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Eigenvalue Problems of Quasilinear Elliptic Systems on $\mathbb{R}^n$

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Abstract

In this paper, we get the existence results of the nontrivial weak solution (λ, u) of the following eigenvalue problem of quasilinear elliptic systems

$$-D_\alpha(a_{\alpha\beta}(x, u)D_\beta u^i) + \frac{1}{2}D_{u^i}a_{\alpha\beta}(x, u)D_\alpha u^j D_\beta u^j + h(x)u^i = \lambda|u|^{p-2}u^i,$$

for $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant N$ and

$$u = (u^1, u^2, \dots, u^N) \in E = \{v = (v^1, v^2, \dots, v^N) \mid v^i \in H^1(\mathbb{R}^n), 1 \leqslant i \leqslant N\},$$

where $a_{\alpha\beta}(x, u)$ satisfy the natural growth conditions. It seems that this kind of problem has never been dealt with before.

1. Introduction

We consider eigenvalue problems of the following quasilinear elliptic systems on $\mathbb{R}^n$

$$(1.1)\quad -D_\alpha(a_{\alpha\beta}(x, u)D_\beta u^i) + \frac{1}{2}D_{u^i}a_{\alpha\beta}(x, u)D_\alpha u^j D_\beta u^j + h(x)u^i = \lambda|u|^{p-2}u^i,$$

for $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant N$ and

$$u = (u^1, u^2, \ldots, u^N) \in E = \{v = (v^1, v^2, \ldots, v^N) \mid v^i \in H^1(\mathbb{R}^N), 1 \leqslant i \leqslant N\}$$

where $R < p < 2\hat{n}/(\hat{n}-2)$, $\hat{n} = n$ if $n > 2$, $2\hat{n}/(\hat{n}-2)$ is any positive number lager than 2 if $n \leqslant 2$,

$$D_\alpha = \frac{\partial}{\partial x_\alpha}, \quad D_{u^i} = \frac{\partial}{\partial u^i} \qquad (1 \leqslant \alpha \leqslant n, \quad 1 \leqslant i \leqslant N)$$

and the summation conventions have been used and will be used in the following, *i.e.* the repeated Greek letters and Latin letters denote the sum from 1 to n and 1 to N respectively.

Problem (1.1) comes from the theory of harmonic mappings. There have been some results of (1.1) in bounded domains ([1], [2]). In [1], the existence of solutions for (1.1) is discussed under the conditions

$$\mu_1|\xi|^2 \leqslant a_{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \leqslant \mu_2|\xi|^2 \qquad \mu_1, \mu_2 > 0$$
$$\lim_{u \to +\infty} uD_u a_{\alpha\beta}(x,u) = 0$$

for every $(u, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n$, $x \in \Omega \subset \mathbb{R}^n$, where $N = 1$, $p = 2n/(n-2)$, $n > 2$ if $n > 2$. In [2] the existence theorem is obtained when $N \geqslant 1$, $h \equiv 0$, $2 < p < 2n/(n-2)$, $n > 2$ under the conditions

$$\begin{cases} a_1|\xi|^2 \leqslant \sigma(|u|)|\xi|^2 \leqslant a_{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \leqslant a_2\sigma(|u|)|\xi|^2 \\ |u^i D_{u^i} a_{\alpha\beta}(x,u)| \leqslant C\sigma(|u|) \\ |D_{u^i} a_{\alpha\beta}(x,u)| \leqslant C\sigma(|u|), \qquad |D_{u^i} a_{\alpha\beta}(x,u)| \leqslant \eta(|u|) \\ -\dfrac{u^i}{2} D_{u^i} a_{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \leqslant a_3 a_{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \qquad (0 < a_3 < 1), \end{cases}$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^n$, where $\sigma(t)$, $\eta(t)$ are nonnegative continuous functions on $[0, +\infty)$ satisfying that for any $c_1 > 1$, there exists c_2, such that $\sigma(c_1 t) \leqslant c_2\sigma(t)$ for all $t \geqslant 0$.

However, there have not been any results for (1.1) in the unbounded domain $\mathbb{R}^n$. Formally, if the minimum of the functional

$$I(u) = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx \tag{1.2}$$

over the set $\left\{u \in E \,\middle|\, \int_{\mathbb{R}^n} |u|^p\, dx = \mu\right\}$ $(\mu > 0)$ were achieved by some u, there should be a $\lambda \in \mathbb{R}^1$ such that (λ, u) solves (1.1) in a weak sense. But there are some difficulties in dealing with the functional $I(u)$. Firstly, because of the unboundedness of $\mathbb{R}^n$, the Sobolev embedding is not compact and the standard convex-compactness techniques can not be used, at least in a straightforward way as in the case of bounded domains, and this makes the problem of the existence of a minimizer more difficult. Secondly, the space where I is differentiable is $L_\infty \cap E$ (see [3]), so even if we had found a minimizer $u \in E$ of I, we could not conclude the existence of (λ, u) $\mathbb{R}^1 \times E$ solving (1.1), unless we had known that $u \in L_\infty$. But, usually, the fact that $\|u\|_\infty$ is finite is obtained because u satisfies the related Euler equation which in turn is a consequence of the differentiability of I at u. This makes the problem complicated.

To overcome the first difficulty, we use the concentration compactness principle, recently developed by P. L. Lions ([4], [5]), when treating the constrained variational problems in unbounded domains. To overcome the second difficulty, we first show that, for any minimizer u of I and some $\varphi \in E$,

$$\frac{d}{dt} I(u + t\varphi)\bigg|_{t=0} = 0$$

i.e. the Euler equation related to the functional I holds in a weak sense for u over special test functions in E. We then use the Nash-Moser methods to show that $\|u\|_\infty$ is finite and finally we get the existence of a nontrivial solution (λ, u) of (1.1).

2. Main Results

In this section, we present the main results of this paper. First of all, we give some notations and conditions.

Let $H^1(\mathbb{R}^n)$ be the usual Sobolev space, $N \geqslant 1$ be a natural number and $E = \{u = (u^1, u^2, \ldots, u^N) \mid u^i \in H^1(\mathbb{R}^n), 1 \leqslant i \leqslant N\}$. The scalar product of $u, v \in E$ is defined by

$$\langle u, v\rangle = \int_{\mathbb{R}^n} [D_\alpha u^i D_\alpha v^i + u^i v^i]\, dx$$

and $(E, <, >)$ is a Hilbert space, the norm of $u \in E$ is $\|u\|_E = (\||Du|\|_2^2 + \|u\|_2^2)^{1/2}$ where hereafter $\|f\|_q$ denotes the $L^q(\mathbb{R}^n)$ norm of the function f and $|f|$ denotes the Euclidean norm of the function f (possibly vector valued). For simplicity, we denote $\|u\|_E$ by $\|u\|$ for $u \in E$.

The main conditions imposed on (1.1) will be the following

(i) $2 < p < 2\hat{n}/(\hat{n} - 2)$ where $\hat{n} = n$ if $n > 2$; and $2\hat{n}/(\hat{n} - 2)$ is any positive number larger than 2 if $n \leqslant 2$.

(ii) $a_{\alpha\beta}(x, u) \in C^1(\mathbb{R}^n \times \mathbb{R}^N)$, $a_{\alpha\beta} = a_{\beta\alpha}$ for any α, β and $a_1 > 0$, $a_2 > 1$ such that for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$

$$(2.1) \qquad a_1|\xi|^2 \leqslant \sigma(|u|)|\xi|^2 \leqslant a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \leqslant a_2\sigma(|u|)|\xi|^2$$

holds, where $\sigma(t)$ is a nonnegative nondecreasing continuous function on $[0, +\infty)$ satisfying: for any $l > 1$, there exists $C_l > 0$, such that

$$\sigma(lt) \leqslant C_l \sigma(t), \qquad \text{for all} \quad t \geqslant 0 \tag{2.2}$$

and C_l are bounded whenever l are bounded. Moreover, there is a constant $C > 0$ with

$$\sigma(t) \leqslant C(1 + |t|^q) \tag{2.3}$$

where $0 \leqslant q \leqslant 4/(n-2)$ if $n > 2$ and $0 \leqslant q$ if $n \leqslant 2$.

(iii) $a_{\alpha\beta}(x, u) \to \bar{a}_{\alpha\beta}(u)$ as $|x| \to +\infty$ uniformly for u bounded.

(iv) There exists, $s \geqslant 0$, $s < p - 2$ such that

$$a_{\alpha\beta}(x, \lambda u)\xi_\alpha \xi_\beta \leqslant \lambda^s a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \tag{2.4}$$

$$a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant \bar{a}_{\alpha\beta}(u)\xi_\alpha \xi_\beta \tag{2.5}$$

for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$, where p is given in (i) and $\bar{a}_{\alpha\beta}$ are defined in (iii), and $\lambda > 1$ is arbitrary.

