}\|\psi\|_{\operatorname{Lip}(\rho)}\sum_m^n \|b_i\|_q\rho(\mu(B_i))\mu(B_i)^{1/q'} \\
&\leqslant c\frac{\sum_m^n \|b_i\|_q\mu(B_i)^{-1/q}}{\omega^{-1}(\mu(B_i)^{-1})} \\
&= c\sum_m^n \alpha_i.
\end{aligned}$$

Applying Lemma 4.7 we obtain the desired result. If $l = 1$, the series $\sum_i b_i$ actually converges in L^1, since $\rho \simeq$ constant. $\square$

In order to prove Theorem 2.1 we shall need the following lemma, which gives a (ρ, ∞) decomposition for a suitable function.

Lemma 4.25. *Let $\rho(t)$ be the function defined by $1/\rho(t) = t\omega^{-1}(1/t)$. Let $h(x)$ be a function in $(L^2 \cap L^\infty)(X, d\mu)$. Suppose that for some p, such that $(1+\gamma)^{-1} < p < l$, $0 < \gamma < \alpha$, the γ-maximal function $h_\gamma^*(x)$ belongs to $L^p(X, d\mu)$. Then there exists a sequence $\{b_n(x)\}$ of multiples of (ρ, ∞) atoms such that*

$$h = \sum_n b_n \quad in \quad E^{\alpha'}$$

and

$$\sum_n \mu(B_n)\omega(\|b_n\|_\infty) \leqslant c\omega(\|h\|_\infty)\|h\|_\infty^{-p} \int h_\gamma^*(x)^p \, d\mu(x), \tag{4.26}$$

where c is a constant independent of $h(x)$.

PROOF. Let ϵ be any number, $0 < \epsilon < 1$. We shall construct, by recurrency a sequence of functions, $\{H_i(x)\}$ in the following way: $H_0(x) = H(x)$. Suppose that $H_{i-1}(x)$ is defined. Then, if

$$\omega(\|h\|_\infty \epsilon^i) \leqslant \int \omega[(H_{i-1})_\gamma^*(x)] \, d\mu(x)/\mu(X),$$

we stop the construction obtaining a finite sequence. If on the contrary

$$\omega(\|h\|_\infty \epsilon^i) > \int \omega[(H_{i-1})_\gamma^*(x)] \, d\mu(x)/\mu(X),$$

we choose $H_i(x)$ to be the function $g(x)$ associated in Lemma 4.18 to $f(x) = H_{i-1}(x)$ and $t = \|h\|_\infty \epsilon^i$. Thus, for those values of $i \geqslant 1$ for which $H_i(x)$ is defined, we have

$$H_i(x) = H_{i-1}(x) - B_i(x) = H_{i-1}(x) - \sum_n b_{i,n}(x), \tag{4.27}$$

and, by (4.22),

$$|H_i(x)| \leqslant c\|h\|_\infty \epsilon^i. \tag{4.28}$$

Moreover, using (4.12), (4.27), (4.28) and proceeding by induction, it can be proved that

$$(H_i)_\gamma^*(x) \leqslant h_\gamma^*(x) + c \sum_{j=1}^{i} \|h\|_\infty \epsilon^j \sum_n \left[\frac{r_{j,n}}{d(x, x_{j,n}) + r_{j,n}} \right]^{1+\gamma}. \tag{4.29}$$

First, we shall study the case when the sequence $\{H_i(x)\}$ is infinite, this is the case if $\mu(X) = \infty$. From (4.27) it follows that

$$h(x) = H_i(x) + \sum_{j=1}^{i} \sum_n b_{j,n}(x).$$

Using (4.28) we obtain that

$$h = \sum_{j=1}^{\infty} \sum_n b_{j,n} \quad \text{in} \quad E^{\alpha'}.$$

Now by (4.19), (4.20) and (4.28) with $i = j - 1$ we have that $b_{j,n}$ are multiples of (ρ, ∞) atoms with $\operatorname{supp}(b_{j,n}) \subset B(x_{j,n}, r_{j,n}) = B_{j,n}$ and

