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The Relation Between the Porous Medium and the Eikonal Equations in Several Space Dimensions

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Abstract

We study the relation between the porous medium equation $u_t = \Delta(u^m)$, $m > 1$, and the eikonal equation $v_t = |Dv|^2$. Under quite general assumptions, we prove that the pressure and the interface of the solution of the Cauchy problem for the porous medium equation converge as $m \downarrow 1$ to the viscosity solution and the interface of the Cauchy problem for the eikonal equation. We also address the same questions for the case of the Dirichlet boundary value problem.

Introduction

In this paper we investigate the relation between the porous medium equation

$$u_t = \Delta(u^m), \qquad m > 1, \tag{0.1}$$

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and the eikonal equation

$$v_t = |Dv|^2, \tag{0.2}$$

for appropriate initial and boundary data. Here $Dv = (v_{x_1}, \ldots, v_{x_N})$ denote the spatial gradient of v and Δ is the Laplace operator. The connection be tween these equations is made apparent when we perform the change o variables

$$v = \frac{m}{m-1} u^{m-1} \tag{0.3}$$

which transforms (0.1) into the «pressure» equation

$$v_t = (m-1) v \Delta v + |Dv|^2. \tag{0.4}$$

Letting now $m \downarrow 1$ we formally obtain (0.2).

Equation (0.1) arises naturally as a mathematical model in several areas o applications (e.g. percolation of gas through porous media [33], radiative hea transfer in ionized plasmas [34], thin liquid films spreading under gravity [12] crowd-avoiding population spreading [26], etc.). Equation (0.2), which is special case of a Hamilton-Jacobi equation, is of main interest in optimal cor trol theory [29], the theory of geometrical optics [29], where it describes th propagation of wave fronts [23], etc.

As far as mathematical properties are concerned, (0.1) exhibits bot parabolic and hyperbolic behavior. In particular, at all points where $u > 0$, is smooth. Moreover, it is known ([9], [10]), that the solution u of (0.1 depends continuously in appropriate norms on both the initial data and on m Thus, as $m \downarrow 1$, (0.1) can be regarded as a perturbation of the heat equatior The hyperbolic behavior of (0.1) is manifested by the existence of a finit speed of propagation and the development of interfaces. (For a detaile discussion of the above as well as a complete list of references, see [37]). O the other hand, (0.2) is a hyperbolic equation with only locally defined smoot solutions but with globally defined weak solutions, namely the viscosity solu tions [17]. A common method for approximating viscosity solutions is th method of artificial viscosity [19], [35]. This method, however, does not giv any information whatsoever about the interface of the hyperbolic problem. I order to control the interface one needs to use approximations which exhib interface. In this context a natural question is whether (0.4) can be regarde as a degenerate viscosity (or diffusion) approximation to (0.2).

In [5] D. G. Aronson and J. L. Vázquez explored the convergence, as $m \downarrow 1$ of the solutions of (0.4) to (0.2) in the case of the Cauchy problem in on space dimension. They proved that not only the solutions but also the inter faces of the solutions of (0.4) converge to the solution and the interfac respectively of (0.2), if the initial data are continuous, nonnegative and converge locally uniformly. The proofs rely on estimates that are very particular to the one-dimensional setting.

In this paper we consider the convergence in N space dimensions with general initial data both for the Cauchy problem in $\mathbb{R}^N$ and for the Dirichlet problem in a bounded domain $O \subset \mathbb{R}^N$. For the Cauchy problem we prove that solutions of (0.4) converge to the unique viscosity solutions of (0.2) (Section 1, Theorem 1). Moreover, we show that the positivity sets of solutions of (0.4) converge (in the sense of sets) to the positivity sets of solutions of (0.2) (Section 2, Theorem 2). The main point for the convergence of the solutions is a new type estimates for the gradient. Gradient estimates are easy to obtain in the case of the one space dimension but not obvious at all in higher dimensions (cf. [1], [14], etc.). For the interfaces we also need some new information. This follows from an important result of L. Caffarelli and A. Friedman [13]. In the case of the Dirichlet problem we investigate the convergence of the solutions. We show that the limit takes on natural boundary conditions, thus giving rise to a boundary layer (Section 3, Theorem 3). Finally, the Appendix is a short survey on (0.2). We examine the existence and uniqueness of viscosity solutions under optimal initial conditions as well as some of their properties (e.g. regularity, growth at infinity, interfaces etc.).

1. The Cauchy Problem

Let us consider the following two problems

$$\begin{cases} v_{mt} = (m-1)v_m \Delta v_m + |Dv_m|^2 & \text{in } \mathbb{R}^N \times (0, T_m) \\ v_m = v_{m0} & \text{on } \mathbb{R}^N \times \{t=0\} \end{cases} \tag{1.1}$$

and

$$\begin{cases} v_t = |Dv|^2 & \text{in } \mathbb{R}^N \times (0, T) \\ v = v_0 & \text{on } \mathbb{R}^N \times \{t=0\} \end{cases} \tag{1.2}$$

with nonnegative initial data $v_{m0}, v_0 \in C(\mathbb{R}^N)$. Here T_m and T denote the maximal time of existence for equations (1.1) and (1.2) respectively.

Problem (1.2) has a unique viscosity solution defined in a time interval $(0, T)$ if the initial data satisfy a quadratic growth condition of the form

$$v_0(x) \leqslant a|x|^2 + b \tag{1.3}$$

with $a, b \geqslant 0$. Moreover if

$$\alpha = \limsup_{|x| \to \infty} \frac{v_0(x)}{|x|^2}, \tag{1.4}$$

we have

$$T = 1/4\alpha, \tag{1.5}$$

so that a global solution exists if and only if $\alpha = 0$. On the other han growth condition like (1.3) on v_{m0} ensures the existence of a unique c tinuous weak solution of problem (1.1) ([11], [21]) for a time

$$T_m \geqslant \frac{1}{2[N(m-1)+2]\alpha}. \tag{1.6}$$

Again v_m is global in time if and only if $\alpha = 0$. Observe that $\liminf_{m\to 1} T_m \geqslant$

Our first Theorem states the convergence of solutions of (1.1) to the o of (1.2) as $m \downarrow 1$.

Theorem 1. *Assume that for m close to* 1 *we are given nonnegative ini data* $v_{m0} \in C(\mathbb{R}^N)$ *satisfying* (1.3) *uniformly in m and such that as m* $v_{m0} \to v_0$ *locally uniformly in* $\mathbb{R}^N$. *Let* v_m *and v be the solutions to probl* (1.1) *and* (1.2). *Then* $v_m \to v$ *as* $m \to 1$ *locally uniformly in* $\mathbb{R}^N \times [0, T)$.

The proof of this result relies on obtaining gradient estimates in the c where the solutions are uniformly bounded from below away from zero a series of delicate approximations which use the uniqueness and continu dependence on the initial data of the solutions to problems (1.1) and (1.2). begin with the gradient estimate. We state the result in a generality that be also useful in Section 3, when dealing with problems in bounded domai The proof is based on a variation of Bernstein's trick ([22], [24], [29]).

Lemma 1.1. *For* $m > 1$ *let* v_m *be a smooth solution of the equation*

$$(v_m)_t = (m-1)v_m \Delta v_m + |Dv_m|^2$$

in O, where O is an open subset of $\mathbb{R}^N \times (0, T]$. *Assume that*

$$\beta \geqslant \sup_O v_m \geqslant \inf_O v_m \geqslant \gamma > 0 \tag{1.7}$$

with β *and* γ *independent of m. Then for every compact subset K of O for* $m-1$ *sufficiently small depending on* K, β *and* γ, *there exists a const* $C = C(K, \beta, \gamma)$ *such that*

$$|Dv_m| \leqslant C \quad in \quad K. \tag{1.8}$$

If Dv_{m0} *is locally bounded, the above estimate holds down to* $t = 0$, *i.e.*

PROOF. Let ζ be a cut-off function supported in O such that: $0 \leqslant \zeta \leqslant 1$ and $\zeta \equiv 1$ on K. We consider the function

$$Z = \zeta^2 |Dv|^2 + \lambda v \tag{1.9}$$

where λ is a constant to be chosen later. Here for notational simplicity we have dropped the subscript m from v_m. If Z has a maximum at some point (x_0, t_0) such that $\zeta(x_0, t_0) > 0$, then at (x_0, t_0) we have

$$\begin{aligned}
Z_t &= 2\zeta\zeta_t |Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i t} + \lambda v_t \geqslant 0, \\
Z_{x_k} &= 2\zeta\zeta_{x_k} |Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i x_k} + \lambda v_{x_k} = 0, \qquad & k = 1, \ldots, N, \\
Z_{x_k x_k} &= (2\zeta_{x_k}^2 + 2\zeta\zeta_{x_k x_k}) |Dv|^2 + 4\zeta\zeta_{x_k} v_{x_i} v_{x_i x_k} \\
&\quad + 2\zeta^2 (v_{x_i x_k})^2 + 2\zeta^2 v_{x_i} v_{x_i x_k x_k} + \lambda v_{x_k x_k} \leqslant 0, \qquad & k = 1, \ldots, N,
\end{aligned}$$

and

$$0 \leqslant Z_t - (m-1) v \Delta Z - 2 Dv \cdot DZ.$$

Substituting in the last inequality and using the equation we obtain

$$\begin{aligned}
0 \leqslant\ & 2(m-1)\zeta^2 |Dv|^2 \Delta v - \lambda |Dv|^2 - 2(m-1) v (|D\zeta|^2 + \zeta \Delta \zeta) |Dv|^2 \\
& + 2\zeta\zeta_t |Dv|^2 - 2(m-1) v \zeta^2 v_{x_i x_k}^2 - 4\zeta\zeta_{x_k} v_{x_k} |Dv|^2 \\
& - 8(m-1)\zeta\zeta_{x_k} v v_{x_i} v_{x_i x_k}.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality together with the elementary inequality

$$(\Delta v)^2 \leqslant N \sum_{i,j=1}^{N} \left[\frac{\partial^2 v}{\partial x_i \partial x_j} \right]^2$$

we get

$$\lambda |Dv|^2 \leqslant C\zeta |Dv|^3 + C|Dv|^2 + (m-1)\zeta^2 \left(2|Dv|^2 \Delta v - \frac{\gamma}{N} (\Delta v)^2 \right)$$

where γ is from (1.7) and C stands for a constant which depends only on $\|D\zeta\|_\infty$, $\|\Delta\zeta\|_\infty$, $\|\zeta_t\|_\infty$ and β from (1.7), and may change from line to line. The last inequality can be transformed into

$$\lambda |Dv|^2 \leqslant C\zeta |Dv|^3 + C|Dv|^2 + (m-1)\frac{N}{\gamma} \zeta^2 |Dv|^4$$

with all the functions evaluated at (x_0, t_0). Let

$$\lambda = \mu \left[\max_O (\zeta^2 |Dv|^2) + 1 \right] \tag{1.10}$$

where $\mu > 0$ is to be chosen. Substituing in the above inequality we obt

$$\mu\left[\max_O \zeta^2|Dv|^2 + 1\right]|Dv|^2 \leqslant C\left[\frac{1}{\gamma}(m-1)\left(\max_O \zeta^2|Dv|^2 + 1\right)^{1/2} + 1\right] \cdot \left(\max_O \zeta^2|Dv|^2 + 1\right)^{1/2}|Dv|^2.$$