(v) $h \in C(\mathbb{R}^n)$ and there are $\bar{h}, c > 0$ such that $h(x) \geqslant c, h(x) \leqslant \bar{h}$ for any $x \in \mathbb{R}^n$ and $\lim_{|x| \to \infty} h(x) = \bar{h}$.

(vi) There is a constant $c > 0$ such that

$$|u^i D_{u^i} a_{\alpha\beta}(x, u)| \leqslant c\sigma(|u|) \tag{2.6}$$

$$|D_{u^i} a_{\alpha\beta}(x, u)| \leqslant c\eta(|u|) \tag{2.7}$$

for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^N$, where $\eta(t)$ is a nonnegative nondecreasing continuous function on $[0, +\infty)$ and $\sigma(t)$ is given in (ii).

(vii) There is a constant a_3 with $0 < a_3 < 1$ such that

$$-\tfrac{1}{2} u^i D_{u^i} a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant a_3 a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \tag{2.8}$$

for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$.

Remark 2.1. If $a_{\alpha\beta}(x, u), h(x)$ satisfy (i)-(vii), then $\bar{a}_{\alpha\beta}(u)$, $\bar{h}$ satisfy (i)-(vii).

If $a_{\alpha\beta}(x, u), h(x)$ satisfy (i)-(v), we set, for any $u \in E$

$$I(u) = \int_{\mathbb{R}^n} (a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)|u|^2)\, dx \tag{2.9}$$

$$I^\infty(u) = \int_{\mathbb{R}^n} (\bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h}|u|^2)\, dx \tag{2.10}$$

For any $\lambda > 0$, we set

$$I_\lambda = \inf\left\{ I(u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\, dx = \lambda \right\} \tag{2.11}$$

$$I_\lambda^\infty = \inf\left\{ I^\infty(u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\, dx = \lambda \right\} \tag{2.12}$$

It is clear that

$$I_\lambda = \inf\left\{I(\lambda^{1/p}u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\,dx = 1\right\} \tag{2.13}$$

$$I_\lambda^\infty = \inf\left\{I^\infty(\lambda^{1/p}u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\,dx = 1\right\} \tag{2.14}$$

The pair $(\lambda, u) \in \mathbb{R}^1 \times E$ will be called a weak solution of (1.1) if

$$\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta \varphi^i + \varphi^i D_{u^j} a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + h(x)u^i\varphi^i]\,dx = \lambda \int_{\mathbb{R}^n} |u|^{p-2}u^i\varphi^i\,dx$$

for any $\varphi \in L_\infty \cap E$.

It is evident that $u = 0$ is a trivial solution of (1.1) for any λ.

The main results of this paper are the following

Theorem 2.1. *Suppose that* (i)-(vi) *hold, then for any* $\lambda > 0$, I_λ^∞ *is achieved by some* $u \in E$.

Theorem 2.2. *Suppose that* (i)-(vi) *hold, then there is a* $\lambda_0 > 0$ *such that* I_{λ_0} *is achieved by some* $u \in E$. *Moreover, if* $I_\lambda < I_\lambda^\infty$ *for any* $\lambda > 0$, *then* I_λ *is achieved by some* $u \in E$ *for any* $\lambda > 0$.

Theorem 2.3. *Suppose that* (i)-(vii) *hold, then* (1.1) *posesses at least a nontrivial weak solution* $(\lambda, u) \in \mathbb{R}^1 \times E$ *and* $\|u\|_\infty < \infty$.

Remark 2.2. By (iv)-(v), it is trivial that $I_\lambda \leqslant I_\lambda^\infty$, and by Theorem 2.1, $I_\lambda < I_\lambda^\infty$ (for all $\lambda > 0$) if

$$\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + h(x)|u|^2]\,dx < \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\,dx \tag{2.15}$$

holds for $u \in E$, $\int_{\mathbb{R}^n} |u|^p\,dx = \lambda$ with $I^\infty(u) = I_\lambda{}^\infty < \infty$. (2.15) is valid, for instance, when $h(x) < \bar{h}$ for any $x \in \mathbb{R}^n$, or $a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta < \bar{a}_{\alpha\beta}(u)\xi_\alpha\xi_\beta$ for any $(x, u, \xi) \in \mathbb{R}^n \times (\mathbb{R}^N - \{0\}) \times (\mathbb{R}^n - \{0\})$.

EXAMPLE 2.1. In (1.1), if $n = 3$, $p = 5$, $h(x)$ satisfies (v), and

$$a_{\alpha\beta}(x, u) = (1 + |u|^2)b_{\alpha\beta}(x) \qquad (\text{or, } a_{\alpha\beta}(x, u) = b_{\alpha\beta}(x)/(1 + |u|^2))$$

where $b_{\alpha\beta}(x) \in C^1(\mathbb{R}^n)$ and $b_{\alpha\beta} = b_{\beta\alpha}$ $(1 \leqslant \alpha, \beta \leqslant n)$ satisfy

$$0 < \lambda|\xi|^2 \leqslant b_{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant M|\xi|^2$$

for any $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\lambda, M > 0$ are constants, and $\lim_{|x|\to\infty} b_{\alpha\beta}(x) = \bar{b}_{\alpha\beta}$, then, it is easy to see that $a_{\alpha\beta}(x, u)$, $h(x)$ satisfy conditions (i)-(vii), and thus we conclude that (1.1) possesses at least a nontrivial weak solution by using Theorem 2.3.

The above is only a simple example, the theorems in this section are applicable to many other cases.

3. Proof of Theorems 2.1 and 2.2

In this section, we prove Theorem 2.1 and Theorem 2.2. We need some lemmata and we always suppose that conditions (i)-(v) hold in this section.

Lemma 3.1. *I_λ, I_λ^∞ are continuous functions of λ on $[0, +\infty)$.*

Proof. It is evident that I_λ, I_λ^∞ are all finite for each $\lambda \geqslant 0$. Let $\lambda_m \to \lambda_0 \in (0, +\infty)$. We may assume that $\lambda_m > 0$ for any $m > 0$. Given $\epsilon > 0$ we have by (2.13), that there are $(u_m) \subset E$ such that $\int_{\mathbb{R}^n} |u_m|^p \, dx = 1$ and

$$I(\lambda_m^{1/p} u_m) \leqslant I_{\lambda_m} + \epsilon.$$

We claim that $|I_{\lambda_m}| \leqslant C$ (hereafter C denotes a constant independent of m). In fact, for fixed $u_0 \in C_0^\infty \subset E$ with $\int_{\mathbb{R}^n} |u_0|^p \, dx = 1$, we have by (2.1), the fact that $|\lambda_m| \leqslant C$ and the continuity of $\sigma(t)$, that

$$\begin{aligned} I_{\lambda_m} &\leqslant I(\lambda_m^{1/p} u_0) = \lambda_m^{2/p} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \lambda_m^{1/p} u_0) D_\alpha u_0^i D_\beta u_0^i + h(x)|u_0|^2] \, dx \\ &\leqslant \lambda_m^{2/p} \int_{\mathbb{R}^n} \sigma(|\lambda_m^{1/p} u_0|)|Du_0|^2 + \lambda_m^{2/p} \int_{\mathbb{R}^n} h(x)|u_0|^2 \, dx \leqslant C < +\infty. \end{aligned}$$

Hence, by (ii) we get

$$\int_{\mathbb{R}^n} [\sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2] \, dx \leqslant I_{\lambda_m} + \epsilon \leqslant C. \tag{3.1}$$

Since $\sigma(t)$ is nondecreasing in t, it is trivial that

$$\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2] \, dx \leqslant C$$

when $\lambda_m \geqslant \lambda_0$, while if $\lambda_m < \lambda_0$, we have by (2.2) and the boundedness of $(\lambda_0/\lambda_m)^{1/p}$, that

$$\begin{aligned} &\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2] \, dx \\ &\qquad = \int_{\mathbb{R}^n} \left[\sigma\left(\left(\frac{\lambda_0}{\lambda_m} \right)^{1/p} \lambda_m^{1/p}|u_m|\right)|Du_m|^2 + h(x)|u_m|^2 \right] dx \\ &\qquad \leqslant C_m \int_{\mathbb{R}^n} [\sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2] \, dx \leqslant C < +\infty. \end{aligned}$$

Thus, we always have

$$\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\, dx \leqslant C. \tag{3.2}$$