$$\|b_{j,n}\|_\infty \leqslant c\|h\|_\infty \epsilon^{j-1}. \tag{4.30}$$

Let us prove (4.26). Denoting

$$\Omega_j = \{x \in X: (H_{j-1})_\gamma^*(x) > \|h\|_\infty \epsilon^j\},$$

from (4.30) and the fact that $\omega(s)$ is of lower type l, we get

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_n \mu(B_{j,n})\omega(\|b_{j,n}\|_\infty) &\leqslant \sum_{j=1}^{\infty} \omega(c\|h\|_\infty \epsilon^{j-1}) \sum_n \mu(B_{j,n}) \\ &\leqslant \frac{c}{\epsilon} \sum_{j=1}^{\infty} \epsilon^{j(l-p)} \epsilon^{jp} \omega(\|h\|_\infty)\mu(\Omega_j). \end{aligned} \tag{4.31}$$

On the other hand, applying Lemma 4.1 and (4.29), we have

$$\mu(\Omega_j)\epsilon^{jp}\|h\|_\infty^p \leqslant \int (H_{j-1})_\gamma^*(x)^p \, d\mu(x) \leqslant (c+2)^j \int h_\gamma^*(x)^p \, d\mu(x). \tag{4.32}$$

So that (4.31) is bounded by

$$\frac{c}{\epsilon} \sum_{j=1}^{\infty} [\epsilon^{l-p}(c+2)]^j \omega(\|h\|_\infty)\|h\|_\infty^{-p} \int h_\gamma^*(x)^p \, d\mu(x).$$

Choosing ϵ small enough, (4.26) holds and the proof of the lemma finishes for the case when $\{H_i(x)\}$ is an infinite sequence.

Assume now that the sequence $\{H_j(x)\}$ is finite, in this case $\mu(X) < \infty$, and we can suppose without loss of generality that $\mu(X) = 1$. Let $H_m(x)$ be the last function of the sequence, thus

$$\omega(\|h\|_\infty \epsilon^{m+1}) \leqslant \int \omega[(H_m)_\gamma^*(x)] \, d\mu(x). \tag{4.33}$$

Moreover, for $j \leqslant m$, the function $H_j(x)$ satisfies (4.27) through (4.29).

Therefore, as before, we get that

$$h(x) = H_m(x) + \sum_{j=1}^{m} \sum_n b_{j,n}(x), \tag{4.34}$$

where $b_{j,n}$ are multiples of (ρ, ∞) atoms for every $j \leqslant m$. Let

$$b_{m+1,1}(x) = \int H_m(y)\, d\mu(y) \chi_X(x)$$

and

$$b_{m+1,2}(x) = H_m(x) - \int H_m(y)\, d\mu(y).$$

These functions have their supports contained in

$$X = B(x_0, 2K_1) \quad \text{and} \quad \int b_{m+1,2}\, d\mu(x) = 0.$$

Therefore, both functions are multiples of (ρ, ∞) atoms. Hence, by (4.34)

$$h(x) = \sum_{j=1}^{m+1} \sum_n b_{j,n}(x).$$

In order to prove (4.26), we first observe that by (4.28) and (4.33)

$$\begin{aligned} \omega(\|b_{m+1,1}\|_\infty) + \omega(\|b_{m+1,2}\|_\infty) &\leqslant \omega(c\|h\|_\infty \epsilon^m) \\ &\leqslant c\epsilon^{-1} \int \omega[(H_m)^*_\gamma(x)]\, d\mu(x). \end{aligned} \tag{4.35}$$

In view of (3.3) and (4.28), it follows that (4.35) is bounded by

$$\begin{aligned} &\frac{c}{\epsilon} \frac{\omega(\|h\|_\infty \epsilon^m)}{(\|h\|_\infty \epsilon^m)^p} \int (H_m)^*_\gamma(x)^p\, d\mu(x) \\ &\qquad \leqslant c\epsilon^{-1} \epsilon^{(l-p)m} \omega(\|h\|_\infty) \|h\|_\infty^{-p} \int (H_m)^*_\gamma(x)^p\, d\mu(x). \end{aligned}$$

On the other hand, as in (4.32) we get

$$\int (H_m)^*_\gamma(x)^p\, d\mu(x) \leqslant (c+2)^m \int h^*_\gamma(x)^p\, d\mu(x).$$