Now, if $|Dv|^2(x_0, t_0) \neq 0$, then

$$\left(\mu - \frac{(m-1)C}{\gamma}\right)(\max \zeta^2|Dv|^2 + 1)^{1/2} \leqslant C,$$

so that, if $\mu > (m-1)C/\gamma$, we have

$$|Dv|^2 \leqslant \left(\frac{C}{\mu - \dfrac{(m-1)C}{\gamma}}\right)^2 \quad \text{for every} \quad (x,t) \in K. \tag{1.11}$$

On the other hand, if $|Dv|^2(x_0, t_0) = 0$, then by the definitions of Z and $(x_0$
we have

$$\zeta^2(\bar{x}, \bar{t})|Dv|^2(\bar{x}, \bar{t}) \leqslant \lambda\beta \tag{1.12}$$

where

$$\zeta^2(\bar{x}, \bar{t})|Dv|^2(\bar{x}, \bar{t}) = \max_O \zeta^2|Dv|^2.$$

Using (1.10) and (1.12) we get

$$|Dv|^2(x, t) \leqslant \frac{\beta\mu}{1 - \mu\beta} \quad \text{for every} \quad (x, t) \in K, \tag{1.13}$$

provided that $1 - \mu\beta > 0$. Choosing μ such that $\mu\beta = 1/2$, then for $m - 1$ ficiently small we have $\mu > (m-1)C/\gamma$, therefore the result follows in the where O is a subset of $\mathbb{R}^N \times (0, T)$. If O intersects the set $\mathbb{R}^N \times \{0\}$ the m mum of Z may take place at $t_0 = 0$. In that case we obtain a local bound $|Dv|$ depending only on β, λ and the sup of $|Dv_{m0}|$ on $K \cap (\mathbb{R}^N \times \{0\})$.

Proof of Theorem.
Step 1. We assume that $0 < \gamma \leqslant v_{m0}(x)$ with γ independent of m. It t follows from known properties of the porous medium equation [1], [11] t for every m, $v_m(x, t) \geqslant \gamma$, $v_m \in C^\infty(\mathbb{R}^N \times (0, T_m))$ and the v_m's are loc bounded in $\mathbb{R}^N \times [0, T_m)$ uniformly in m. Therefore we can apply Lemma on any compact subset K of $\mathbb{R}^N \times (0, T)$ and obtain a bound for $|Dv_m|$ o that is uniform in m for m sufficiently close to 1. By [25] it follows that

v_m's are also locally Hölder-continuous in t with exponent $1/2$ and coefficient independent of m if m is again sufficiently close to 1. The family $\{v_m\}_{m>1}$ is therefore relatively compact in $C(K)$. By a standard diagonal process we may extract a subsequence from every sequence $m_n \to 1$, which we again denote by m_n for simplicity, such that the v_{m_n}'s converge locally uniformly in $\mathbb{R}^N \times (0, T)$ to a function $v \in C(\mathbb{R}^N \times (0, T))$, which is locally Lipschitz continuous in x, Hölder continuous with exponent $1/2$ in t and a viscosity solution of (0.2) (cf. [15], [17]).

If, moreover, $Dv_{m0} \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ uniformly in m, then the gradient estimates hold in compact subsets of $\mathbb{R}^N \times [0, T_m)$ and the same argument implies that the convergence $v_{m_n} \to v$ holds locally uniformly in $\mathbb{R}^N \times [0, T)$. Since $v_{m0} \to v_0$ locally in $\mathbb{R}^N$ we conclude that $v \in C(\mathbb{R}^N \times [0, T))$ takes on the initial value v_0. Therefore, in view of Theorem A.1 of the Appendix, v is the unique viscosity solution of problem (1.2) and the whole family $\{v_m\}_{m>1}$ converges to v.

To prove that v is continuous down to $t = 0$ and $v(x, 0) = v_0(x)$ for $x \in \mathbb{R}^N$ in the case where we do not have a control on $|Dv_{m0}|$ we proceed by approximation. Indeed, we approximate v_{m0} by sequences $\{v^n_{m0}\}$, $\{v_{m0,n}\}$ such that:

(i) the functions v^n_{m0} and $v_{m0,n}$ are smooth in $\mathbb{R}^N$, and for fixed n the gradients are locally bounded in $\mathbb{R}^N$ uniformly in m.

(ii) for each fixed m we have the monotone convergence $v^n_{m0} \downarrow v_{m0}$ and $v_{m0,n} \uparrow v_{m0}$ uniformly in m and $x \in \mathbb{R}^N$.

(Such approximations can be easily obtained by partition of unity and convolution with a smooth kernel).

We conclude as follows: For each fixed n, v^n_{m0} and $v_{m0,n}$ converge along subsequences to some functions v^n_0 and $v_{0,n}$ respectively, which have gradients locally bounded in $\mathbb{R}^N$ and converge, as $n \to \infty$, locally uniformly to v_0. The ordering properties of the porous medium equation imply that $v^n_m \geqslant v_m \geqslant v_{m,n}$ in $\mathbb{R}^N \times [0, T_m)$ where v^n_m and $v_{m,n}$ are the solutions of problem (1.1) in $\mathbb{R}^N \times [0, T_m)$ with initial data v^n_{m0} and $v_{m0,n}$ respectively. The argument above then implies that for each fixed n, as $m \downarrow 1$, $v^n_m \to v^n$ and $v_{m,n} \to v_n$ locally uniformly in $\mathbb{R}^N \times [0, T)$ where v^n, v_n are the unique viscosity solutions of problem (1.2) in $\mathbb{R}^N \times [0, T]$ with initial data v^n_0 and $v_{0,n}$ respectively. Moreover,

$$v^n \geqslant \overline{\lim_{m\downarrow 1}}\, v_m \geqslant \underline{\lim_{m\downarrow 1}}\, v_m \geqslant v_n \quad \text{in} \quad \mathbb{R}^N \times [0, T).$$

Letting $n \to \infty$ and using the uniqueness result of Theorem A.1 we obtain

$$\lim_{n\to\infty} v^n = \lim_{n\to\infty} v_n = v$$

where $v \in C(\mathbb{R}^N \times [0, T])$ is the unique viscosity solution of (1.2) in $\mathbb{R}^N \times [0, T)$. The result follows.

Step 2. The general case. Let

$$v_{m0}^n = v_{m0} + 1/n.$$

If v_m^n is the solution of (0.1) in $\mathbb{R}^N \times [0, T_m)$, then the maximum princi yields $v_m \leqslant v_m^n$ in $\mathbb{R}^N \times [0, T)$. Using Step 1 and Theorem A.1 we get

$$\overline{\lim_{m \to 1}}\, v_m \leqslant v$$

uniformly on compact subsets of $\mathbb{R}^N \times [0, T)$. To conclude, we need to estab the inequality

$$\text{(1.14)} \qquad \underline{\lim_{m \to 1}}\, v_m \geqslant v \quad \text{locally uniformly in} \quad \mathbb{R}^N \times [0, T).$$

We first prove (1.14) in the case where the v_{m0}'s satisfy the inequalities

$$0 \leqslant v_{m0} \leqslant C \quad \text{and} \quad \Delta v_{m0} \geqslant -C \quad \text{in} \quad \mathbb{R}^N$$

where C is a constant independent of m.

Let v_m^n be the solution of (1.1) with initial data $v_{m0} + 1/n$. Tl $v_m^n \in C^\infty(\mathbb{R}^N \times [0, T_m))$, $0 < v_m^n \leqslant C + 1/n$ and $\Delta v_m^n \geqslant -C$ in $\mathbb{R}^N \times [0, 7$ Using (1.1) we see that the function

$$w_m^n = v_m^n + C\left(C + \frac{1}{n}\right)(m-1)t$$

is a smooth solution of

$$\begin{cases} w_{mt}^n \geqslant |Dw_m^n|^2 & \text{in} \quad \mathbb{R}^N \times [0, T_m) \\ w_m^n \geqslant v_{m0} & \text{on} \quad \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

It then follows that $w_m^n \geqslant V_m$, the solution of (1.2) with initial data $v_{m($ $\mathbb{R}^N \times [0, T_m)$. Now we let $n \to \infty$. The continuous dependence of the soluti of the porous medium equation on the initial data ([11]) yields

$$v_m(x, t) + C^2(m-1)t \geqslant v(x, t).$$

Letting $m \downarrow 1$ and using Proposition A.10 we obtain (1.14).

Next we prove (1.14) under only the assumption that v_0 is bounded. $\delta > 0$. We can find functions $\tilde{v}_0$, $\tilde{v}_{m0}$ bounded in $W^{2,\infty}(\mathbb{R}^N)$ uniformly in m ; such that $\tilde{v}_0, \tilde{v}_{m0} \geqslant 0$, $\tilde{v}_{m0} \leqslant v_{m0}$, $\tilde{v}_{m0} \geqslant \tilde{v}_0$ in $\mathbb{R}^N$ and $\tilde{v}_0(x) \geqslant v_0(x) - \delta$ in ! Then

$$\underline{\lim_{m \downarrow 1}}\, v_m \geqslant \underline{\lim_{m \downarrow 1}}\, \tilde{v}_m \geqslant \tilde{v} \quad \text{locally uniformly in} \quad \mathbb{R}^N \times [0, T).$$

Since $v - \delta$ is a solution of (1.2) with data $v_0 - \delta \leqslant \tilde{v}_0$ we have $v - \delta \leqslant \tilde{v}$. Letting $\delta \to 0$ we conclude (1.14).