It is clear that

$$\begin{aligned} I_{\lambda_m} + \epsilon &\geqslant I(\lambda_m^{1/p}u_m) \\ &= I(\lambda_m^{1/p}u_m) - I(\lambda_0^{1/p}u_m) + I(\lambda_0^{1/p}u_m) \\ &\geqslant I(\lambda_m^{1/p}u_m) - I(\lambda_0^{1/p}u_m) + I_{\lambda_0}, \end{aligned}$$

but

$$\begin{aligned} I(\lambda_m^{1/p}u_m) - I(\lambda_0^{1/p}u_m) &= \lambda_m^{2/p} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \lambda_m^{1/p}u_m) - a_{\alpha\beta}(x, \lambda_0^{1/p}u_m)] D_\alpha u_m^i D_\beta u_m^i\, dx \\ &\quad + (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, \lambda_0^{1/p}u_m) D_\alpha u_m^i D_\beta u_m^i\, dx \\ &\quad + (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} h(x)|u_m|^2\, dx \\ &\equiv I_m^1 + I_m^2 + I_m^3. \end{aligned}$$

It is trivial that $\lim_{m\to\infty} I_m^3 = 0$ and by (2.1) and (3.2) we have that $\lim_{m\to\infty} I_m^2 = 0$. On the other hand, by the mean value theorem, we have

$$\begin{aligned} &|[a_{\alpha\beta}(x, \lambda_m^{1/p}u_m) - a_{\alpha\beta}(x, \lambda_0^{1/p}u_m)] D_\alpha u_m^i D_\beta u_m^i| \\ &\qquad = |(\lambda_m^{1/p} - \lambda_0^{1/p}) u_m^j D_{u^j} a_{\alpha\beta}(x, \xi_m(x)u_m) D_\alpha u_m^i D_\beta u_m^i|, \end{aligned}$$

where $\xi_m(x)$ is between $\lambda_m^{1/p}$ and $\lambda_0^{1/p}$, hence $|\xi_m(x)| \geqslant C > 0$. So, by (2.6), (3.1) and (3.2) we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u_m^j D_{u^j} a_{\alpha\beta}(x, \xi_m(x)u_m) D_\alpha u_m^i D_\beta u_m^i\, dx \right| &\leqslant C \int_{\mathbb{R}^n} \sigma(\xi_n(x)|u_m|)|Du_m|^2\, dx \\ &\leqslant \max_{0\leq m} C \int_{\mathbb{R}^n} \sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2\, dx \\ &\leqslant C \end{aligned}$$

from which $\lim_{m\to\infty} I_m^1 = 0$ and hence $\liminf_{m\to\infty} I_{\lambda_m} + \epsilon \geqslant I_{\lambda_0}$. Thus we have $\liminf_{m\to\infty} I_{\lambda_m} \geqslant I_{\lambda_0}$ which shows that I_λ is lower-semi continuous on $(0, +\infty)$. On the other hand, it is trivial to see that $\limsup_{m\to\infty} I_{\lambda_m} \leqslant I_{\lambda_0}$, which gives that I_λ is upper-semi continuous on $(0, +\infty)$. So we see that I_λ is continuous on $(0, +\infty)$. It is trivial that I_λ is continuous at $\lambda = 0$ and the lemma is proved. □

Lemma 3.2. *For any* $\lambda > 0$, *we have*

$$(3.3) \qquad I_\lambda \leqslant I_\lambda^\infty$$

$$(3.4) \qquad I_\lambda^\infty < I_\alpha^\infty + I_{\lambda-\alpha}^\infty \quad \text{for every} \quad \alpha \in (0,\lambda)$$

$$(3.5) \qquad I_\lambda < I_\alpha + I_{\lambda-\alpha} \quad \text{for every} \quad \alpha \in (0,\lambda)$$

If $I_\beta < I_\beta^\infty$ *for any* $\beta > 0$, *then*

$$(3.6) \qquad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty \quad \text{for every} \quad \alpha \in [0,\lambda).$$

PROOF. By (iv) and (v), it is trivial that (3.3) holds. To prove (3.5), we only need to show that

$$(3.7) \qquad I_{\theta\gamma} < \theta I \quad \text{for every} \quad \gamma \in (0,\lambda), \theta \in \left(1, \frac{\lambda}{\gamma}\right)$$

(see Lemma II.1 of [4]). Given $\gamma \in (0,\lambda)$, $\theta \in \left(1, \frac{\lambda}{\gamma}\right)$, we have by (2.13) and (2.4), that

$$\begin{aligned} I_{\theta\gamma} &= (\theta\gamma)^{2/p} \inf\left\{\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, (\theta\gamma)^{1/p}u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx \colon u \in E, \right. \\ &\qquad \left. \int_{\mathbb{R}^n} |u|^p\, dx = 1\right\} \\ &\leqslant \theta^{2/p}\gamma^{2/p}\theta^{s/p} \inf\left\{\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \gamma^{1/p}u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx \colon u \in E, \right. \\ &\qquad \left. \int_{\mathbb{R}^n} |u|^p\, dx = 1\right\} \\ &= \theta^{(2+s)/p} I_\gamma < \theta I_\gamma \end{aligned}$$

here we have made use of $I_\gamma > 0$ (for all $\gamma > 0$) which can easily be derived from the definition. Thus (3.7) holds and hence (3.5) holds. Similarly, by Remark 2.1 we see that (3.4) holds. By (3.3), (3.5) and $I_\beta < I_\beta^\infty$ (for all $\beta > 0$), we see that (3.6) holds. □

PROOF OF THEOREM 2.1 AND THEOREM 2.2. Let $(u_m) \subset E$ be a minimizing sequence of I_λ (or I_λ^∞) with

$$\int_{\mathbb{R}^n} |u_m|^p\, dx = \lambda > 0$$

and

$$I(u_m) < I_\lambda + 1/m \qquad (\text{or } I_\lambda^\infty(u_m) < I_\lambda^\infty + 1/m).$$

Since I_λ is finite, by (ii) we have

$$(3.8) \qquad \int_{\mathbb{R}^n} [\sigma(|u_m|)|Du_m|^2 + h(x)|u_m|^2]\, dx \leqslant C$$

(or

$$\int_{\mathbb{R}^n} [\sigma(|u_m|)|Du_m|^2 + \bar{h}|u_m|^2]dx \leqslant C$$

in the case of I_λ^∞) and $\|u_m\| \leqslant C$.

By the Sobolev embedding theorem, we may assume the existence of a $u_0 = (u_0^1, u_0^2, \ldots, u_0^N) \in E$ such that

$$\begin{aligned}
&u_m \rightharpoonup u_0 \quad \text{in} \quad E \\
&u_m^i \rightharpoonup u_0^i \quad \text{in} \quad H^1(\mathbb{R}^n), \qquad 1 \leqslant i \leqslant N \\
&u_m \to u_0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^n \\
&u_m^i \to u_0^i \quad \text{in} \quad L_{\text{loc}}^t(\mathbb{R}^n), \qquad 2 \leqslant t < \frac{2\hat{n}}{\hat{n}-2}
\end{aligned} \tag{3.9}$$

where « $\rightharpoonup$ » designates weak convergence, while « $\to$ » means strong convergence.

Let

$$\rho_m = a_{\alpha\beta}(x, u_m)D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2$$

(respectively

$$\rho_m = \bar{a}_{\alpha\beta}(u_m)Du_m^i Du_m^i + \bar{h}|u_m|^2$$

in the case of I_λ^∞), and

$$\lambda_m = \int_{\mathbb{R}^n} \rho_m \, dx,$$

we easily see that $\lambda_m \geqslant C > 0$. We need the following concentration compactness lemma:

Lemma 3.3. *Let u_m, ρ_m, λ_m be as above, then there exists a subsequence of (ρ_m), still denoted by (ρ_m), satisfying one of the three following possibilities:*

(i) (*Compactness*) *There exists $y_m \in \mathbb{R}^n$ such that $\rho_m(x + y_m)$ is tight, i.e. for every $\epsilon > 0$, there exists R such that*

$$\int_{y_m + B_R} \frac{\rho_m(x)}{\lambda_m} dx \geqslant 1 - \epsilon,$$

where

$$y_m + B_R = \{x \in \mathbb{R}^n : |x - y_m| \leqslant R\}.$$

(ii) (*Vanishing*) $\lim_{m\to\infty} \sup_{y \in \mathbb{R}^n} \int_{y + B_R} \rho_m(x)\, dx = 0$ *for all* $R < +\infty$.