Then, (4.35) is less than or equal to

$$\epsilon^{-1} c[\epsilon^{l-p}(c+2)]^m \omega(\|h\|_\infty) \|h\|_\infty^{-p} \int h^*_\gamma(x)^p\, d\mu(x).$$

Thus, arguing as in the case of an infinite sequence, we have

$$\sum_{j=1}^{m+1} \sum_n \mu(B_{j,n}) \omega(\|b_{j,n}\|_\infty) \leqslant c\omega(\|h\|_\infty) \|h\|_\infty^{-p} \int h^*_\gamma(x)^p\, d\mu(x). \qquad \square$$

We are in position to prove Theorem 2.1. The inclusion of $H^{\rho,q}$ in H_ω is, as we shall see, an immediate consequence of Lemma 4.3. The proof of the converse is based on the Calderón-Zygmund type Lemmas 4.9 and 4.18, which are applied to obtain the atomic decomposition of a distribution belonging to H_ω.

PROOF OF THEOREM 2.1. *First inclusion*: $H^{\rho,q} \subset H_\omega$. Let f be a distribution in $H^{\rho,q}$. So, by (1.24), for every $\epsilon > 0$ there exists a sequence $\{b_i(x)\}_i$ of multiples of (ρ, q) atoms such that $f = \sum_i b_i$ in $(E^\alpha)'$ and

$$(1 + \epsilon)\|f\|_{H^{\rho,q}} > \Lambda_q(\{b_i\}). \tag{4.36}$$

On the other hand, let η be a positive real constant. Then, by Lemma 4.3, we obtain

$$\begin{aligned}\int \omega\left[\frac{f_\gamma^*(x)}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right] d\mu(x) &\leqslant \sum_j \int \omega\left[\frac{(b_j)_\gamma^*(x)}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right] d\mu(x) \\ &\leqslant \sum_j \mu(B_j)\omega\left[\frac{c^{1/l}\|b_j\|_q \mu(B_j)^{-1/q}}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right].\end{aligned}$$

Taking $\eta = c$, by (1.23) and (1.19) we get

$$\|f\|_{H_\omega} \leqslant c\Lambda_q(\{b_i\}),$$

which by (4.36), is bounded by $c\|f\|_{H^{\rho,q}}$, as we wanted to prove.

Second inclusion: $H_\omega \subset H^{\rho,q}$. Since $H^{\rho,\infty}$ is continuously included in $H^{\rho,q}$, $1 < q < \infty$, it is enough to show $H_\omega \subset H^{\rho,\infty}$. Assume that $\mu(X) = \infty$, the case $\mu(X) < \infty$ follows the same lines. Given $f \in H_\omega$, we shall prove that there exists a sequence $\{b_n(x)\}$, of multiples of (ρ, ∞) atoms satisfying

$$f = \sum_n b_n \tag{4.37}$$

in the sense of $(E^\alpha)'$, and

$$\|f\|_{H^{\rho,\infty}} \leqslant c\|f\|_{H_\omega}, \tag{4.38}$$

where c is a constant independent of f.

We first assume that f is a distribution in H_ω such that $f_\gamma^*(x)$ belongs to $L^2(X, d\mu)$. Thus, by Theorem 4.15, f can be represented in $(E^\alpha)'$ by a function $f(x)$ belonging to $L^2(X, d\mu)$ satisfying $|f(x)| \leqslant cf_\gamma^*(x)$. For $k \in \mathbb{Z}$, let us consider $\Omega_k = \{x: f_\gamma^*(x) > 2^k\}$. Applying Lemma 4.18 with $t = 2^k$ and $t = 2^{k+1}$, we obtain

$$f(x) = B_k(x) + G_k(x) = B_{k+1}(x) + G_{k+1}(x).$$

So, we can write

$$G_{k+1}(x) - G_k(x) = B_k(x) - B_{k+1}(x) = H_k(x). \tag{4.39}$$

Then, from (4.22) we have

$$|H_k(x)| \leqslant c2^k. \tag{4.40}$$

Therefore, the inequality

$$(H_k)^*_\gamma(x) \leqslant c2^k \sum_{j=k}^{k+1} \sum_i [r_{j,i}/(d(x, x_{j,i}) + r_{j,i})]^{1+\gamma}, \tag{4.41}$$

follows from (4.12) if $x \notin \Omega_k$ and from (4.40) if $x \in \Omega_k$. Consequently, by Lemma 4.1, for any p satisfying $(1+\gamma)^{-1} < p \leqslant 1$, we obtain

$$\int (H_k)^*_\gamma(x)^p \, d\mu(x) \leqslant c2^{kp}\mu(\Omega_k). \tag{4.42}$$