For the general unbounded case we truncate the initial data at height n. If v_m^n and v^n are the solutions of (1.1) and (1.2) with the truncated initial data, the above and the maximum principle yield

$$\underline{\lim}_{m \to 1} v_m \geqslant \underline{\lim}_{m \to 1} v_m^n \geqslant v^n \quad \text{locally uniformly in } \mathbb{R}^N \times [0, T).$$

Letting $n \to \infty$ we obtain $v^n \to v$. The result follows. □

We continue with a remark concerning Lemma 1.1. In fact gradient estimates can be obtained in a similar way for general classes of equations like for instance

$$u_t^\epsilon - \epsilon F(x, t, u^\epsilon, Du^\epsilon, D^2 u^\epsilon) + H(x, t, u^\epsilon, Du^\epsilon) = 0 \tag{1.15}$$

under suitable assumptions on F and H and provided that the family of smooth solutions $\{u^\epsilon\}_{\epsilon > 0}$ is locally bounded from above and below away from zero uniformly in ϵ. Such bounds allow to pass to the limit $\epsilon \to 0$ and thus obtain viscosity solutions of the limit problem

$$u_t + H(x, t, u, Du) = 0. \tag{1.16}$$

General equations of the form (1.15) have a certain usefulness. For instance, in some numerical codes the approximation of shocks is improved with the addition of some nonlinear artificial viscosity (so called numerical viscosity, cf. [32]). The assumptions that one has to make on F and H are rather cumbersome although quite general. We leave it to the reader to fill in the details in particular applications.

Our next remark deals with an alternative and simpler proof of the gradient estimate of Lemma 1.1. Though it needs stronger assumpions on the initial data, it can be of interest for some applications.

Lemma 1.2. *Assume that for every $m > 1$ the continuous functions v_{m0} satisfy $0 < \gamma \leqslant v_{m0} \leqslant \beta$ and $|Dv_{m0}| \leqslant M_0$ where β, γ and M_0 are positive constants. Then there exists a bound for $|Dv_m|$ of the form*

$$|Dv_m(x, t)|^2 \leqslant \frac{M_0}{1 - (t/T_m)} \quad \textit{for} \quad 0 \leqslant t < T_m \tag{1.17}$$

where

$$T \;=\; \frac{2\gamma}{\qquad}.$$

PROOF. Let $w_m = |Dv_m|^2$. Using equation (1.1) we obtain

$$\begin{aligned} w_t - (m-1)v\,\Delta w - 2Dv\cdot Dw &= 2(m-1)w\,\Delta v - 2(m-1)v \sum_{i,j=1}^{N}\left(\frac{\partial^2 v}{\partial x_i\,\partial x_j}\right. \\ &\leqslant 2(m-1)w\,\Delta v - \frac{2(m-1)\gamma}{N}(\Delta v)^2 \\ &\leqslant \frac{(m-1)N}{2\gamma}w^2, \end{aligned}$$

where we have dropped the m's for simplicity and have used the inequa

$$(\Delta v)^2 \leqslant N\sum_{i,j=1}^{N}\left(\frac{\partial^2 v}{\partial x_i\,\partial x_j}\right)^2.$$

We compare w_m to the explicit solution

$$W_m(t) = \frac{M_0 C}{C-t}, \qquad 0<t<C=\frac{2\gamma}{M_0 N(m-1)}$$

of the problem

$$\begin{cases} W' = \dfrac{N(m-1)}{2\gamma}W^2 \\ W(0)=M_0 \end{cases}$$

The maximum principle implies that $w_m \leqslant W_m$ in $\mathbb{R}^N\times[0,C)$. If γ,β and do not depend on m, then $C\uparrow\infty$ as $m\downarrow 1$. Thus we obtain a bound for $|D$ on bounded time intervals which is uniform in m for m close to 1. □

We conclude with a further remark about a sharper gradient estimate solutions of (1.1) under the assumption that $0<\gamma\leqslant v_{m0}\leqslant\beta$. Indeed a resul Ph. Bénilan [8] implies that if u_m is a solution of (0.1) and $m\leqslant 1+(N-1)$ then there exists a gradient bound of the form

(1.18) $$|D(u_m)^{m-1/2}| \leqslant Ct^{-1/2}$$

where C depends on β but not on m. If $v_{m0}\geqslant\gamma>0$, we conclude that

$$|Dv_m|^2 \leqslant \tilde{c}_1 t^{-1}$$

where $\tilde{c}_1$ depends on β and γ.

2. Convergence of Interfaces

In this section we prove that under the assumptions of Theorem 1 the interf of the solution of problem (1.1) converges to the interface of the solution of (1.2) as $m \downarrow 1$. The interfaces are described in terms of the functions

$$S_m(x) = \inf\{t \geqslant 0 : v_m(x, t) > 0\}, \tag{2.1}$$

and

$$S(x) = \inf\{t \geqslant 0 : v(x, t) > 0\}, \tag{2.2}$$

The so-called retention property implies that $v_m(x, t) > 0$ for every $t > S_m(x)$. On the other hand it is proved in [12] that, if v_{m0} has compact support, the function S_m is Hölder continuous in the open set

$$A_m = \mathbb{R}^N \setminus \bar{\Omega}_m(0), \tag{2.3}$$

where for $m > 1$ and $t \geqslant 0$, $\Omega_m(t) = \{x \in \mathbb{R}^N : v_m(x, t) > 0\}$. The above restriction is essential in view of the fact that S_m is discontinuous at points of the boundary of $\Omega_m(0)$ whenever a positive waiting time occurs. This phenomenon may appear even in one space dimension, (cf. [3], [4]). The restriction of compact support in [13] is inessential. The interface to problem (1.2) has similar properties as we show in the Appendix.

In view of these observations we prove the convergence of S_m to S away from the initial sets $\Omega_m(0)$. More precisely, let

$$A = \mathbb{R}^N \Big/ \bigcap_{\epsilon > 0} \operatorname{closure}\Big(\bigcup_{1 < m < 1 + \epsilon} \Omega_m(0) \Big) \tag{2.4}$$

The set A consists of points $x \in \mathbb{R}^N$ such that for some $\epsilon > 0$ and $r > 0$ and all $1 < m < 1 + \epsilon$, v_{m0} vanishes dentically on $B(x, r)$, the open ball centered at x with radius r.

Theorem 2. *S_m converges to S as $m \downarrow 1$ uniformly on compact subsets of A.*

PROOF. Let K be a compact subset of A and suppose that S_m does not converge uniformly to S on K. Then there exist $\epsilon > 0$, $x_n \in K$ and $m_n \downarrow 1$ such that

$$|S_{m_n}(x_n) - S(x_n)| \geqslant \epsilon. \tag{2.5}$$

Suppose first (upon passing to a subsequence if necessary) that we have

$$S_{m_n}(x_n) \geqslant S(x_n) + \epsilon. \tag{2.6.a}$$

A contradiction follows then easily from the uniform convergence of v_m to v on K and the continuity of S (see Appendix). In fact the definition of S_m (2.6.a) implies

$$v_{m_n}(x_n, S(x_n) + \epsilon) = 0,$$

so that for any limit point $\bar{x}$ of $\{x_n\}$ we have

$$v(\bar{x}, S(\bar{x}) + \epsilon) = 0,$$

against the definition of S.

Let us now discuss the case where, as $m_n \downarrow 1$,

$$S_{m_n}(x_n) \leqslant S(x_n) - \epsilon. \tag{2.6.b}$$

This case is significantly more difficult. To exclude it, we need to use a precis information about the growth of solutions and interfaces of (1.1) based on th results of [13]. This information is summarized in the following lemmata.

Lemma 2.1. *Let K be a compact set where $v_{m0} = 0$, $1 < m < 1 + \epsilon \leqslant 2$. The for every compact set $K' \subset K$ there exists a $\tau > 0$ depending only on K, K' bu not on m or ϵ, such that for every $m \in (1, 1 + \epsilon)$*

$$v_m \equiv 0 \quad on \quad K' \times [0, \tau].$$

Lemma 2.2. *Let $0 < \tau < t_0$, $x_0 \in \mathbb{R}^N$ and $R_0 > 0$. There exist $\delta = \delta(\tau, R_0) >$ and $R = R(\tau, t_0, R_0) > 0$ such that whenever $v_m(\bullet, \tau) \equiv 0$ on $\overline{B(x_0, R_0)}$ an $v_m \leqslant \delta$ on $\overline{B(x_0, R_0)} \times [\tau, t_0]$, then $v_m(\bullet, t_0) \equiv 0$ on $\overline{B(x_0, R)}$.*

We postpone the proof of the lemmata to the end of the section and cor tinue with the proof of Theorem 2. Let $\bar{x}$ be a limit point of $\{x_n\}$. The propei ties of S (cf. Appendix) imply that there exists $t_0 \geqslant S(\bar{x}) - \epsilon/2$ and $r > 0$ suc that $v(\bullet, t_0) \equiv 0$ on $B(\bar{x}, r)$ and, since $v_t \geqslant 0$ by (1.2),

$$v \equiv 0 \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0].$$

On the other hand, Lemma 2.1 yields the existence of a $\tau = \tau(\bar{x}, r) < t_0$ suc that, for m_n sufficiently close to 1,

$$v_{m_n} \equiv 0 \quad \text{on} \quad B\left(\bar{x}, \frac{r}{2}\right) \times \{\tau\}.$$

Finally, in view of Theorem 1 and the above there exists n_0 such that for $n \geqslant r$

$$v_{m_n} \leqslant \delta \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0],$$

where $\delta = \delta(\tau, r)$ is given by Lemma 2.2. Since all the assumptions of Lemm 2.2 are satisfied, we obtain

$$v_{m_n}(\bullet, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x, r')},$$

where $r' = r'(\tau, t_0, r)$. This contradicts (2.6.b). $\square$

We continue with a discussion of the convergence in the complement of the set A. Firstly, for every $\bar{x} \in \Omega(0)$ we have $v_0(x) > 0$ and $v_{m0}(x) > 0$ for m near 1 and x near $\bar{x}$. Therefore $S(x) = S_m(x) = 0$; i.e., $S_m \to S$ locally uniformly in $\Omega(0)$. In the case where $\Omega_m(0) \subset \bar{\Omega}(0)$ for all m near 1, then $A \supset \mathbb{R}^N \setminus \overline{\Omega(0)}$. So the only place where the convergence may fail is the boundary of $\Omega(0)$. It is easy to construct examples with waiting times where this happens. Finally, we cannot expect convergence on the set

$$B = \limsup_{m \downarrow 1} \Omega_m(0) \setminus \overline{\Omega(0)}.$$

In fact for each $x \in B$ there exists a subsequence $m_n \downarrow 1$ such that $v_{m_n 0}(x) > 0$ and $S_{m_n}(x) = 0$. However, $S(x) > 0$. In particular, it may happen that $\Omega_m(0) = \mathbb{R}^N$ for every $m > 1$ so that $B = \mathbb{R}^N \setminus \overline{\Omega(0)}$ and the only convergence that we get is the trivial convergence on $\Omega(0)$.