(iii) (*Dichotomy*) *There exist* $\alpha \in (0, 1)$ *and a positive function* $\mu(\epsilon)$, *with* $\lim_{\alpha \to 0} \mu(\epsilon) = 0$, *such that for every* $\epsilon > 0$ *there exist* $m_0 \geqslant 1$ *and* $u_m^1, u_m^2 \in E$ *with* $\|u_m^1\|, \|u_m^2\| \leqslant C$, *so that*

$$\lim_{m \to \infty} \operatorname{dist}(\operatorname{supp} u_m^1, \operatorname{supp} u_m^2) = +\infty \tag{3.10}$$

$$\|u_m - (u_m^1 + u_m^2)\|_2 \leqslant \mu(\epsilon) \tag{3.11}$$

$$\|u_m - (u_m^1 + u_m^2)\|_p < \mu(\epsilon) \tag{3.12}$$

$$\left| \frac{I(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon) \tag{3.13}$$

$$\left| \frac{I(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.14}$$

$$I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon) \tag{3.15}$$

or, respectively, in the case of I_λ^∞,

$$\left| \frac{I^\infty(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon) \tag{3.16}$$

$$\left| \frac{I^\infty(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.17}$$

$$I^\infty(u_m) \geqslant I^\infty(u_m^1) + I^\infty(u_m^2) - \mu(\epsilon). \tag{3.18}$$

PROOF. For any $t \geqslant 0$, let

$$Q_m(t) = \sup_{y \in \mathbb{R}^n} \int_{y + B_t} \frac{\rho_m}{\lambda_m} dx.$$

Then $Q_m(t)$ is nondecreasing in t and $|Q_m(t)| \leqslant 1$, so by Helly's principle there is a subsequence of $Q_m(t)$, still denoted by $Q_m(t)$ with $\lim_{m \to \infty} Q_m(t) = Q(t)$ for any $t \geqslant 0$, where $Q(t)$ is a nondecreasing function on $[0, +\infty)$ and $|Q(t)| \leqslant 1$.

Let $\lim_{t \to \infty} Q(t) = \alpha \in [0, 1]$. If $\alpha = 0$, then $Q(t) \equiv 0$, hence $\lim_{m \to \infty} Q_m(t) = 0$ and (ii) (vaninshing) occurs.

If $\alpha = 1$, we can easily show that (i) (compactness) occurs by using the same method as in the proof of Lemma I.1 of [4].

Now, letting $\alpha \in (0, 1)$, we want to show that (iii) (dichotomy) occurs.

Given $\epsilon > 0$, there exists $R_0 = R_0(\epsilon) > 0$ such that

$$\alpha - \epsilon < Q(R_0) < \alpha + \epsilon$$
$$\alpha - 2\epsilon < Q(2R_0) < \alpha + 2\epsilon$$

hence there exists $m_0(\epsilon) > 0$ with

$$\alpha - \epsilon < Q_m(R_0) < \alpha + \epsilon \tag{3.19}$$

$$\alpha - 2\epsilon < Q_m(2R_0) < \alpha + 2\epsilon \tag{3.20}$$

whenever $m \geqslant m_0$.

We may choose $R_m \to +\infty$ such that

$$Q_m(2R_m) < \alpha + 1/m. \tag{3.21}$$

By the absolute continuity of Lebesgue integrals, there are $(z_m) \subset \mathbb{R}^n$ such that

$$Q_m(R_0) = \int_{z_m + B_{R_0}} \frac{\rho_m}{\lambda_m} \, dx. \tag{3.22}$$

Let $\xi, \varphi \in C_b^\infty(R^n)$, $0 \leqslant \xi, \varphi \leqslant 1$, $\xi \equiv 1$ and $\varphi \equiv 0$ if $|x| \leqslant 1$; $\xi \equiv 0$ and $\varphi \equiv 1$ if $|x| \geqslant 2$ and set $\xi_m = \xi[(x - z_m)/\tilde{R}]/\tilde{R}(\tilde{R} \geqslant R_0$ is to be determined) $\varphi_m = \varphi[(x - 3m)/R_m]$ and $u_m^1 = \xi_m u_m$, $u_m^2 = \varphi_m u_m$. It is trivial that (3.10) holds and that $\|u_m^1\|$, $\|u_m^2\| \leqslant C$.

By (3.22) we have

$$\begin{aligned} Q_m(R_0) &= \frac{1}{\lambda_m} \int_{z_m + B_{R_0}} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2] \, dx \\ &= \frac{1}{\lambda_m} \int_{z_m + B_{R_0}} [a_{\alpha\beta}(x, \xi_m u_m) D_\alpha(\xi_m u_m^i) D_\beta(\xi_m u_m^i) \\ &\qquad + h(x)|\xi_m u_m|^2] \, dx \\ &= \frac{1}{\lambda_m} I(u_m^1) \\ &\quad - \frac{1}{\lambda_m} \int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i \\ &\qquad + h(x)|u_m^1|^2] \, dx \end{aligned} \tag{3.23}$$

We want to show that

$$\frac{1}{\lambda_m} \int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2] \, dx < \mu(\epsilon). \tag{3.24}$$

Since

$$\begin{aligned} &\frac{1}{\lambda_m} \int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2] \, dx \\ &\leqslant \frac{1}{\lambda_m} \int_{R_0 \leq |x - z_m| \leq 2\tilde{R}} [a_{\alpha\beta}(x, u_m^1)(u_m^i D_\alpha \xi_m + \xi_m D_\alpha u_m^i)(u_m^i D_\beta \xi_m + \xi_m D_\beta u_m^i) \\ &\qquad + h(x)|u_m|^2] \, dx \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \xi_m^2 a_{\alpha\beta}(x, u_m^1) D_\alpha u_m^i D_\beta u_m^i \, dx \\
&\quad + \frac{2}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \xi_m u_m^i a_{\alpha\beta}(x, u_m^1) D_\alpha \xi_m D_\beta u_m^i \, dx \\
&\quad + \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} a_{\alpha\beta}(x, u_m^1) D_\alpha \xi_m D_\beta \xi_m \cdot u_m^i u_m^i \, dx \\
&\quad + \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} h(x) |u_m|^2 \, dx \\
&\equiv J_m^1 + J_m^2 + J_m^3 + J_m^4.
\end{aligned}
$$

By (3.19), (3.21) and the fact that $Q_m(t)$ is nondecreasing, it is evident that

$$
|J_m^4| \leqslant Q_m(2\tilde{R}) - Q_m(R_0) < \alpha + 1/m - (\alpha - \epsilon) = 1/m + \epsilon < \mu(\epsilon)
$$

(for m large enough).

By (2.1), (2.2) and (2.3) and since $\|u_m\| \leqslant C$, we have that

$$
\begin{aligned}
|J_m^3| &\leqslant 2a_2 \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|\xi_m u_m|) |D\xi_m|^2 |u_m|^2 \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|u_m|) |u_m|^2 \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} \int_{\mathbb{R}^n} (|u_m|^2 + |u_m|^{q+2}) \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} < \mu(\epsilon),
\end{aligned}
$$

for $\tilde{R}(\epsilon)$ large enough. In the same way, using (2.3) and (3.8) we have that

$$
\begin{aligned}
|J_m^2| &\leqslant \frac{C}{\tilde{R}} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} |a_{\alpha\beta}(x, \xi_m u_m) D_\alpha u_m^i D_\beta u_m^i| \, dx \\
&\leqslant \frac{C}{\tilde{R}} \int_{\mathbb{R}^n} \sigma(|u_m|) |Du_m| \, |u_m| \\
&\leqslant \frac{C}{\tilde{R}} \int_{\mathbb{R}^n} \sigma(|u_m|)(|Du_m|^2 + |u_m|^2) \, dx \\
&< \frac{C}{\tilde{R}} < \mu(\epsilon)
\end{aligned}
$$

for $\tilde{R}(\epsilon)$ large enough. By (2.1), (3.19), (3.21) and (3.22) we have that

$$\begin{aligned} 0 \leqslant J_m^1 &\leqslant C \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|u_m|)|Du_m|^2 \\ &\leqslant C \int_{R_0 \le |x - z_m| \le 2\tilde{R}} a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i \\ &\leqslant Q_m(2R_m) - Q_m(R_0) < \alpha + 1/m - (\alpha - \epsilon) \\ &= 1/m + \epsilon < \mu(\epsilon) \end{aligned}$$

(for m large enough).

Combining the above estimates, we see that (3.24) holds and (3.13) holds by (3.23). Similarly, (3.16) holds.

It is easy to show (see *e.g.* Lemma I.1 of [4]) that

$$\left| \int_{|x - z_m| \ge 2R_m} \frac{1}{\lambda_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.26}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{\lambda_m} I(u_m^2) &= \frac{1}{\lambda_m} \int_{|x - z_m| \ge R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx \\ &= \frac{1}{\lambda_m} \int_{R_m \le |x - z_m| \le 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx \\ &\quad + \frac{1}{\lambda_m} \int_{|x - z_m| \ge 2R_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \end{aligned} \tag{3.27}$$

Similarly to (3.24), we can prove that

$$\frac{1}{\lambda_m} \int_{R_m \le |x - z_m| \le 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m|^2]\, dx \leqslant \mu(\epsilon) \tag{3.28}$$

Thus (3.26) and (3.27) imply that (3.14) holds. Similarly, (3.17) holds.