Let us see that $\sum_{k\in\mathbb{Z}} H_k$ converges to f in $(E^\alpha)'$. In fact, by (4.39) we have

$$f - \sum_{k=-n}^{n} H_k = f - G_{n+1} + G_{-n} = B_{n+1} + G_{-n}.$$

From (4.13) it follows that

$$\int \omega[(B_{n+1})^*_\gamma(x)] \, d\mu(x) \leqslant c \int_{\Omega_{n+1}} \omega[f^*_\gamma(x)] \, d\mu(x).$$

Thus, B_n converges to zero in H_ω when n tends to infinity and consequently, B_n converges to zero in $(E^\alpha)'$. On the other hand, since by (4.22) $|G_{-n}(x)| \leqslant c2^{-n}$, G_{-n} converges to zero in $(E^\alpha)'$ as n tends to infinity. Then,

$$f = \sum_k H_k, \quad \text{in} \quad (E^\alpha)'. \tag{4.43}$$

Let us now observe that H_k satisfies the hypothesis of Lemma 4.25. Since $f(x)$ belongs to $L^2(X, d\mu)$, $H_k(x)$ is in $L^2(X, d\mu)$ and by (4.40), $H_k(x)$ also belongs to $L^\infty(X, d\mu)$. Furthermore, from (4.42), $(H_k)^*_\gamma(x)$ is in $L^p(X, d\mu)$, for any p satisfying $(1+\gamma)^{-1} < p < l$. Then, Lemma 4.25 implies that there exists a sequence $\{b_i^k\}_i$ of multiples of (ρ, ∞) atoms such that

$$H_k = \sum_i b_i^k \quad \text{in} \quad (E^\alpha)'$$

and

$$\sum_i \mu(B_i^k)\omega(\|b_i^k\|_\infty) \leqslant c\omega(\|H_k\|_\infty)\|H_k\|_\infty^{-p} \int (H_k)^*_\gamma(x)^p \, d\mu(x).$$

Therefore, using (3.3), (4.40) and (4.42), we get

$$\sum_i \mu(B_i^k)\omega(\|b_i^k\|_\infty) \leqslant c\omega(c2^k)(c2^k)^{-p} \int (H_k)_\gamma^*(x)^p \, d\mu(x) \leqslant \omega(c2^k)\mu(\Omega_k). \tag{4.44}$$

On the other hand, from (4.43), we have

$$f = \sum_k \Big(\sum_i b_i^k\Big). \tag{4.45}$$

Let $\eta \geqslant 1$ be a constant to be determined later, and denote $\lambda = \eta \|f\|_{H_\omega}$. We now estimate the sum

$$\sum_{k\in\mathbb{Z}} \sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{\lambda^{1/l}}\right).$$

By (4.44) applied to $\lambda^{-1/l}H_k$ this sum is bounded by

$$\sum_{k\in\mathbb{Z}} \mu(\Omega_k)\omega\left(\frac{c2^k}{\lambda^{1/l}}\right) = \sum_{k\in\mathbb{Z}} \omega\left(\frac{c2^k}{\lambda^{1/l}}\right) \int_{\{x: f_\gamma^*(x) > 2^k\}} d\mu(x) \leqslant \int_X \sum_{k < \log_2(f_\gamma^*(x))} \omega\left(\frac{c2^k}{\lambda^{1/l}}\right),$$

applying that $\omega(s)/s$ is non increasing this is bounded by

$$\leqslant 2\int_X \left[\sum_{k<\log_2(f_\gamma^*(x))} \int_{c\lambda^{-1/l}2^{k-1}}^{c\lambda^{-1/l}2^k} \frac{\omega(s)}{s}\, ds\right] d\mu(x) \leqslant 2\int_X \left[\int_0^{c\lambda^{-1/l}f_\gamma^*(x)} \frac{\omega(s)}{s}\, ds\right] d\mu(x),$$

which by (3.4) is less than or equal to

$$\int_X \omega\left[\frac{cf_\gamma^*(x)}{\lambda^{1/l}}\right] d\mu(x).$$

Choosing $\eta = c^l$, we get

$$\sum_{k\in\mathbb{Z}} \sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{c\|f\|_{H_\omega}^{1/l}}\right) \leqslant \int_X \omega\left(\frac{f_\gamma^*(x)}{\|f\|_{H_\omega}^{1/l}}\right) d\mu(x) \leqslant 1.$$