We next formulate the convergence of the interfaces in terms of the positivity sets $\Omega_m(t)$ and $\Omega(t)$. Since the proof of this result is only a variation of the proof of Theorem 2 we leave it up to the reader to fill in the details.

Theorem 2′. *Under the assumptions of Theorem* 1 *we have*

(i) $\liminf_{m \downarrow 1} \Omega_m(t) \supset \Omega(t)$

(ii) $\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t) \cup (\mathbb{R}^N \setminus A)$.

As explained above an inclusion of the type

$$\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t)$$

cannot be true in general. It may happen e.g. that $\Omega_m(t) = \mathbb{R}^N$ for every $m > 1$, $t > 0$, while $\Omega(t)$ is bounded.

We conclude with the proof of the lemmata.

PROOF OF LEMMA 2.1. Without any loss of generality we may assume that $K = \overline{B(0, R_1)}$ and $K' = \overline{B(0, R)}$ with $R < R_1$.

We proceed by constructing a barrier function $V: B(0, R_1) \times [0, \tau] \to \mathbb{R}$ for an appropriate choice of τ. It is given by the formula

$$V(r, t) = \lambda[a^2 t + a(r - R - \theta)]^+ \tag{2.7}$$

where $r = |x|$, $s^+ = \max(s, 0)$, $R < R + \theta < R_1$ and $a, \lambda > 0$. We choose a, λ

(i) Whenever $V > 0$, V satisfies

$$V_t \geqslant (m-1)V\Delta V + |DV|^2 \tag{2.8}$$

(ii) $V(r, \tau) = 0$ for $r \leqslant R$.
(iii) $V(R_1, t) \geqslant A$ for $t \in [0, \tau]$, where A is the L^∞-bound of v_m o $K \times [0, T-\epsilon]$, which in view of the proof of Theorem 1, is indepe dent of m.

If all the above are satisfied, setting

$$U = \left(\frac{m-1}{m} V\right)^{1/(m-1)}, \qquad u = \left(\frac{m-1}{m} v_m\right)^{1/(m-1)}$$

we have $U_t \geqslant \Delta U^m$, $u_t = \Delta u^m$ in $B(0, R_1) \times (0, \tau]$ and $U \geqslant u$ on the parabol boundary of $B(0, R_1) \times (0, \tau]$. By the standard comparison principle for t porous medium equation, it follows that $U \geqslant u$ throughout $B(0, R_1) \times [0, \tau$ hence, in particular, $v(x, \tau) \leqslant V(|x|, \tau) = 0$ if $|x| < R$ and thus the result.

We conclude by establishing (i), (ii) and (iii) above. We begin by observi that $V > 0$ if and only if

$$r \geqslant R + \theta - at. \tag{2.9}$$

To satisfy (ii) it suffices to have

$$a\tau \leqslant \theta \tag{2.10}$$

For (iii) we need

$$\lambda[a^2 t + a(R_1 - (R+\theta))] \geqslant A$$

which requires

$$\lambda a \geqslant \frac{A}{R_1 - (R+\theta)}. \tag{2.11}$$

Finally, V satisfies (2.8), whenever $V > 0$, if and only if

$$\lambda\left[(m-1)(N-1)\frac{at + r - R - \theta}{r} + 1\right] \leqslant 1.$$

For the latter to be satisfied, in view of (2.9) and (2.10), it suffices to ha

$$\lambda \leqslant \frac{1}{1 + N(1 - (R/R_1))}. \tag{2.12}$$

To conclude we choose λ so that the inequality holds in (2.12). Then for a su ficiently large and τ sufficiently small (2.10) and (2.11) can be achieved. ∎

For the proof of Lemma 2.2 we need the following result.

Lemma 2.3 [13]. *For any $\tau > 0$ and $m > 1$ there exist positive constants η, c depending only on m, N and τ such that the following is true: Let $t_0 > \tau$, $R > 0$, $0 < \sigma < \eta$. If*

$$v_m(\cdot, t_0) \equiv 0 \quad on \quad B(x_0; R) \qquad x_0 \in \mathbb{R}^N \tag{2.13}$$

and

$$\fint_{B(x_0, R)} v_m(x, t_0 + \sigma)\, dx \leqslant \frac{cR^2}{\sigma}, \tag{2.14}$$

then

$$v_m(\cdot, t_0 + \sigma) \equiv 0 \quad on \quad B(x_0, R/6), \tag{2.15}$$

where $\fint_{B(x_0, R)} v_m(x, s)\, dx$ denotes the average of $v_m(\cdot, s)$ over the ball $B(x_0, R)$.

A careful scrutiny of the proof shows that the constants *do not depend* on m in the range $1 < m < 2$.

PROOF OF LEMMA 2.2. Let η, c be the constants which correspond to τ via Lemma 2.3, let M be so large that $\sigma = (t_0 - \tau)/M < \eta$ and let $\delta > 0$ be such that $\delta < c\, 6^{-2(M-1)} R_0^2/\sigma$. For every $i = 1, \ldots, M$, we then have

$$\fint_{B(x_0, 6^{-(i-1)} R_0)} v_m(x_0, \tau + i\sigma)\, dx \leqslant \frac{c}{\sigma} 6^{-2(i-1)} R_0^2.$$

Using Lemma 2.3 and arguing inductively we obtain

$$v_m(\cdot, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x_0, 6^{-M} R_0)}. \quad \square$$

3. The Initial-Boundary Value Problem

Here we focus our attention to the initial-boundary value problems

$$\begin{cases} v_{mt} = (m-1) v_m \Delta v_m + |Dv_m|^2 & \text{in} \quad O \times (0, T] \\ v_m = 0 & \text{on} \quad \partial O \times [0, T] \\ v_m = v_{m0} & \text{on} \quad O \times \{t = 0\} \end{cases} \tag{3.1}$$

and

$$\begin{cases} v_t = |Dv|^2 & \text{in} \quad O \times (0, T] \\ v = 0 & \text{on} \quad \partial O \times [0, T] \\ v = v_0 & \text{on} \quad O \times \{t = 0\}. \end{cases} \tag{3.2}$$

Problem (3.2) does not have a globally defined viscosity solution which is c tinuous up to the boundary. There exists, however, a *minimal* viscosity sc tion v (the value function of the underlying control problem) which assuı some natural boundary conditions, not necessarily zero ([27]). We begin discussing this minimal viscosity solution. To make some of the formı clearer, we will occasionally refer to their form when O is convex. To this e for $x, y \in O$ and $t > 0$ we define

$$(3.3)\quad L(x,y,t) = \inf\left\{\int_0^t \frac{1}{4}|\xi_s|^2\,ds\colon \xi(0) = x,\ \xi(t) = y,\ \xi(s) \in \bar{\Omega} \text{ for } s \in [0, t\right.$$

If O is convex, then it is easy to see that

$$(3.4)\qquad L(x, y, t) = \frac{|x-y|^2}{4t}.$$

In order to have a viscosity solution of (3.2) which is continuous up to boundary, one needs certain compatibility conditions which restrict the c of allowed initial data and the time of existence [29]. In particular, to h a viscosity solution $v \in C(\bar{O} \times [0, T])$ of the problem

$$(3.5)\qquad \begin{cases} v_t = |Dv|^2 & \text{in} \quad O \times (0, T] \\ v = \phi & \text{on} \quad \partial O \times (0, T] \\ v = v_0 & \text{on} \quad O \times \{t = 0\} \end{cases}$$

we need to assume

$$(3.6)\qquad \begin{cases} \phi(x, t) \geqslant \phi(y, s) - L(x, y, t-s) & \text{for all } x, y \in \partial O,\ t \geqslant s > 0 \\ \phi(x, t) \geqslant v_0(y) - L(x, y, t) & \text{for all } x \in \partial O,\ t > 0 \text{ and } y \in (\end{cases}$$

Next we define

$$(3.7)\qquad v(x, t) = \sup_{y \in \bar{O}} \{v_0(y) - L(x, y, t)\}.$$

Arguments similar to the ones of [29, Chapter 11] yield that v is a visco solution of $v_t = |Dv|^2$ in $O \times (0, \infty)$. More precisely, (cf. [29]) v is minimum element of the set of Lipschitz-continuous solutions of

$$(3.8)\qquad \begin{cases} v_t - |Dv|^2 = 0 & \text{in} \quad O \times (0, \infty), \\ v \geqslant 0 & \text{on} \quad \partial O \times (0, \infty), \\ v = v_0 & \text{on} \quad O \times \{t = 0\}. \end{cases}$$

Moreover, on $\partial O \times (0, \infty)$ $v = \tilde{\phi}$, where $\tilde{\phi}$ is the minimum element of the of functions $\psi \in C([\partial O \times (0, \infty)] \cup [O \times \{t = 0\}])$ satisfying (3.6) and ψ on $\partial O \times [0, \infty)$.

In the case when O is convex and $\phi = 0$, (3.4) and (3.6) yield

$$v_0(x) \leqslant \frac{1}{4T} \operatorname{dist}(x, \partial O)^2. \tag{3.9}$$

Then v defined by (3.7) satisfies $v = 0$ on $\partial O \times [0, T]$ and it is the unique viscosity solution of (3.2) in $O \times (0, T)$. The maximal time T for which such a solution exists is given by

$$T^* = \inf_{x \in \bar{O}} \frac{\operatorname{dist}(x, \partial O)^2}{4v_0(x)}. \tag{3.10}$$

We remark that this is precisely the waiting time T for the interface of the Cauchy problem in $\mathbb{R}^N$ (cf. Proposition A.15. See also (1.5)). In general we say that v is the minimal viscosity solution to (3.2).

The relation between (3.1) and (3.2) in the interior of $\bar{O} \times [0, T]$ is the same as the one of (1.1) and (1.2). At the boundary, however, *boundary layers* appear. This is due to the fact that although we are forcing Dirichlet data on (3.1), the solution of (3.2) takes on natural boundary values as explained above.

Theorem. 3. *Assume that $v_{m0} \to v_0$ uniformly on $\bar{O}$ as $m \downarrow 1$ and let v be the viscosity solution of* (3.2) *in $O \times [0, \infty)$ given by* (3.7) *above. Then, as $m \downarrow 1$, $v_m \to v$ locally uniformly in $O \times [0, \infty)$.*

PROOF.