By (3.19) and (3.21) we have that

$$\begin{aligned} \|u_m - (u_m^1 + u_m^2)\|_2^2 &= \int_{\mathbb{R}^n} |1 - \xi_m - \varphi_m|^2 |u_m|^2\, dx \\ &\leqslant C \int_{\tilde{R} \le |x - z_m| \le 2R_m} |u_m|^2 \\ &\leqslant C[Q_m(2R_m) - Q_m(R_0)] < \mu(\epsilon). \end{aligned}$$

So we have (3.11). Similarly, by $\|u_m\| \leqslant C$ and $\|u_m^1\| \leqslant C$, $\|u_m^2\| \leqslant C$, we see that (3.12) holds.

Finally we prove (3.15). Since

$$
\begin{aligned}
I(u_m) \geqslant & \int_{|x-z_m| \leq \tilde{R}} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\
& + \int_{|x-z_m| \geq 2R_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\
= & \ I(u_m^1) + I(u_m^2) \\
& - \int_{\tilde{R} \leq |x-z_m| \leq 2\tilde{R}} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2]\, dx \\
& - \int_{R_m \leq |x-z_m| \leq 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx
\end{aligned}
$$

and because of (3.24) and (3.28), we deduce that

$$I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon).$$

Thus (3.15) holds. Similarly (3.18) holds. □

Lemma 3.4. (*cf. Lemma* 1.1 *of* [5].) *Let* $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, *with* $q \neq Np/(N-p)$ *if* $p < N$. *Assume that* (u_m) *is bounded in* $L^q(\mathbb{R}^N)$, $|Du_m|$ *is bounded in* $L^p(\mathbb{R}^N)$ *and*

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_m|^q \, dx \to 0 \quad \text{as } m \to \infty, \quad \text{for some} \quad R > 0.$$

Then $u_m \to 0$ *in* $L^t(\mathbb{R}^N)$ *for any* t *between* q *and* $Np/(N-p)$.

We now turn to prove Theorem 2.1 and Theorem 2.2. We already know that there is a minimizing sequence (u_m) of I_λ (or I_λ^∞) such that Lemma 3.3 holds.

If «vanishing» occurs, then

$$\lim_{m\to\infty} \sup_{y \in \mathbb{R}^n} \int_{y+B_R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx = 0 \tag{3.29}$$

for all R. We know also that (Du_m) is bounded in $L^2(\mathbb{R}^n)$ and by (3.29) we know that

$$\lim_{m\to\infty} \sup_{y \in \mathbb{R}^n} \int_{y+B_R} |u_m|^2 \, dx = 0 \qquad \text{(for any } R > 0).$$

So Lemma 3.4 gives that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^n}|u_m|^p\,dx = 0$$

and this contradicts

$$\int_{\mathbb{R}^n}|u_m|^p\,dx = \lambda.$$

Thus we have ruled out «vanishing».

If «dichotomy» occurs, then Lemma 3.3 shows that for any $\epsilon > 0$, there are $u_m^1, u_m^2 \in E$ such that (3.10)-(3.15) hold (or (3.10), (3.12), (3.5) and (3.18) hold in the case of I_λ^∞). Therefore we would have that

$$\begin{aligned}
I_\lambda + \epsilon &\geqslant I(u_m)\\
&\geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon)\\
&\geqslant I_{\int_{\mathbb{R}^n}|u_m^1|^p dx} + I_{\int_{\mathbb{R}^n}|u_m^2|^p dx} - \mu(\epsilon).
\end{aligned} \tag{3.30}$$

We may assume that

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m^1|^p\,dx = \lambda_1(\epsilon), \qquad \lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m^2|^p\,dx = \lambda_2(\epsilon).$$

Now

$$\lambda = \int_{\mathbb{R}^n}|u_m|^p\,dx$$

and

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}|u_m|^p\,dx - \int_{\mathbb{R}^n}|u_m^1|^p\,dx - \int_{\mathbb{R}^n}|u_m^2|^p\,dx\right| &\leqslant \int_{\mathbb{R}^n}|1-\varphi_m^p-\xi_m^p|\,|u_m|^p\,dx\\
&\leqslant C\int_{R_0\le|x-z_m|\le 2R_m}|u_m|^p\,dx\\
&\leqslant C\left(\int_{R_0\le|x-z_m|\le 2R_m}|u_m|^2\,dx\right)^{p/2}\\
&< \mu(\epsilon),
\end{aligned}$$

(where we have made use of notations in the proof of Lemma 3.3.)

We conclude that

$$|\lambda - (\lambda_1(\epsilon)+\lambda_2(\epsilon))| \leqslant \mu(\epsilon) \tag{3.31}$$

Letting $m\to\infty$ in (3.30) and using Lemma 3.1 we obtain that

$$I_\lambda + \epsilon \geqslant I_{\lambda_1(\epsilon)} + I_{\lambda_2(\epsilon)} - \mu(\epsilon).$$

We assume now that $\lambda_1(\epsilon) \to \lambda_1$, $\lambda_2(\epsilon) \to \lambda_2$ as $\epsilon \to 0$. Then we have by Lemma 3.1, that

$$I_\lambda \geqslant I_{\lambda_1} + I_{\lambda_2}. \tag{3.32}$$

By Lemma 3.3 and the fact that $\lambda_m \geqslant c > 0$ we have that

$$|I(u_m^1) - \tilde{\alpha}| < \mu(\epsilon), \quad \text{where} \quad \tilde{\alpha} > 0$$
$$|I(u_m^2) - \tilde{\beta}| < \mu(\epsilon), \quad \text{where} \quad \beta > 0.$$

Thus, if $\lambda_1 = 0$ then by (3.31) $\lambda_2 = \lambda$. Since

$$I_\lambda + \epsilon \geqslant I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon)$$

we obtain that

$$I_\lambda \geqslant \tilde{\alpha} + I_{\lambda_2(\epsilon)} - \mu(\epsilon).$$

Hence

$$I_\lambda \geqslant \tilde{\alpha} + I_\lambda.$$

This is a contradiction and so $\lambda_1 > 0$; similarly $\lambda_2 > 0$. And now $\lambda_1 + \lambda_2 = \lambda$ and (3.32) contradict (3.5). Thus we have ruled out the «dichotomy» for I_λ. Similarly we can rule out the «dichotomy» for I_λ^∞ using (3.4).

So we only have «compactness» *i.e.* there exists $(y_m) \subset \mathbb{R}^n$ such that for any $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ with

$$\int_{|x-y_m| \leq R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \geqslant \lambda_m(1-\epsilon).$$

Hence

$$\int_{|x-y_m| \geq R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \leqslant \lambda_m \epsilon \tag{3.33}$$
$$\int_{|x-y_m| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \mu(\epsilon)$$

or, in the case of I_λ^∞, we have

$$\int_{|x-y_m| \geq R} [\bar{a}_{\alpha\beta}(u_m) D_\alpha u_m^i D_\beta u_m^i + \bar{h}|u_m|^2]\, dx \leqslant \lambda_m \epsilon \tag{3.34}$$
$$\int_{|x-y_m| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \mu(\epsilon)$$

We first prove Theorem 2.1. Let $\bar{u}_m(x) = u_m(x + y_m)$, then $\|\bar{u}_m\| \leqslant C < +\infty$ and by (3.34) and the Sobolev embedding theorem we may assume the existence of a $u = (u^1, u^2, \ldots, u^N) \in E$ such that

$$
\left\{\begin{array}{llll}
\bar{u}_m \rightharpoonup u & \text{in} & E & \\
\bar{u}_m^i \rightharpoonup u^i & \text{in} & H^1(R^n) & \\
\bar{u}_m^i \to u^i & \text{in} & L^t(R^n) & 2 \leqslant t < 2\hat{n}/(\hat{n}-2) \\
\bar{u}_m \to u & \text{a.e.} & \text{in } R^n, &
\end{array}\right.
\tag{3.35}
$$

for $1 \leqslant i \leqslant N$, and

$$\lambda = \int_{\mathbb{R}^n} |u_m|^p \, dx = \int_{\mathbb{R}^n} |\bar{u}_m|^p \, dx \to \int_{\mathbb{R}^n} |u|^p \, dx \qquad (\text{as} \quad m \to \infty).$$

Also

$$I_\lambda^\infty = \lim_{m \to \infty} \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i + \bar{h}|u_m|^2] \, dx.$$

By (3.35) and (ii), (iii) of Section 2 we see that

$$\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u) \quad \text{a.e. in} \quad \mathbb{R}^n.$$

So for any bounded domain $\Omega \subset \mathbb{R}^n$ and $\delta > 0$, there is a $\Omega_\delta \subset \Omega$ with

$$|\Omega - \Omega_\delta| < \delta$$

and

$$\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u)$$

uniformly for $x \in \Omega_\delta$ where $|A|$ denotes the Lebesgue measure of A for any $A \subset \mathbb{R}^n$. So that for any $\epsilon > 0$ and m large enough we have, by (2.2), that

$$
\begin{aligned}
\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx &\geqslant \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx \\
&\geqslant \int_{\Omega_\delta} [\bar{a}_{\alpha\beta}(\bar{u}_m^i) - \bar{a}_{\alpha\beta}(u)] D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx \\
&\quad + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx \\
&\geqslant -\epsilon \int_{\mathbb{R}^n} |D\bar{u}_m|^2 \, dx + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx \\
&\geqslant -\epsilon C + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx.
\end{aligned}
$$