This proves that $\Lambda_\infty(\{b_i^k\}_{i,k}) \leqslant c\|f\|_{H_\omega}$. Applying Lemma 4.23 and using (4.45), the Theorem follows under the assumption that $f_\gamma^*(x)$ belongs to $L^2(X, d\mu)$. Next, we shall remove that assumption. Let f be a distribution in

H_ω. In view of Theorem 4.16, we have that, for any positive integer k, there exists a function $f_k(x)$ in $L^2(X, d\mu)$ such that

$$\|f - f_k\|_{H_\omega} \leqslant 2^{-k}\|f\|_{H_\omega}. \tag{4.46}$$

Defining $f_0(x) = 0$, since H_ω is immerse in $(E^\alpha)'$, we get that

$$f = \sum_{k=1} f_k - f_{k-1},$$

in $(E^\alpha)'$. On the other hand, since $f_k - f_{k-1}$ is a distribution in H_ω satisfying that $(f_k - f_{k-1})^*_\gamma(x)$ belongs to $L^2(X, d\mu)$, we have an atomic decomposition for $f_k - f_{k-1}$, *i.e.* $f_k - f_{k-1} = \sum_i b_i^k$, where $\{b_i^k\}_i$ is a sequence of multiples of (ρ, ∞) atoms such that

$$\Lambda_\infty(\{b_i^k\}_i) \leqslant (1 + \epsilon)\|f_k - f_{k-1}\|_{H^{\rho,\infty}} \leqslant c\|f_k - f_{k-1}\|_{H_\omega}$$

for every $\epsilon > 0$. Therefore, by (4.73),

$$\Lambda_\infty(\{b_i^k\}_i) \leqslant c2^{-k}\|f\|_{H_\omega}.$$

Thus

$$\begin{aligned}\sum_{k=1}^\infty \sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{(c\|f\|_{H_\omega})^{1/l}}\right) &\leqslant \sum_{k=1}^\infty \sum_i \mu(B_i^k)2^{-k}\omega\left(\frac{\|b_i^k\|_\infty}{(2^{-k}c\|f\|_{H_\omega})^{1/l}}\right)\\ &\leqslant \sum_{k=1}^\infty 2^{-k}\sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{\Lambda_\infty(\{b_i^k\}_i)^{1/l}}\right)\\ &\leqslant \sum_{k=1}^\infty 2^{-k} = 1.\end{aligned}$$

This implies that

$$f = \sum_{k=1}^\infty \sum_i b_i^k,$$

in $(E^\alpha)'$ and $\|f\|_{H^{\rho,\infty}} \leqslant c\|f\|_{H_\omega}$, as we wanted to prove. □

Remark. Observe that the statement of the theorem implies in particular that all the spaces $H^{\rho,q}$ are equivalent, for $1 < q \leqslant \infty$. In fact the original proof of the first inclusion was obtained by proving directly this equivalence and using the inclusion $H^{\rho,\infty} \subset H_\omega$ proved by applying Lemma 4.3 restricted to $q = \infty$. This path is longer than the approach presented here. I want to thank the referee for suggesting this shortcut based on the extension of Lemma 4.3 to the general case $1 < q \leqslant \infty$.

5. Dual Spaces

In this section we use the atomic decomposition obtained in Theorem 2.1 to show that BMO (ρ) is the dual space of H_ω. Let us point out that as by product of the proof of this characterization we get that BMO (ρ) coincides with BMO (ρ, q) for $1 < q < \infty$.