Step 1. We begin by assuming that $v_0 > 0$ in O. We may also assume that the v_{m0}'s are Lipschitz continuous with gradients bounded uniformly in m; the general case follows by approximating v_{m0} from above and below by Lipschitz-continuous functions much as in Theorem 1. Let $B(0, R)$ be a ball strictly included in O, and let $\overline{B(0, R_1)} \subset O$ for some $R_1 > R$. Since $v_{m0} \to v_0$ uniformly on $\overline{B(0, R_1)}$ as $m \downarrow 1$ and $v_0 > 0$ in O, there exist $m_0 = m_0(R_1)$ and $\gamma > 0$ such that

$$\min_{\overline{B(0, R_1)}} v_{m0} > \gamma > 0 \quad \text{for} \quad m < m_0.$$

We claim that for every $T > 0$ there exist $m_1 = m_1(T) > 1$ and $\beta = \beta(R, R_1) > 0$ such that for $m < m_1$

$$v_m \geqslant \beta \quad \text{on} \quad \overline{B(0, R)} \times [0, T]. \tag{3.11}$$

Indeed we consider the similarity solutions

$$V_m(x, t; a, \tau) = \frac{1}{2[N(m-1)+2]} \frac{1}{(t+\tau)} (a^2(t+\tau)^{2/(N(m-1)+2)} - |x|^2)^+$$

of $v_t = (m-1)v\Delta v + |Dv|^2$ in $\mathbb{R}^N \times (0,\infty)$, where $a, \tau > 0$. We choose a and τ so that for m near 1 we have

$$\begin{cases} \operatorname{supp} V_m(\bullet, t) \subset B(0, R_1) & \text{for } t \in [0, T] \\ \overline{B(0,R)} \subset \operatorname{supp} V_m(\bullet, 0) \\ V_m \leqslant \gamma \quad \text{on } \overline{B(0,R_1)} \times \{t=0\}. \end{cases}$$

But then

$$v_m \geqslant V_m \quad \text{on} \quad (\{|x| = R_1\} \times [0, T]) \cup (\{|x| \leqslant R_1\} \times \{t = 0\}),$$

therefore

$$v_m \geqslant V_m \quad \text{on} \quad B(0, R_1) \times [0, T].$$

We conclude by observing that there exists a constant $\beta > 0$, independen m, such that

$$\min V_m \geqslant \frac{1}{2[N(m-1)+2]}\,\frac{1}{T+\tau}\,(a^2\tau^{2/(N(m-1)+2)} - R^2) \geqslant \beta.$$

Using Lemma 1.1 and the results fo [25], we obtain that, along subseque $m \downarrow 1$, $v_m \to \bar{v} \geqslant 0$ locally uniformly in $O \times [0, \infty)$, where $\bar{v}$ is a viscosity solu of (3.8). Since the function v given by (3.7) is the minimal viscosity solu of (3.8), we immediately have $\bar{v} \geqslant v$.

For the other inequality, $v \geqslant \bar{v}$, we use several approximations. To this (let O^δ be the δ-neighborhood of O defined by

$$O^\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \bar{O}) \leqslant \delta\}. \tag{3.12}$$

We extend v_{m0}, v_0 to be zero in $O^\delta \setminus O$ and we denote by $\tilde{v}_{m0}$, $\tilde{v}_0$ their ex sions respectively. Let $\underline{v}_m^\delta$ be the minimal viscosity solution of

$$\begin{cases} \underline{v}_{mt}^\delta = |D\underline{v}_m^\delta|^2 & \text{in} \quad O^\delta \times (0, \infty) \\ \underline{v}_m^\delta = \delta & \text{on} \quad \partial O^\delta \times (0, \infty) \\ \underline{v}_m^\delta = \tilde{v}_{m0} + \delta & \text{on} \quad O^\delta \times \{t = 0\}, \end{cases} \tag{3.13}$$

and define the function $w\colon O^{\delta/2} \times [0, \infty) \to \mathbb{R}$

$$w = \underline{v}_m^\delta * \rho_\alpha = (\underline{v}_m^\delta)_\alpha \tag{3.14}$$

where $*$ denotes the standard convolution with a smooth kernel ρ_α. immediate that, for $\alpha < \alpha_0 = \alpha_0(\delta)$, w satisfies

$$\begin{cases} w_t \geqslant (m-1)w\Delta w - (m-1)C_\alpha w & \text{in} \quad O \times (0, T) \\ w \geqslant v_{m0} & \text{on} \quad O \times \{t = 0\} \\ w > 0 & \text{on} \quad \partial O \times (0, \infty), \end{cases}$$

where C_α is such that $\Delta w \leqslant C_\alpha$ on $O \times [0, T]$. Next for $\mu > 0$ we define

$$z = e^{\mu t} w\left(x, \frac{e^{\mu t} - 1}{\mu}\right).$$

A simple calculation shows that, for $\alpha < \alpha_0(\delta)$, z satisfies

$$z_t - (m-1)z\,\Delta z - |Dz|^2 \geqslant (\mu - (m-1)C_\alpha e^{\mu t})z.$$

For any $T > 0$ choose m_1 so small that $(m-1)C_\alpha e^T < 1$ for $m < m_1$. Then there exists $\mu = \mu(\alpha, m_1)$ such that $\mu \geqslant (m-1)C_\alpha e^{\mu T}$. Using the standard comparison argument for the porous medium equation we obtain

$$v_m \leqslant e^{\mu t}(\underline{v}_m^\delta)_\alpha\left(x, \frac{e^{\mu t} - 1}{\mu}\right) \quad \text{in} \quad O \times [0, T].$$

Now we let $m \downarrow 1$ keeping μ, α, δ fixed and we get

$$\bar{v} \leqslant e^{\mu t}(\underline{v}^\delta)_\alpha\left(x, \frac{e^{\mu t} - 1}{\mu}\right) \quad \text{in} \quad O \times [0, T], \tag{3.15}$$

where $\underline{v}^\delta$ is the minimal viscosity solution of

$$\begin{cases} \underline{v}_t^\delta = |D\underline{v}^\delta|^2 & \text{in} \quad O^\delta \times [0, \infty) \\ \underline{v}^\delta = \delta & \text{on} \quad \partial O^\delta \times [0, \infty), \\ \underline{v}^\delta = \tilde{v}_0 + \delta & \text{on} \quad O^\delta \times \{t = 0\}. \end{cases}$$

Sending $\mu \to 0$, $\alpha \to 0$ and $\delta \to 0$ we obtain the following sequence of inequalities in $O \times [0, T]$

$$\bar{v} \leqslant (\underline{v}^\delta)_\alpha, \qquad \bar{v} \leqslant \underline{v}^\delta, \qquad \bar{v} \leqslant v.$$

The result follows.

Step 2. We next consider the general case where $v_0 \geqslant 0$ in $\bar{O}$. The main problem here is that, since we cannot bound the v_m's from below away from zero, we are unable to obtain local Lipschitz estimates. To circumvent this difficulty (i.e. the apparent lack of estimates), we will employ some of the recent ideas of H. Ishii [27] and G. Barles and B. Perthame [7]. To this end, we define the functions

$$v_*(x, t) = \liminf_{\substack{m \downarrow 1 \\ (y,s) \to (x,t)}} v_m(y, s)$$

and

$$v^*(x, t) = \limsup_{\substack{m \downarrow 1 \\ (y,s) \to (x,t)}} v_m(y, s).$$

It is known ([7]) that v_* is a lower-semicontinuous viscosity solution of

$$(3.16)\quad \begin{cases} v_{*t} - |Dv_*|^2 \geqslant 0 & \text{in} \quad O \times (0, \infty) \\ \max(v_{*t} - |Dv_*|^2, v_* - \psi) \geqslant 0 & \text{on} \quad [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\} \end{cases}$$

and v^* is an upper semicontinuous viscosity solution of

$$(3.17)\quad \begin{cases} v^*_t - |Dv^*|^2 \leqslant 0 & \text{in} \quad O \times (0, \infty) \\ \min(v^*_t - |Dv^*|^2, v^* - \psi) \leqslant 0 & \text{on} \quad [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\} \end{cases}$$

where $\psi: [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\}] \to \mathbb{R}$ is given by

$$(3.18)\quad \psi = \begin{cases} 0 & \text{on} \quad \partial O \times (0, \infty) \\ v_0 & \text{on} \quad \bar{O} \times \{t = 0\}. \end{cases}$$

Our goal is to show that

$$v^* = v_* = v \quad \text{in} \quad O \times (0, \infty).$$

Since, by definition, $v_* \leqslant v^*$, we only have to show that

$$(3.19)\quad v \leqslant v_* \quad \text{and} \quad v^* \leqslant v \quad \text{in} \quad O \times (0, \infty).$$

We begin with the right-hand side of (3.19), which is more or less immedia Indeed, let $v_0^n > 0$ be such that v_{m0}, $v_0 \leqslant v_0^n$ and $v_0^n \to v_0$ as $n \to \infty$ uniforr on $\bar{O}$. If v_m^n and v^n are the solutions of (3.1) and (3.2) with initial datum then the first part of this proof yields

$$\lim_{m \downarrow 1} v_m^n = v^n \quad \text{uniformly on compact subsets of} \quad O \times (0, \infty).$$

By the maximum principle we have that $v_m \leqslant v_m^n$ on $\bar{O} \times [0, \infty)$. Moreov it follows from the formulae that $v^n \to v$ uniformly on $O \times (0, \infty)$ as $n \to$ Combining all the above we obtain $v^* \leqslant v$ in $O \times (0, \infty)$.