By (3.35), Mazur's theorem (see [6]) and Fatou's lemma, we see that

$$\liminf_{m \to \infty} \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i \, dx \geqslant \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i \, dx,$$

and hence we get, for any N, that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i\,dx \geqslant \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx$$

$$\geqslant \int_{\Omega_\delta} [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N\,dx$$

where the function $[f]_N$ for any $f \geqslant 0$ is given by

$$[f]_N = \begin{cases} f & \text{if } f \leqslant N \\ N & \text{if } f \geqslant N \end{cases}$$

Since $[\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N \in L^1(\Omega)$, and since $|\Omega_\delta| \to |\Omega|$ we have that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i\,dx \geqslant \int_\Omega [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N\,dx.$$

Letting $N \to \infty$, we have that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u_m^i D_\beta u_m^i \geqslant \int_\Omega \bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx \tag{3.36}$$

Thus, since the supremum of any sequence of lower-semicontinuous functions is still lower-semicontinuous, we have that

$$\liminf_{m\to\infty}\int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u_m^i D_\beta u_m^i\,dx \geqslant \int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx \tag{3.37}$$

On the other hand, by (3.35) we have that

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} \bar{h}|u_m|^2\,dx = \int_{\mathbb{R}^n} \bar{h}|u|^2\,dx$$

and so we get that

$$I_\lambda^\infty \geqslant \liminf_{m\to\infty}\int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u_m^i D_\beta u_m^i\,dx + \lim_{m\to\infty}\int_{\mathbb{R}^n} \bar{h}|u_m|^2\,dx$$

$$\geqslant \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\,dx.$$

But

$$\int_{\mathbb{R}^n} |u|^p\,dx = \lambda$$

and so

$$I_\lambda^\infty = \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\, dx$$

so I_λ^∞ is achieved and Theorem 2.1 is proved.

In the case of I_λ, by (3.33) we still have (3.35) with $\bar{u}_m(x) = u_m(x + y_m)$. If there is $\lambda_0 \in (0, \lambda]$ such that $I_{\lambda_0} = I_{\lambda_0}^\infty$, then by Theorem 2.1 there exists $u_0 \in E$ with $\int_{\mathbb{R}^n} |u_0|^p\, dx = \lambda_0$ and such that $I_{\lambda_0}^\infty = I^\infty(u_0)$, and hence $I_{\lambda_0} \leqslant I(u_0) \leqslant I^\infty(u_0) = I_{\lambda_0}^\infty = I_{\lambda_0}$ implies that $I(u_0) = I_{\lambda_0}$ and therefore I_{λ_0} is achieved by u_0 Theorem 2.2 is proved.

Now we assume that for any $0 < \mu \leqslant \lambda$ we have $I_\mu < I_\mu^\infty$. If (y_m) is unbounded, say $|y_m| \to \infty$, we have, by (ii) of Section 2 and (3.35), that

$$a_{\alpha\beta}(x + y_m, \bar{u}_m) \to \bar{a}_{\alpha\beta}(u) \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^n.$$

So we have, as in (3.37), that

$$\liminf_{i\to\infty} \int_{\mathbb{R}^n} a_{\alpha\beta}(x + y_m, \bar{u}_m) D_\alpha u_m^i D_\beta u_m^i\, dx \geqslant \int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i\, dx. \tag{3.38}$$

By (v) of Section 2, (3.35) and the Lebesgue's theorem we have that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} h(x + y_m)|\bar{u}_m|^2\, dx = \int_{\mathbb{R}^n} \bar{h}|u|^2\, dx. \tag{3.39}$$

Combining (3.38), (3.39) and

$$\int_{\mathbb{R}^n} |u|^p\, dx = \lambda$$

we have that

$$\begin{aligned} I_\lambda &= \lim_{m\to\infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x + y_m, \bar{u}_m) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i + h(x + y_m)|\bar{u}_m|^2]\, dx \\ &\geqslant \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\, dx \\ &\geqslant I_\lambda^\infty \end{aligned}$$

which contradicts that $I_\mu < I_\mu^\infty$ for any $0 < \mu \leqslant \lambda$. Thus we have $|y_m| \leqslant C$ and by (3.34) we see that for any $\epsilon > 0$, there is a $R(\epsilon) > 0$ such that

$$\int_{|x| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \epsilon \tag{3.40}$$

and hence we may assume the existence of a $u_0 \in E$ such that

$$
(3.41) \qquad \begin{cases} u_m \rightharpoonup u_0 \quad \text{in} \quad E \\ u_m^i \rightharpoonup u_0^i \quad \text{in} \quad H^1(\mathbb{R}^n), \qquad (1 \leqslant i \leqslant N), \\ u_m^i \to u_0^i \quad \text{in} \quad L^t(\mathbb{R}^n) \qquad 2 \leqslant t < \dfrac{2\hat{n}}{\hat{n}-2} \qquad (1 \leqslant i \leqslant N), \\ u_m \to u_0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^n \\ \int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda \end{cases}
$$

Thus, similarly to (3.38) and (3.39) we can prove that

$$
\liminf_{m \to \infty} \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i \, dx \geqslant \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_0) D_\alpha u_0^i D_\beta u_0^i \, dx
$$

$$
\lim_{m \to \infty} \int_{\mathbb{R}^n} h(x) |u_m|^2 \, dx = \int_{\mathbb{R}^n} h(x) |u_0|^2 \, dx.
$$

Since $\int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda$ we have

$$
\begin{aligned}
I_\lambda &\geqslant \liminf_{m \to \infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2] \, dx \\
&\geqslant \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_0) D_\alpha u_0^i D_\beta u_0^i + h(x)|u_0|^2] \, dx \\
&\geqslant I_\lambda
\end{aligned}
$$

and hence I_λ is achieved by $u_0 \in E$. Theorem 2.2 is proved. □

4. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. The main difficulty is that I_λ is in general not in $C^1(E, \mathbb{R})$. To overcome this difficulty, we first prove that

$$
\frac{d}{dt} I\left(\frac{\lambda(u + t\varphi)}{\|u + t\varphi\|_p} \right) \Bigg|_{t=0}
$$

exists for special $\varphi \in E$ and then show that $\|u\|_\infty$ is finite where u is a minimizer of I_λ for some $\lambda > 0$. Finally we prove the theorem.

PROOF OF THEOREM 2.3. By Theorem 2.2 we may assume without loss of generality the existence of $u \in E$, with $\int_{\mathbb{R}^n} |u|^p \, dx = 1$ and such that

$$
I_1 = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)|u|^2] \, dx.
$$

We first prove that for any $\tau \geqslant 0$,

$$\frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\bigg|_{t=0} = 0 \tag{4.1}$$

where

$$|u|_L = \begin{cases} |u| & \text{if} \quad |u| \leqslant L \\ L & \text{if} \quad |u| \geqslant L \end{cases}$$

It is easy to see that $u + t|u|_L^\tau u = (1 + t|u|_L^\tau)u \in E$ for any $t \geqslant 0$ and since u achieves I_1, (4.1) will hold if

$$\frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\bigg|_{t=0}$$

exists.

Because $0 \leqslant |u|_L^\tau \leqslant L^\tau$, there is a $M > 0$, depending on β and L, such that

$$\tfrac{1}{2} \leqslant \|u + t|u|_L^\tau u\|_p \leqslant M \tag{4.2}$$

for t small enough.