We shall work, as before, on a normal space $(X, d\mu)$ of order α with the additional assumption that μ is a regular measure. It is well known the density of Lip (β), $0 < \beta < \alpha$ in $L^p(X, d\mu)$, $1 \leqslant p < \infty$ (see [MS]). Consequently, if g belongs to BMO (ρ), for every ball B and $\epsilon > 0$ there exists a bounded continuous function h satisfying

$$(5.1) \qquad \int_B |g(x) - h(x)|\, d\mu(x) < \epsilon.$$

We shall denote by $g_t(x)$ the function defined by

$$(5.3) \qquad g_t(x) = \int \phi(x, y, t) g(y)\, d\mu(y),$$

where $\phi(x, y, t)$ is the function constructed in Lemma 3.15 of [MS]. In the proof of Theorem 2.2 we shall need the following two lemmas.

Lemma 5.2. *Let g belongs to* BMO (ρ). *Then*

$$(5.3) \qquad \|g_t\|_{\mathrm{BMO}(\rho)} \leqslant c\|g\|_{\mathrm{BMO}(\rho)}$$

and for every ball B

$$(5.4) \qquad \lim_{t \to 0} \int_B |g_t(x) - g(x)|\, d\mu(x) = 0,$$

Proof. The proof is similar to Lemma 5.3 in [MS] and it makes use of remark 5.1. □

Lemma 5.5. *Let $\{b_i\}_i$ be as in Lemma* 4.7 *with $q = \infty$ and such that $\sum b_i$ converges to zero in $(E^\alpha)'$. Then $\sum b_i$ converges to zero in* BMO (ρ).

Proof. Following the same argument given in Theorem 5.9 of [MS], it is easy to see that we only need to prove the convergence for functions g in BMO (ρ) with bounded support and non-negative. In fact, by Lemma 4.7, for any $\epsilon > 0$, there exists N such that $\sum_{i>N} \alpha_i < \epsilon$. Let $B = B(x_0, r)$ be a ball containing the support of $b_i(x)$ for every $1 \leqslant i \leqslant N$. Now, if g is as above, g_t is in E^α for $0 < t \leqslant 1$, so that $\sum b_i(g_t) = 0$. Thus,

$$\sum_i b_i(g) = \sum_i b_i(g - g_t) = \sum_{i \leq N} b_i(g - g_t) + \sum_{i > N} b_i(g - g_t).$$

By (1.16) and (1.11), we get

$$\left|\sum_{i>N} b_i(g - g_t)\right| = \left|\sum_{i>N} \int b_i(x)[g(x) - g_t(x) - m_{B_i}(g - g_t)]\, d\mu\,(x)\right|$$
$$\leqslant \|g - g_t\|_{\mathrm{BMO}\,(\rho)} \sum_{i>N} \mu(B_i)\rho(\mu(B_i))\,\|b_i\|_\infty .$$

Using (5.3) and the definition of $\rho(t)$, we have

$$\left|\sum_{i>N} b_i(g - g_t)\right| \leqslant c\|g\|_{\mathrm{BMO}\,(\rho)} \sum_{i>N} \alpha_i < c\|g\|_{\mathrm{BMO}\,(\rho)}\epsilon.$$

On the other hand, we obtain

$$\left|\sum_{i\leq N} b_i(g - g_t)\right| \leqslant \sum_{i\leq N} \|b_i\|_\infty \int_B |g(x) - g_t(x)|\, d\mu\,(x),$$

which by (5.4), tends to zero with t. This proves the lemma. □

PROOF OF THEOREM 2.2. *First inclusion*: BMO (ρ) $\subset (H_\omega)'$. Let f be a distribution in H_ω with $\|f\|_{H_\omega} \leqslant 1$, represented by $f = \sum b_i$, where b_i are multiples of (ρ, ∞) atoms. Given g in BMO (ρ), we first prove that the series $\sum b_i(g)$ is absolutely convergent. By (1.16), (1.11) and Lemma 4.7, we get

$$\sum_i |b_i(g)| \leqslant \|g\|_{\mathrm{BMO}\,(\rho)} \sum_i \alpha_i \leqslant c\|g\|_{\mathrm{BMO}\,(\rho)}. \tag{5.6}$$