To obtain the left-hand side of (3.19) we have to work a bit harder. begin by regularizing the v_*'s using the *inf-convolutions* introduced by J. Lasry and P.-L. Lions [28]. For $\alpha > 0$, let $O_\alpha = \{x \in O: \operatorname{dist}(x, \partial O) \geqslant \alpha\}$ a consider the functions

$$(3.20)\quad v_{*\alpha}(x, t) = \inf_{(y, s) \in \bar{O} \times (0, \infty)} \left\{ v_*(y, s) + \frac{|x - y|^2 + |t - s|^2}{2\alpha} \right\}.$$

It turns out (cf. P.-L. Lions and P. E. Souganidis [31]) that for each $\alpha >$ $v_{*\alpha}$ is a Lipschitz continuous viscosity solution of $w_t - |Dw|^2 \geqslant 0$ in $O_\alpha \times (\alpha,$ and $v_{*\alpha} \downarrow v_*$ as $\alpha \downarrow 0$. Next we consider the minimal viscosity solution of problem

$$\begin{cases} w_t - |Dw|^2 = 0 & \text{in} \quad O_\alpha \times (\alpha, \infty) \\ w = 0 & \text{on} \quad \partial O_\alpha \times (\alpha, \infty) \\ w = v^n_{*\alpha} & \text{on} \quad \bar{O}_\alpha \times \{t = \alpha\}, \end{cases}$$

where $v^n_{*\alpha} \in C(\bar{O}_\alpha)$, $v^n_{*\alpha|\partial O\alpha} = 0$ and $v^n_{*\alpha} \uparrow v_{*\alpha}(\cdot, t)$ as $n \to \infty$. The definition of w (given at the beginning of this section) yields

$$v_{*\alpha} \geqslant w \quad \text{in} \quad O_\alpha \times (\alpha, \infty). \tag{3.21}$$

Let $(x, t) \in O_\alpha \times (\alpha, \infty)$. Since

$$w(x, t) = \sup_{y \in O_\alpha} \{v^n_{*\alpha}(y, \alpha) - L(x, y, t - \alpha)\},$$

the properties of $v_{*\alpha}$ and (3.21) yield

$$v_*(x, t) \geqslant v_{*\alpha}(x, t) \geqslant v^n_{*\alpha}(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all} \quad y \in \bar{O}_\alpha.$$

Upon letting $n \to \infty$ we obtain

$$v_*(x, t) \geqslant v_{*\alpha}(x, t) \geqslant v_{*\alpha}(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all} \quad y \in \bar{O}_\alpha.$$

To conclude, we need to examine the behaviour of $v_{*\alpha}(x, \alpha)$ as $\alpha \downarrow 0$. Since $v_{*t} \geqslant 0$, (3.21) yields

$$v_{*\alpha}(x, \alpha) \geqslant \inf_{(y, s) \in \bar{O} \times [0, \infty)} \left\{ v_*(y, 0) + \frac{|x - y|^2 + |\alpha - s|^2}{\alpha} \right\}.$$

Using the lower semicontinuity of $v_*(\cdot, 0)$ we then see that

$$\varliminf_{\alpha \downarrow 0} v_{*\alpha}(s, \alpha) \geqslant v_*(x, 0).$$

Combining all the above we get

$$v_*(x, t) \geqslant v_*(x, 0) - L(x, y, t) \quad \text{for all} \quad y \in \bar{O}. \tag{3.22}$$

Finally, it follows from (3.16) and the definition of v_* (cf. [7]) tha

$$v_*(\cdot, 0) = v_0 \quad \text{on} \quad \bar{O}.$$

This together with (3.22) yields

$$\begin{aligned} v_*(x, t) &\geqslant \sup_{y \in \bar{O}} \{v_0(y) - L(x, y, t)\} \\ &= v(x, t); \end{aligned}$$

APPENDIX

We consider the questions of existence and uniqueness as well as o qualitative properties of viscosity solutions of $v_t = |Dv|^2$ defined in a s $Q_T = \mathbb{R}^N \times (0, T)$ for some $T > 0$. The uniqueness results we obtain genera the results of [16] and [18], in the sense that they allow more general in data. Some of the properties of the viscosity solutions we are intereste here are growth at infinity, reguralizing effects, domain of dependence, ir face, etc. Several of the results presented have also appeared in similar f in ([6], [16], [18], [29], [30], etc.); for this reason a lot of proofs are ra sketchy.

We recall here for the reader's convenience the definition of *viscosity solu* A continuous function u defined in a domain $\Omega \subset \mathbb{R}^{N+1}$ is called a visco solution of equation $v_t - |Dv|^2 = 0$ if for any function $\varphi \in C^1(\Omega)$ we h $\varphi_t - |D\varphi|^2 \leqslant 0$ at all points $P_0 = (x_0, t_0) \in \Omega$ at which $v - \varphi$ attains a l maximum and $\varphi_t - |D\varphi|^2 \geqslant 0$ where $v - \varphi$ attains a local minimum. We r the interested reader to the references at the end of this paper, especially and [17], for the theory of viscosity solutions.

1. Growth at Infinity and Initial Trace

Proposition A.1. *Let v be a viscosity solution of* (0.2) *defined in Q_T. every (x_1, t_1), $(x_2, t_2) \in Q_T$ with $0 < t_1 < t_2 < T$ we have*

$$v(x_1, t_1) \leqslant v(x_2, t_2) + \frac{|x_1 - x_2|^2}{4(t_2 - t_1)}. \tag{A.1}$$

Therefore, if $t \in (0, T)$, then

$$\limsup_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \leqslant \frac{1}{4(T - t)} \tag{A.2}$$

and

$$\liminf_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \geqslant -\frac{1}{4t}. \tag{A.3}$$

PROOF. We begin assuming that v is bounded below. Let $C, \delta > 0$ define the function $\phi \in C^\infty(\mathbb{R}^N \times [t_1, T))$ by

$$\phi(x, t) = v(x_1, t_1) - \frac{|x - x_1|^2}{4(t - t_1 + \delta)} - C(t + 1).$$

If we fix $C > 0$ and choose δ small enough, it is immediate that $\phi(\cdot, t_1) < v(x, t_1)$ on $\mathbb{R}^N$. We want to prove that $v \geqslant \phi$ in $\Omega = \mathbb{R}^N \times [t_1, t_2]$. In fact if $v - \phi$ attains a minimum in Ω at a point $(\bar{x}, \bar{t})$, then, by the definition of the viscosity solution, we must have $\phi_t - |D\phi|^2 \geqslant 0$ at $(\bar{x}, \bar{t})$. However, $\phi_t - |D\phi|^2 = -C < 0$ in Ω. Therefore the minimum of $v - \phi$ either is attained at $t = t_1$ and then $v > \phi$ in Ω or it is approached as $|x| \to \infty$. But v is bounded from below and $\phi \to -\infty$ as $|x| \to \infty$, therefore the latter cannot happen. Letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we obtain (A.1), from which (A.2) and (A.3) follow easily.

If v is not bounded from below we have to suitably modify our test function ϕ. To simplify notation we assume that $t_1 = v(x_1, t_1)$ $x_1 = 0$ and $|x_2| \leqslant 1$. We consider the rectangle $R = \{(x, t) \colon |x| \leqslant 2, 0 \leqslant t \leqslant t_2\}$ and we define the function

$$\phi(x, t) = -\frac{1}{4}\psi\left(\frac{x^2}{t + \delta}\right) - C(t + 1)$$

where C and δ are positive constants and $\psi \in C^\infty(\mathbb{R}^+)$ satisfies $\psi(0) = 0$, $\psi'(s) \geqslant 1$ for $s > 0$, $\psi(s) = s$ for $0 < s \leqslant s_2 = 1/(t_2 + \delta)$ and $-(1/4)\psi(s) \leqslant v(x, t)$ for every $|x| = 2$, $0 \leqslant t \leqslant t_2$ and $s = 4/t + \delta$. With these assumptions ϕ satisfies $\phi_t - |D\phi|^2 \leqslant -C < 0$ in R. Repeating the argument of the first part of the proof, we see that the minimum of $v - \phi$ is attained either at $t = 0$ or at $|x| = 2$. It follows from the properties of ψ that $v - \phi \geqslant 0$ for $|x| = 2$, $0 \leqslant t \leqslant t_2$. Moreover, choosing δ very small for fixed $C > 0$ we have $v(x, 0) \geqslant \phi(x, 0)$. Therefore $v \geqslant \phi$ in R and, in particular,

$$v(x_2, t_2) \geqslant \phi(x_2, t_2) = -C(t_2 + 1) - \frac{1}{4}\psi\left(\frac{|x_2|^2}{t_2 + \delta}\right).$$

The properties of ψ imply, however, that

$$\psi\left(\frac{|x_2|^2}{t_2 + \delta}\right) = \frac{|x_2|^2}{t_2 + \delta},$$

hence letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we conclude. □

Next we turn our attention to the question of the initial trace of viscosity solutions of (0.2). Since $v_t \geqslant 0$, the family $\{v(\cdot, t)\}_t$ is nondecreasing as $t \downarrow 0$ ([17]). Therefore the initial trace

$$v_0(\cdot) = \lim_{t \downarrow 0} v(\cdot, t)$$

exists. The following proposition is immediate.

Proposition A.2. *Every viscosity solution of* (0.2) *in* Q_T *has an initial tr* v_0, *which is an upper semicontinuous function* v_0: $\mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ *satisfy*

(A.4) $$\limsup_{|x|\to\infty} \frac{v_0(x)}{|x|^2} \leqslant \frac{1}{4T}.$$

2. Existence of Solutions

It follows from Proposition A.1 that for every viscosity solution v we h

(A.5) $$v \geqslant \underline{v} \quad \text{on} \quad Q_T$$

where $\underline{v}$ is given by the Lax-Oleinik formula

(A.6) $$\underline{v}(x, t) = \sup_{y\in\mathbb{R}^N} \left\{ v_0(y) - \frac{|x-y|^2}{4t} \right\}.$$

In fact, as we will see below, this last formula provides with the unique s(tion of the Cauchy problem (1.2) in $\mathbb{R}^N \times [0, T)$, where T depends on

$$\limsup_{|x|\to\infty} v_0(x)|x|^{-2}.$$

The Lax-Oleinik formula has been studied rather extensively at least in case where v_0 is bounded ([6], [29], [30]). Next, in a series of propositions summarize the properties of (A.6) under assumption (A.4). The proofs (lot of these propositions are slight modifications of the ones for bounde(therefore we omit them.

Proposition A.3. *For every function* v_0: $\mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *such that*

(A.7) $$-\infty \not\equiv v_0(x) \leqslant A|x|^2 + B \quad in \quad \mathbb{R}^N$$

for some $A, B > 0$, *the Lax-Oleinik formula* (A.6) *provides with a continu viscosity solution of* (1.2) *in* Q_T, *where the maximal* T (*blow-up time*) *is g by*

(A.8) $$T = 1/4\alpha$$

with

(A.9) $$\alpha = \limsup_{|x|\to\infty} \{v_0(x)|x|^{-2}\}.$$

In particular, v *exists for all time if and only if* $v_0(x) \leqslant o(|x|^2)$. *Moreover, every* $t \in (0, T)$, $v(\cdot, t) \geqslant v_0(\cdot)$ *and*

(A.10) $$\liminf_{|x|\to\infty} \frac{v(x,t)}{|x|^2} \geqslant -\frac{1}{4t} \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{v(x,t)}{|x|^2} \leqslant \frac{\alpha}{1-4\alpha t}.$$

Let $\mathfrak{F}_\alpha$ be the set of functions $v_0: \mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ such that

$$-\infty \not\equiv v_0(x) \leqslant A|x|^2 + B \quad \text{for some} \quad A, B > 0$$

and

$$\limsup_{|x|\to\infty} v_0(x)|x|^{-2} \leqslant \alpha.$$

For $t \in (0, 1/4\alpha)$ and $\beta = \alpha(1-4\alpha t)^{-1}$, let $L_t: \mathfrak{F}_\alpha \to \mathfrak{F}_\beta$ be the nonlinear operator defined by the Lax-Oleinik formula.