It is easy to prove that

$$\frac{d}{dt}(\|u + t|u|_L^\tau u\|_p)\bigg|_{t=0} = \int_{\mathbb{R}^n} |u|^p |u|_L^\tau \, dx \tag{4.3}$$

and hence

$$\begin{aligned} \frac{d}{dt}\left[\int_{\mathbb{R}^n} \frac{h(x)|u + t|u|_L^\tau u|^2}{\|u + t|u|_L^\tau u\|_p} dx\right]\bigg|_{t=0} \\ = 2\int_{\mathbb{R}^n} h(x)|u|^2|u|_L^\tau \, dx - 2\int_{\mathbb{R}^n} h(x)|u|^2 \, dx \int_{\mathbb{R}^n} |u|^p |u|_L^\tau \, dx. \end{aligned} \tag{4.4}$$

On the other hand

$$\begin{aligned} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) &= \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha u^i D_\beta u^i}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ 2t \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t|u|_L^2 u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha u^i D_\beta |u|_L^\tau u^i}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ t^2 \int_{\mathbb{R}^n} a_{\alpha\beta}\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha(|u|_L^\tau u^i) D_\beta(|u|_L^\tau u^i)}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ \int_{\mathbb{R}^n} \frac{h(x)|u + t|u|_L^\tau u|^2}{\|u + t|u|_L^\tau u\|_p^2} dx \end{aligned}$$

$$(4.5)\quad I\left(\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau u\|_p}\right)=I^1(t)+I^2(t)+I^3(t)+\int_{\mathbb{R}^n}\frac{h(x)|u+t|u|_L^\tau u|^2}{\|u+t|u|_L^\tau u\|_p^2}\,dx$$

Using (ii), (iii) of Section 2, (4.2) and (2.1), the inequality

$$\left|a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau\|_p}\right)\frac{u^it|u|^{\tau-1}D_\alpha u^2D_\beta|u|_L}{\|u+t|u|_L^\tau u\|_p^2}\right|\leqslant C|D_\alpha u^iD_\beta|u|_L|u|^\tau|,L^1(\mathbb{R}^n)$$

(which holds if $|u|\leqslant L$) and the Dominated Convergence Theorem, we have that

$$(4.6)\qquad \frac{d}{dt}[I^2(t)]\bigg|_{t=0}=\lim_{t\to0}\frac{I^2(t)}{t}=2\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta(|u|_L^\tau u^i)\,dx.$$

Similarly, we have that

$$(4.7)\qquad \frac{d}{dt}[I^3(t)]\bigg|_{t=0}=0$$

On the other hand

$$\begin{aligned}\frac{d}{dt}[I^1(t)]\bigg|_{t=0}&=\lim_{t\to0}\int_{\mathbb{R}^n}\frac{1}{t}\left[a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau\|_p}\right)\|u+t|u|_L^\tau u\|_p^{-2}\right.\\&\qquad\left.-a_{\alpha\beta}(x,u)\right]D_\alpha u^iD_\beta u^i\,dx\\&=\lim_{t\to0}\int_{\mathbb{R}^n}\frac{1}{t}\left[a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau u\|_p}\right)-a_{\alpha\beta}(x,u)\right]\\&\qquad\|u+t|u|_L^\tau u\|_p^{-2}D_\alpha u^iD_\beta u^i\,dx\\&\qquad+\lim_{t\to0}\frac{1}{t}(\|u+t|u|_L^\tau u\|_p^{-2}-1)\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta u^i\,dx\\&\equiv\lim_{t\to0}I^4(t)+\lim_{t\to0}I^5(t).\end{aligned}$$

By (4.3), we have

$$(4.8)\qquad \lim_{t\to0}I^5(t)=-2\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta u^i\,dx.$$

Using the mean value theorem we get that

$$\begin{aligned}\lim_{t\to0}I^4(t)=\lim_{t\to0}\int_{\mathbb{R}^n}&D_{u^j}a_{\alpha\beta}\left(x,\frac{u+t'|u|_L^\tau u}{\|u+t'|u|_L^\tau u\|_p}\right)\\&\left[\frac{|u|_L^\tau u^j}{\|u+t'|u|_L^\tau u\|_p}-\frac{u^j+t'|u|_L^\tau u^j}{\|u+t'|u|_L^\tau u\|_p^2}\frac{d}{dt}\|u+t|u|_L^\tau u\|_{p|t=t'}\right]\\&\|u+t|u|_L^\tau u\|_p^{-2}D_\alpha u^iD_\beta u^i\,dx\end{aligned}$$

$$= \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p}$$

$$\|u + t|u|_L^\tau u\|_p^{-2} D_\alpha u^i D_\beta u^i\, dx$$

$$- \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)$$

$$\frac{u^j + t'|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p^2}\frac{d}{dt}\|u + t|u|_L^\tau u\|_{p|t=t'}\|u + t|u|_L^\tau u\|_p^{-2} D_\alpha u^i D_\beta u^i\, dx$$

$$\equiv \lim_{t\to 0} I^6(t) - \lim_{t\to 0} I^7(t) \qquad (0 < t' = t'(x) < t) \tag{4.9}$$

By (vi) of Section 2 and (3.2) we have that

$$\left| D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p} D_\alpha u^i D_\beta u^i \|u + t|u|_L^\tau u\|_p^{-2}\right|$$

$$\leqslant C\sigma\left(\frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau}{1 + t'|u|_L^\tau}|Du|^2$$

$$\leqslant C\sigma(|u|)|Du|^2 \in L^1(\mathbb{R}^n),$$

hence by the Dominated Convergence Theorem

$$\lim_{t\to 0} I^6(t) = \int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx. \tag{4.10}$$

Similarly, by (vi) of Section 2, (4.2) and (4.3) we get

$$\lim_{t\to 0} I^7(t) = \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j\, dx. \tag{4.11}$$

Combining (4.4)-(4.11) we see that (4.1) holds and that

$$0 = \frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\Bigg|_{t=0}$$

$$= 2\int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta(|u|_L^\tau u^i)\, dx$$

$$+ \int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx$$

$$- \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx$$

$$- 2\int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx + 2\int_{\mathbb{R}^n} h(x)|u|^2 |u|_L^\tau\, dx$$

$$- 2\int_{\mathbb{R}^n} h(x)|u|^2\, dx \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx$$

which implies that

$$\int_{\mathbb{R}^n} a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta(|u|_L^\tau u^i)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j} a_{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^i\,dx + \int_{\mathbb{R}^n} h(x)|u|^2|u|_L^\tau\,dx = \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx \quad (\text{for every } \tau \geqslant 0 \text{ and } L \geqslant 0), \tag{4.12}$$

where

$$\lambda = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i + \tfrac{1}{2}u^j D_{u^j} a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i + h(x)|u|^2]\,dx.$$

Now we are ready to prove that $\|u\|_\infty < +\infty$. By (4.12), we have for any $\tau \geqslant 0$, that

$$\int_{\mathbb{R}^n} |u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j\,dx + \tau\int_{\mathbb{R}^n}|u|_L^{\tau-1}u^i a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta |u|_L\,dx + \frac{1}{2}\int_{\mathbb{R}^n}|u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx + \int_{\mathbb{R}^n} h(x)|u|^2|u|_L^\tau\,dx = \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx. \tag{4.13}$$

It is easy to see that

$$\begin{aligned}\int_{\mathbb{R}^n}|u|_L^{\tau-1}u^i a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta|u|_L\,dx &= \int_{\mathbb{R}^n}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha|u|D_\beta|u|_L\,dx\\ &= \int_{\{|u|\leq L\}}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha|u|D_\beta|u|\,dx\\ &\geqslant 0.\end{aligned}$$

So by (2.8) we have

$$(1-a_3)\int_{\mathbb{R}^n}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx \tag{4.14}$$

hence

$$\mu(1-a_3)\int_{\mathbb{R}^n}|Du|^2|u|_L^\tau\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx.$$

It is easy to see that

$$|D|u||^2 \leqslant |Du|^2$$

and from this and (4.14) we get that

$$\mu(1-a_3)\int_{\mathbb{R}^n}|D|u||^2|u|_L^\tau\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx.$$

Thus, there is a $C>0$ such that for any $\tau \geqslant 0$

$$\int_{\mathbb{R}^n}|D|u|\,|u|_L^{\tau/2}|^2\,dx \leqslant C\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx \tag{4.15}$$

holds.

By (4.15) we have, for any $\tau \geqslant 1$, that

$$\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx \leqslant C\int_{\mathbb{R}^n} |u|^p|u|_L^{2(\tau-1)}\,dx. \tag{4.16}$$

Let $w_L = |u|\,|u|_L^{\tau-1}$, then we have

$$Dw_L = D|u|\,|u|_L^{\tau-1} + (\tau-1)|u|_L^{\tau-2}D|u|_L^{\tau-2}D|u|_L|u|.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n} |Dw_L|^2\,dx &\leqslant C\Big[\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx + (\tau-1)^2\int_{\mathbb{R}^n} |\,|u|_L^{\tau-2}|u|D|u|_L|^2\,dx\Big]\\ &\leqslant C\Big[\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx + (\tau-1)^2\int_{\{|u|\leq L\}} |D|u|\,|u|^{\tau-1}|^2\,dx\Big]\\ &\leqslant C(1+(\tau-1)^{2)}\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx\\ &\leqslant C\tau^2\int_{\mathbb{R}^n} |u|^p|u|_L^{2\tau-2}\,dx.\end{aligned}$$

So we get

$$\int_{\mathbb{R}^n} |Dw_L|^2\,dx \leqslant C\tau^2\int_{\mathbb{R}^n} |u|^{p-2}|w_L|^2\,dx \tag{4.17}$$