On the other hand, by Lemma 5.5, $\sum b_i(g)$ is independent of the representation of f. Therefore, we obtain that the linear functional

$$L_g(f) = \sum_i b_i(g)$$

is well defined and from (5.6) it satisfies

$$|L_g(f)| \leqslant c\|g\|_{\mathrm{BMO}\,(\rho)}\|f\|^{1/l}.$$

Second inclusion: $(H_\omega)' \subset$ BMO (ρ). By Hölder inequality, we have that BMO (ρ, q) $\subset$ BMO (ρ), for every $1 < q < \infty$. Thus, it is enough to prove that $(H_\omega)' \subset$ BMO (ρ, q). Let us only consider the case $\mu(X) = \infty$. Let $1 < q < \infty$ and $q' = q/(q-1)$. Given a ball B, we define

$$L_0^q(B) = \left\{f \in L^q(B) \colon \int_B f = 0\right\}.$$

Therefore, if f is in $L_0^q(B)$, then

$$b(x) = \frac{f(x)\mu(B)^{1/q}}{\|f\|_q \mu(B)\rho(\mu(B))} = \frac{f(x)}{\|f\|_q \mu(B)^{1/q'}\rho(\mu(B))}$$

is a (ρ, q) atom. Let L be an element of $(H_\omega)'$. Hence, for f in $L_0^q(B)$, we have that $L(f)$ is defined and

$$|L(f)| \leqslant \|L\|_{(H_\omega)'} \mu(B)^{1/q'} \rho(\mu(B)) \|f\|_q. \tag{5.3}$$

Consequently, L is a bounded linear functional on $L_0^q(B)$. Applying the Hahn-Banach Theorem L can be extended to $L^q(B)$ with the same norm. By the Riesz's Representation Theorem there exists a $h \in L^{q'}(B)$ such that

$$L(f) = \int_B f(x)h(x)\, d\mu(x), \tag{5.4}$$

for every f in $L^q(B)$. It is easy to check that h is determined in $L^{q'}(B)$ up to constants. Let $C(B)$ be the space of constant functions on B. Then, there exists g in $L^{q'}(B)/C(B)$ such that (5.4) holds for every $h \in g$ and every f in $L_0^q(B)$. Consider now an increasing sequence of balls, $\{B_k\}_{k=1}^\infty$, such that $\bigcup_k B_k = X$ and denote by T_k the operator which takes a function on X and restrict it to B_k. Thus, if $g_{k+1} \in L^{q'}(B_{k+1})/C(B_{k+1})$ and for every $h \in g_{k+1}$, (5.4) holds with $B = B_{k+1}$, then

$$L(f) = \int_{B_k} hf\, d\mu,$$

for every f in $L_0^q(B_k)$. Consequently $T_k(g_{k+1}) = g_k$. Therefore, there exists g in $L_{\text{loc}}^q(X)/C(X)$ such that $T_k(g) = g_k$, for every k. Let L_g be the operator defined on atoms by

$$L_g(a) = \int_X h(x)a(x)\, d\mu(x),$$

for every $h \in g$. From Lemma 4.7 we see that L_g is continuous on the space spanned by the set of atoms. So that there is only one continuous extension of L_g to the space $H^{\rho,q}$. Clearly $L_g \equiv L$.

It remains to show that every $h \in g$ belongs to BMO (ρ, q'). Let B be a ball, then, using (5.3) we have

$$\begin{aligned}
\|(g - m_B(g))\chi_B\|_{q'} &= \sup_{\|f\chi_B\|_q = 1} \left| \int_B (g - m_B(g)) f\, d\mu \right| \\
&= \sup_{\|f\chi_B\|_q = 1} \left| \int_B (g - m_B(g))(f - m_B(f))\, d\mu \right| \\
&= \sup_{\|f\chi_B\|_q = 1} |L[(f - m_B(f))\chi_B]| \\
&\leqslant \sup_{\|f\chi_B\|_q = 1} \|L\|_{(H_\omega)'} \mu(B)^{1/q'} \rho(\mu(B)) \|(f - m_B(f))\chi_B\|_q \\
&\leqslant 2\|L\|_{(H_\omega)'} \mu(B)^{1/q'} \rho(\mu(B)).
\end{aligned}$$

Consequently,

$$\|g\|_{\mathrm{BMO}(\rho, q')} \leqslant 2\|L\|_{(H_\omega)'}. \quad \square$$

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