Proposition A.4. *Let $\alpha > 0$, $t \in (0, 1/4\alpha)$, $\beta = \alpha/(1-4\alpha t)$ and $s \in (0, 1/4\beta)$. For any $v_0 \in \mathfrak{F}_\alpha$ we have*

$$L_s(L_t v_0) = L_{g+s}(v_0)$$

i.e. L_t has the semigroup property.

We have remarked in Proposition A.2 that a viscosity solution can only take on upper-semicontinuous initial data. On the other hand, we have not discussed yet about whether $L_t v_0$ assumes the initial datum v_0. The next proposition addresses this question and gives more precise information.

Proposition A.5. *Let v_0 be an upper-semicontinuous function in $\mathfrak{F}_\alpha$ for some α and let v given by* (A.6). *Then v takes on the initial value v_0. More precisely.*

(A.11) $$\limsup_{(x,t)\to(x_1,0)} v(x,t) \leqslant v_0(x_1)$$

If $v_0(x_1) > -\infty$, then

(A.12) $$\liminf_{\substack{(x,t)\to(x_1,0)\\ |x-x_1| = 0(t^{1/2})}} v(x,t) \geqslant v_0(x_1).$$

PROOF. We only prove (A.11). For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_1| \leqslant \delta$ then

$$v_0(x) \leqslant v_0(x_1) + \epsilon.$$

If $|x - x_1| < \delta/2$ and $|y - x| \leqslant \delta/2$, then for $t > 0$ we have

$$v_0(y) - \frac{|x-y|^2}{4t} \leqslant v_0(x_1) + \epsilon.$$

On the other hand, if $|x - x_1| < \delta/2$ and $|y - x| \geqslant \delta/2$, then

$$v_0(y) - \frac{|x-y|^2}{4t} \downarrow -\infty$$

uniformly in y as $t \downarrow 0$. □

Corollary. *If $v_0 \in \mathfrak{F}_\alpha$ the $L_t v_0$ converges as $t \downarrow 0$ to the upper semicontinu envelope of v_0 i.e. the minimal of the upper semicontinuous functi $w: \mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ which are larger than v_0.*

3. Regularity properties of the Lax-Oleinik Formula

Proposition A.6. *Let $v_0 \in \mathfrak{F}_\alpha$ for some $\alpha > 0$. Then for every $\tau > 0$, ι Lipschitz continuous uniformly on compact subsets of $\mathbb{R}^N \times (\tau, T)$. If ($v_0(x) \leqslant (a|x| + b)^2$ in $\mathbb{R}^N$, then for almost every $x \in \mathbb{R}^N$ and $t \in (0, 1/4\alpha)$*

$$v_t = |Dv|^2 \leqslant \frac{(a|x|+b)^2}{t(1-2at^{1/2})^2}. \tag{A.13}$$

On the other hand, if v_0 is bounded from above by M, then

$$v_t = |Dv|^2 \leqslant \frac{M-v}{t} \leqslant \frac{M-v_0(x)}{t}. \tag{A.14}$$

The proofs are easy consequences of (A.6). Another regularity type qu tion is related to the optimality of the bounds (A.10).

Proposition A.7. *For every $t \in (0, T)$, we have ($\alpha = 1/4T$)*

$$\limsup_{|x|\to\infty} \frac{v(x,t)}{|x|^2} = \frac{\alpha}{1-4\alpha t} = \frac{1}{4(T-t)}. \tag{A.15}$$

PROOF. The inequality $\leqslant$ was proved in Proposition A.2. For the conver assume that for some $t_1 \in (0, T)$ we have

$$\limsup_{|x|\to\infty} \frac{v(x,t_1)}{|x|^2} = \alpha_1 < \frac{\alpha}{1-4\alpha t_1}.$$

Then the solution with initial value $v(\bullet, t_1)$ exists for a time

$$t_2 = \frac{1}{4\alpha_1} > \frac{1-4\alpha t_1}{4\alpha}.$$

By the semigroup property the original solution would then exist for a time

$$t_1 + t_2 > \frac{1}{4\alpha} = T,$$

which is a contradiction. □

As far as lower bounds are concerned we have the following result.

Proposition A.8. *Let $\beta \in [-\infty, \infty)$ be defined by*

$$\beta = \liminf_{|x| \to \infty} \frac{v_0(x)}{|x|^2}. \tag{A.16}$$

For every $t \in (0, T)$ we have

$$\liminf_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \geqslant \frac{\beta}{1 - 4t\beta}. \tag{A.17}$$

The equality is false in general.

PROOF. If $\beta = -\infty$, (A.17) reduces to (A.10). If $\beta \in (-\infty, 0)$, then for every $\epsilon > 0$ there exists a $B_\epsilon \in \mathbb{R}$ such that $v_0(x) \geqslant (\beta - \epsilon)|x|^2 - B_\epsilon$. We compare $v(x, t)$ to the explicit solution

$$\phi(x, t) = -B_\epsilon - \frac{|x|^2}{4t + \tau}$$

with $\tau = -1/(\beta - \epsilon)$ and $\beta - \epsilon \neq 0$. Using the Lax-Oleinik formula we conclude that $v \geqslant \phi$ in Q_T, hence as $\epsilon \to 0$ we obtain the inequality $\geqslant$ in (A.18). To show that equality does not hold in general we consider a v_0 defined as follows: Let $B(y_n, r_n)$ be a sequence of balls such that $r_n \to 0$ and $y_n \to \infty$ where v_0 is negative, continuous and $v(y_n)|y_n|^{-2} \to -\infty$. Outside these balls $v_0 \equiv 0$. Therefore $\beta = -\infty$. If $(x, t) \in \mathbb{R}^N \times (0, \infty)$ we have $v(x, t) = 0$ if $x \notin \bigcup_n B(y_n, r_n)$. If $x \in B(y_n, r_n)$, then there exists $y = y(x)$ such that $v_0(y) = 0$ and $|y - x| = r_n$. Hence

$$v(x, t) \geqslant v_0(y) - \frac{|x - y|^2}{4t} \geqslant -\frac{r_n^2}{4t}.$$

Therefore,

$$\lim_{|x| \to \infty} \frac{v(x, t)}{|x|^2} = 0. \quad \square$$

We conclude the presentation of regularity-related properties of the L Oleinik formula with a result concerning their semiconvexity. Since th properties are an immediate consequence of the formula, we again omit proof.

Proposition A.9. (1) *Let* $\chi \in \mathbb{R}^N$, $|\chi| = 1$. *Then*

(i)
$$\frac{\partial^2 v}{\partial \chi^2} \geqslant -\frac{1}{2t}.$$

(ii) *If for every* $\chi \in \mathbb{R}^N$, $|\chi| = 1$, $\partial^2 v_0/\partial \chi^2 \geqslant -\alpha$ *then for every* $t > 0$,

$$\frac{\partial^2 v}{\partial \chi^2} \geqslant -\frac{\alpha}{1 + 2\alpha t}.$$

(2) *If* $\Delta v_0 \geqslant -\alpha$ *then for* $t > 0$,

$$\Delta v(x, t) \geqslant -\frac{N\alpha}{N + 2\alpha t}.$$

All the above inequalities should be interpreted in the sense of distributio

4. Uniqueness and continuous dependence

We begin with a proposition concerning the domain of dependence of Lax-Oleinik formula. The proof of this result is based on the gradi estimates from Proposition A.6 and the proofs of M. G. Crandall and Newcomb [20] and P. E. Souganidis [36] concerning viscosity solutions on boundary. See [5] for $N = 1$. Since it is a long exercise, we omit it.

Proposition A.10. *Let* v_{01}, v_{02} *be two initial data in* $\mathbb{R}^N$ *such that*

$$v_{01}(x), v_{02}(x) \leqslant (a|x| + b)^2$$

for every $x \in \mathbb{R}^N$ *and some constants* $a, b \geqslant 0$. *Let* $t \in (0, 1/4a^2)$. *Then*

$$v_1(0, t) - v_2(0, t) \leqslant \sup_{y \in I_0} \{v_{01}(y) - v_{02}(y)\}$$

where

$$I_0 = \left\{ y \in \mathbb{R}^N : |y| \leqslant \frac{b}{a}\left[\exp\left(\frac{\lambda}{1 - \lambda}\right) - 1\right], \lambda = 2at^{1/2} \right\}$$

if $a, b > 0$ *and*

$$I_0 = \{ y \in \mathbb{R}^N : |y| \leqslant \sqrt{4bt} \}$$

if $a = 0$ *and* $b > 0$.

Corollary. *If* $v_0^n \to v_0$ *locally uniformly in* $\mathbb{R}^N$, *then* $v^n \to v$ *locally uniformly in* $\mathbb{R}^N \times [0, T)$.

Next we prove the uniqueness of viscosity solutions of (A.7) with upper-semicontinuous initial datum $v_0 \colon \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$. This implies that the viscosity solution of (A.7) is given by the Lax-Oleinik formula, therefore it enjoys all the regularity presented above.