By (4.17), the Sobolev embedding theorems and Hölder's inequality we have that

$$\begin{aligned}\|w_L\|_{2^*}^2 &\leqslant C\|\,|Dw_L|\,\|_2^2\\ &\leqslant C\tau^2\Big(\int_{\mathbb{R}^n} |u|^{(p-2)\frac{2^*}{p-2}}\,dx\Big)^{\frac{p-2}{2^*}}\Big(\int_{\mathbb{R}^n} |w_L|^{2\frac{2^*}{2^*-(p-2)}}\,dx\Big)^{\frac{2^*-(p-2)}{2}}\\ &= C\tau^2\|u\|_{2^*}^{p-2}\|w_L\|_{\frac{2q}{2-2}}^2\end{aligned} \tag{4.18}$$

where $2q/(q-2) = 2\cdot 2^*/(2^*-(p-2))$, *i.e.* $q = 2\cdot 2^*/(p-2)$. It is easy to see that $q > n$ when $n > 2$ or $n \leqslant 2$ by choosing 2^* large enough, hence $2^* > 2^* > 2q/(q-2)$. If $|u|^2 \in L^{2q/(q-2)}(\mathbb{R}^n)$, letting $L \to +\infty$ in (4.18) and using the Dominated Convergence Theorem and Fatou's lemma together with the fact that $|w_L| \leqslant |u|^\tau$ we get that

$$\|\,|u|^\tau\|_{2^*}^2 \leqslant C\tau^2\|\,|u|^\tau\|_{\frac{2q}{q-2}}^2.$$

Thus $u \in L^{2\tau q/(q-2)}(\mathbb{R}^n)$ implies that $u \in L^{\tau 2^*}(\mathbb{R}^n)$. If we set $q^* = 2q/(q-2)$, $\chi = 2^*/q^*$ then $\tau\chi q^* = \tau 2^*$ and we have that

$$\|\,|u|\,\|_{\tau\lambda q^*}^\tau \leqslant C\tau\|\,|u|\,\|_{\tau q^*}^\tau$$

that is

$$\|\,|u|\,\|_{\tau\chi q^*} \leqslant C^{1/\tau}\tau^{1/\tau}\|\,|u|\,\|_{\tau q^*}.$$

Let $\tau = \chi^m$, $m = 0, 1, \ldots$, then we have

$$\|u\|_{\chi^N q^*} \leqslant \prod_{m=0}^{N-1} (C\chi^m)^{-\chi^m} \|u\|_{q^*} \leqslant C^\sigma \chi^\tau \| |u| \|_{q^*} \leqslant C\|u\|_{q^*} \tag{4.19}$$

where

$$\sigma = \sum_{m=0}^{N-1} \chi^{-m}, \qquad \tau = \sum_{m=0}^{N-1} m\chi^{-m}$$

and C is independent of N for $\sum_{m=0}^{\infty} \chi^{-m}$, $\sum_{m=0}^{\infty} m\chi^{-m}$ are all convergent. Letting $N \to \infty$ in (4.19) we get

$$\|u\|_\infty \leqslant C\|u\|_{q^*} < +\infty. \tag{4.20}$$

Thus $u \in L_\infty \cap E$.

Finally, we show that for any $\varphi \in L_\infty \cap E$, we have

$$\frac{d}{dt} I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right)\bigg|_{t=0} = 0. \tag{4.21}$$

Note that we only need to show that

$$\frac{d}{dt} I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right)\bigg|_{t=0}$$

exists for any $\varphi \in L_\infty \cap E$. (4.21) can be proved by using the same method for proving (4.1). In fact, similarly to (4.2), (4.3), (4.4) and (4.5) we may obtain

$$\frac{1}{2} \leqslant \|u + t\varphi\|_p \leqslant M \qquad \text{(for } t \text{ small enough)} \tag{4.22}$$

$$\frac{d}{dt} \|u + t\varphi\|_{p|t=0} = \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \tag{4.23}$$

$$\frac{1}{dt} \int_{\mathbb{R}^n} \frac{h(x)|u + t\varphi|^2}{\|u + t\varphi\|_p^2} dx \bigg|_{t=0} = 2\int_{\mathbb{R}^n} h(x) u^i \varphi^i \, dx - 2\int_{\mathbb{R}^n} h(x)|u|^2 \, dx \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \tag{4.24}$$

$$I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right) = \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) \frac{D_\alpha u^i D_\beta u^i}{\|u + t\varphi\|_p^2} dx + 2t \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) \frac{D_\alpha u^i D_\beta \varphi^i}{\|u + t\varphi\|_p^2} dx \tag{4.25}$$

$$+ \frac{t^2}{\|u + t\varphi\|_p^2} \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) D_\alpha \varphi^i D_\beta \varphi^i \, dx$$

$$+ \int_{\mathbb{R}^n} \frac{h(x)|u + t\varphi|^2}{\|u + t\varphi\|_p^2} dx$$

$$\equiv J^1(t) + J^2(t) + J^3(t) + \int_{\mathbb{R}^n} \frac{h(x)|u + t\varphi|^2}{\|u + t\varphi\|_p^2} dx.$$

Using that $\|u\|_\infty \leqslant C$, $\|\varphi\|_\infty \leqslant C$, (4.22) and (ii) of Section 2, and similarly to (4.6) and (4.7) we obtain that

$$\frac{d}{dt} J^2(t)\Big|_{t=0} = 2 \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta \varphi^i \, dx, \qquad \frac{d}{dt} J^3(t)\Big|_{t=0} = 0.$$

On the other hand, we have that

$$\frac{d}{dt} J^1(t)\Big|_{t=0} = \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) \|u + t\varphi\|_p^{-2} - a_{\alpha\beta}(x, u) \right]$$
$$D_\alpha u^i D_\beta u^i \, dx$$
$$= \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) - a_{\alpha\beta}(x, u) \right]$$
$$\|u + t\varphi\|_p^{-2} D_\alpha u^i D_\beta u^i \, dx$$
$$+ \lim_{t\to 0} \frac{1}{t} (\|u + t\varphi\|_p^{-2} - 1) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i \, dx$$
$$= \lim_{t\to 0} J^4(t) + \lim_{t\to 0} J^5(t).$$

By (4.23) and similarly to (4.8) we obtain that

$$\lim_{t\to 0} J^5(t) = -2 \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i \, dx.$$

Using the mean value theorem we have that

$$\lim_{t\to 0} J^4(t) = \lim_{t\to 0} \int_{\mathbb{R}^n} D_{u^j} a_{\alpha\beta}\left(x, \frac{u + t'\varphi}{\|u + t'\varphi\|_p}\right)$$
$$\left[\frac{\varphi^j}{\|u + t'\varphi\|_p} - \frac{u^j + t'\varphi^j}{\|u + t'\varphi\|_p^2} \frac{d}{dt} \|u + t\varphi\|_p \Big|_{t=t'} \right]$$
$$\|u + t\varphi\|_p^{-2} D_\alpha u^i D_\beta u^i \, dx$$

$$= \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{\varphi^j}{\|u+t'\varphi\|_p}$$

$$\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i\,dx$$

$$- \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{u^j+t'\varphi^j}{\|u+t'\varphi\|_p^2}$$

$$\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i \frac{d}{dt}\|u+t\varphi\|_p\Big|_{t=t'}\,dx$$

$$= \lim_{t\to 0} J^6(t) - \lim_{t\to 0} J^7(t),$$

where $0 < t'(x) < t$. By (2.7) and (4.22) we see that

$$\left|D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{\varphi^j}{\|u+t'\varphi\|_p}\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i\right|$$

$$\leqslant C\eta\left(\frac{|u|+t'|\varphi|}{\|u+t'\varphi\|_p}\right)\|u+t\varphi\|_p^{-2}|Du|^2$$

$$\leqslant C|Du|^2 \in L^1(\mathbb{R}^n).$$

So, by the Dominated Convergence Theorem we have that

$$\lim_{t\to 0} J^6(t) = \int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx. \tag{4.26}$$

Similarly to (4.11), we have that

$$\lim_{t\to 0} J^7(t) = \int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\,dx\int_{\mathbb{R}^n}u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx. \tag{4.27}$$

Combining (4.24)-(4.27) we have that

$$\begin{aligned}0 &= 2\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta\varphi^i\,dx + \int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx\\ &\quad - \int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\,dx\int_{\mathbb{R}^n}u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx\\ &\quad - 2\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\,dx\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx\\ &\quad + 2\int_{\mathbb{R}^n}h(x)u^i\varphi^i\,dx - 2\int_{\mathbb{R}^n}h(x)|u|^2\,dx\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\,dx\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta\varphi^i\,dx + \tfrac{1}{2}\int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx\\ &\qquad + \int_{\mathbb{R}^n}h(x)u^i\varphi^i\,dx = \lambda\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\,dx\end{aligned} \tag{4.28}$$

for every $\varphi \in L_\infty \cap E$ where

$$\lambda = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + \tfrac{1}{2} u^j D_{u^j} a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx$$

i.e. u is a weak solution of (1.1) with $\|u\|_\infty < \infty$ and Theorem 2.3 is completely proved. $\square$

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