Theorem A.1. *The viscosity solution of* (1.2) *in* Q_T *with upper-semicontinuous initial datum* $v_0 \colon \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *is unique.*

PROOF. If $\underline{v}(x, t) = L_t(v_0)(x)$, in view of (A.5), we only have to show that $v \leqslant \underline{v}$. We argue as follows: Since v is defined in Q_T, for every $t \in [0, T)$ we have

$$\limsup_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \leqslant \frac{1}{4(T - t)}.$$

Let $v_{0n} \in C(\mathbb{R}^N, \mathbb{R})$ be such that $v_{0n} > v_0$ and

$$\limsup_{|x| \to \infty} \frac{v_{0n}(x)}{|x|^2} = \frac{1}{4\left(T - \dfrac{1}{n}\right)}.$$

Then $v_n = L_t(v_{0n})$ exists for a time $T_n = T - 1/n$. Moreover, for $t \in (0, T_n)$,

$$\limsup_{|x| \to \infty} \frac{v_n(x, t)}{|x|^2} = \frac{1}{4\left(T - \dfrac{1}{n} - t\right)}.$$

If $v_{n\epsilon} = v_n * \rho_\epsilon$, where $*$ denotes the standard convolution, then for $t > \epsilon$ we have

$$(v_{n\epsilon})_t = |Dv_n|^2 * \rho_\epsilon \geqslant |Dv_{n\epsilon}|^2.$$

Let $w_{\epsilon n} \colon \mathbb{R}^N \times [0, T_n - \epsilon]$ be defined by

$$w_{n\epsilon}(x, t) = v_{n\epsilon}(x, t + \epsilon) + \epsilon t.$$

The sup of $v - w_{n\epsilon}$ in $\mathbb{R}^N \times [0, t_1)$ with $t_1 < t_n - \epsilon$ cannot be taken ir interior of $\mathbb{R}^N \times [0, t_1)$. Therefore, either it is approached as $|x| \to \infty$ or taken at $t = 0$. In either case, it is negative. It then follows that

$$v < w_{n\epsilon} \quad \text{in} \quad \mathbb{R}^N \times [0, T_n - \epsilon].$$

Letting $\epsilon \to 0$ yields

$$v \leqslant v_n \quad \text{in} \quad \mathbb{R}^N \times [0, T_n).$$

Sending $n \to \infty$ and using the continuous dependence of the Lax-Oleinik : tions on the initial data in local norms we conclude. □

5. Free Boundaries

In the case of solutions which are bounded from either above or belo makes sense to consider the boundary of the sets where the largest or sm: values are attained. Let us consider first the case of an upper-semicontin initial datum $v_0: \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ which is bounded from above by a con M. Let

$$\text{(A.18)} \qquad \begin{aligned} D_+ &= D_+(v_0) \\ &= \{x \in \mathbb{R}^N: v_0(x) = M\}. \end{aligned}$$

This is a closed, possibly empty, set. It is immediate from $v_t \geqslant 0$ that if $x \epsilon$ then for every $t > 0$, $v(x, t) = M$. Therefore the set D_+ is invariant in tim‹ so is its boundary. On the contrary, if v_0 is bounded from below by a con which without any loss of generality we may assume to be zero, then th

$$\text{(A.19)} \qquad \Omega_0 = \{x \in \mathbb{R}^N: v_0(x) > 0\}$$

is not necessarily open or closed. We define:

$$\text{(A.20)} \qquad \begin{cases} \Omega = \{(x, t) \in \mathbb{R}^N \times [0, T): v(x, t) > 0\} \\ \Omega(t) = \{x \in \mathbb{R}^N: (x, t) \in \Omega\} \\ \Gamma = \text{boundary of } \Omega \text{ in } \mathbb{R}^N \times [0, T) \\ \Gamma(t) = \{x \in \mathbb{R}^N: (x, t) \in \Gamma\}. \end{cases}$$

Γ is called the *free boundary* of v. Since $v_t \geqslant 0$, the following result is imme‹

Proposition A.11. *For every* $t_2 > t_1$ *in* $(0, T)$, $\Omega_0 \subset \Omega(t_1) \subset \Omega(t_2)$.

Next we examine the behavior of v on Ω.

Proposition A.12. *Let $t \in (0, T)$. For almost every $x \in \Omega(t) \setminus \Omega_0$ there exists a point $y = y(x) \in \Omega_0$ such that*

$$Dv = -\frac{x - y}{2t}.$$

For all points $x \in \Omega(t) \setminus \Omega_0$, $-\dfrac{x - y}{2t}$ is a subdifferential of v at x.

PROOF. Since $v(x, t) > 0$ there exist $y_n \in \mathbb{R}^N$ such that

$$v_0(y_n) - \frac{|x - y_n|^2}{4t} \uparrow v(x, t).$$

It follows that $v_0(y_n) > 0$, i.e. $y_n \in \Omega_0$ and

$$|x - y_n|^2 \leqslant 4tv_0(y_n) < 4t(\alpha + \epsilon)(|y_n| + b_\epsilon)^2. \tag{A.21}$$

Since $4t(\alpha + \epsilon) < 1$ if ϵ is small enough, $|y_n| \leqslant C$ and, upon passing to a subsequence, we may assume that $y_n \to y$. The upper semicontinuity of v_0 yields $y \in \Omega_0$ and

$$v(x, t) = v_0(y) - \frac{|x - y|^2}{4t}, \tag{A.22}$$

Next let $h \in \mathbb{R}^N$ with $|h|$ small. Then

$$\begin{aligned} v(x + h, t) - v(x, t) &\geqslant v_0(y) - \frac{|x + h - y|^2}{4t} - v_0(y) - \frac{|x - y|^2}{4t} \\ &\geqslant -\frac{1}{2t} h \cdot (x - y) - \frac{|h|^2}{4t}. \quad \square \end{aligned}$$

Since $-Dv$ is the local velocity of propagation of the solutions of (0.2), this result controls the speed with which the interface moves. In fact the interface consists of a stationary part Γ_0, a union of vertical segments $\{(x, t): 0 \leqslant t \leqslant t_1\}$ with $x \in \partial\Omega_0$ fixed, and the moving interface

$$\Gamma_1 = \{(x,t) \in \Gamma: x \notin \Omega_0\}.$$

Proposition A.13. *The moving interface Γ_1 can be described by a Lipschitz continuous function $t = S(x)$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}_0$. More precisely, for every $(\bar{x}, \bar{t}) \in \Gamma_1$ there is a conical region $K = \{(x, t): |x - \bar{x}| < h, |x - \bar{x}| < c|t - \bar{t}|\}$ with $0 < c <$ dist $(\bar{x}, \Omega_0)(2\bar{t})^{-1}$ and h small depending on c, $\bar{x}$, such that*

$$K_+ = \{(x, t) \in K: t > \bar{t}\} \subset \Omega \quad \textit{and} \quad K_- = \{(x, t) \in K: t < \bar{t}\}$$

is disjoint with Ω.

PROOF. We only prove the result concerning K_+. The result about K follows in a similar way. To this end, let $(x, t) \in K_+$ and set $x - \bar{x} = z$ a $t - \bar{t} = \tau$. If there exists a $y \in \Omega_0$ with the properties described in Propositi A.12 (since $v(\bar{x}, \bar{t}) = 0$ this is not necessarily the case) we have

$$
\begin{aligned}
v(x,t) &\geqslant v_0(y) - \frac{|x-y|^2}{4t} \\
&= \frac{|\bar{x}-y|^2}{4\bar{t}} - \frac{|x-y|^2}{4t} \\
&= \frac{|\bar{x}-y|^2}{4\bar{t}} - \frac{|\bar{x}-y|^2}{4t} - \frac{(\bar{x}-y)\cdot z}{2t} - \frac{|z|^2}{4t} \\
&= \frac{|\bar{x}-y|}{4t}\left(\frac{|\bar{x}-y|}{\bar{t}}\tau - 2z\cdot\theta\right) - 0(|z|^2)
\end{aligned}
$$

where $\theta = \bar{x} - y/|\bar{x} - y|$. Therefore if $z/\tau \leqslant d(\bar{x}, D_0)/2\bar{t} \leqslant |\bar{x} - y|/2\bar{t}$ a $0 < |z| < h$ with h small we have $v(x, t) > 0$.

If such a y does not exist we select a sequence of points $x_n \in \Omega_t$, $x_n \to x$, find $y_n \in \Omega_0$, construct a cone K_n with vertex $(x_n, \bar{t})$ and let $n \to \infty$ to obta K_+. We define

$$S(x) = \sup\{t \geqslant 0 : v(x, t) = 0\}. \tag{A.23}$$

The Lipschitz continuity of S at $(\bar{x}, \bar{t})$ follows from the fact for every x su that $|x - \bar{x}| < h$ then $(x, t) \in \Gamma_1$ implies $t - \bar{t} \leqslant C|x - \bar{x}|$. □

Corollary. *Let $x \in \Omega_t \setminus \Omega_0$. If $d(x) = \operatorname{dist}(x, \Omega_0)$, then*

$$\frac{d(x)}{2t} < |Dv| < \frac{c(|x|)}{2t} \tag{A.24}$$

where $c(r)$ is a continuous function of r.

PROOF. Take $y \in \Omega_0$ as in (A.22). We have $|x - y| \geqslant d(x)$ and, from (A.2

$$|y| \leqslant \frac{|x| + b_\epsilon k}{1 - k}, \qquad k = (4t(\alpha + \epsilon))^{1/2}.$$

We conclude. □

The above proof also shows that at every point x where S is differentiab we have $|DS| < 1/c$ for any c as in Proposition A.13. Therefore

$$|DS| \cdot \frac{d(x)}{2t} < 1. \tag{A.25}$$

In other words, $(2t)^{-1}d(x)$ is a lower bound for the velocity with which Ω_t grows. Thus if $\operatorname{dist}(\Omega_t, \Omega_0) = d > 0$ then for every $\tau > 0$ small enough $\Omega_{t+\tau}$ contains an ϵ-neighborhood of Ω_t, if $\epsilon < d\tau/2t$. In fact Ω_t moves with speed bounded from above. More precisely, we have:

Proposition A.14. *If $\tau > 0$ is small enough, then for every $\bar{x} \in \partial Q_{\bar{t}+\tau}$ there exists $C = C(|\bar{x}|)$ such that*

$$\operatorname{dist}(\bar{x}, \Omega_{\bar{t}}) \leqslant \frac{C\tau}{2\bar{t}}. \tag{A.26}$$

Moreover, if v_0 is bounded from above, the bound on (A.26) *is independent of $\bar{x}$ and*

$$\operatorname{dist}(\Gamma_{t+\tau}, \Omega_t) \leqslant \frac{C\tau}{2t}. \tag{A.27}$$

PROOF. Let $r = \operatorname{dist}(\bar{x}, \Omega_{\bar{t}})$ and c be an upper bound on v in $B(x, 2r)$ (which should be separated from Ω_0). The function

$$V(x,t) = c(c(t-\bar{t}) + |x-\bar{x}| - r)^+$$

is a supersolution of (1.2). The result follows. □

We conclude by characterizing the existence of a stationary interface Γ_0. A careful look at the Lax-Oleinik formula yields the following proposition.

Proposition A.15. *Let $x \in \Omega_0$. Then $v(x,t) = 0$ for $t \in [0, t^*]$ if and only if the quantity*

$$\gamma(x) = \sup_{y \in \mathbb{R}^N} \frac{v_0(y)}{|x-y|^2} \tag{A.28}$$

is finite. The starting time t^* is given by $1/4\gamma$.

6. Generalizations

All the above can be easily generalized to the Cauchy problems

$$\begin{cases} v_t = |Dv|^p & \text{in } \mathbb{R}^N \times (0,T) \\ v = v_0(x) & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases} \tag{A.29}$$

with $p > 1$. The formula for viscosity solutions of (A.29) is

(A.30) $$\underline{v}(x) = \sup \left\{ v_0(y) - C_p \frac{|x-y|^{p/(p-1)}}{t^{1/(p-1)}} \right\}$$

for $C_p = (p-1)p^{-p/(p-1)}$.

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