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Homeomorphisms Preserving A_p

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Introduction

In a recent paper, Benedetto, Heinig and Johnson [1] showed that if w is a monotone A_2 weight, $w(1/x)$ is also an A_2 weight. Here A_2 denotes the set of weights satisfying the condition found by Muckenhoupt, defined in general by

$$w \in A_p \quad \text{iff} \quad \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} < +\infty,$$

$1 < p < +\infty$, which was shown by Muckenhoupt [6] to characterize the weights w for which the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q f(y)\,dy, \qquad f \geqslant 0,$$

satisfies

$$\int Mf(x)^p w(x)\,dx \leqslant C^p \int f(x)^p w(x)\,dx.$$

Hunt, Muckenhoupt and Wheeden showed that A_p is the condition that characterizes weights for which the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\lim_{\epsilon \to 0}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\,dy$$

satisfies

$$\int |Hf(x)|^p w(x)\,dx \leqslant C^p \int |f(x)|^p w(x)\,dx$$

(see [4] for the proof). The Hilbert transform commutes with the unitary operator on $L^2(\mathbb{R})$,

$$Uf(x) = \frac{1}{x} f(1/x),$$

and this gives the result that for any $w \in A_2$, $w(1/x) \in A_2$. We have considered the more general question: which homeomorphisms of $\mathbb{R}$ preserve the A_p class? We answer this question, (and its counterpart on $\mathbb{R}^n$) and apply it to determine the pointwise multipliers of A_p. In view of the close connection between A_p, reverse Hölder inequalities and $A_\infty = \bigcup_{p<\infty} A_p$, we also investigate the corresponding questions for these conditions, by means of a precise connection between reverse Hölder inequalities and A_∞. We also have results for double weights $(u, v) \in A_p$ though not so complete.

1. Notation and Preliminary Results

In addition to the A_p-classes we define for $1 < p < \infty$,

$$A_p(w) = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

We shall also need A_1, the class of all w for which $Mw \leqslant cw$ with

$$A_1(w) = \inf \{ c : Mw \leqslant cw \}.$$

We let

$$A_\infty = \bigcup_{p<\infty} A_p,$$

and write

$$A_\infty(w) = \lim_{p \to \infty} A_p(w).$$

This limit exists since for $q \geqslant p$, $A_q(w) \leqslant A_p(w)$. We say that $w \in RH_{p_0}$ (reserve Hölder) if

$$\left(\frac{1}{|Q|} \int_Q w^{p_0} \right)^{1/p_0} \leqslant \frac{C}{|Q|} \int_Q w,$$

and we abbreviate by $RH_{p_0}(w)$ the infimum of all such C. It is easily seen by Hölder's inequality that $A_p(w) \geqslant 1$ and $RH_p(w) \geqslant 1$.

We will repeatedly use some common properties of A_p, namely,

(*a*) $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$,
(*b*) $w \in A_p$ implies $w \in A_q$, $q \geqslant p$, and $w^\alpha \in A_p$, $0 \leqslant \alpha \leqslant 1$,
(*c*) if $w_1, w_2 \in A_p$, then $w_1^\alpha w_2^{1-\alpha} \in A_p$, $0 \leqslant \alpha \leqslant 1$,
(*d*) $w \in A_p$, $1 < p < \infty$, if and only if there exists $u, v \in A_1$, so that $w = uv^{1-p}$,
(*e*) $w \in A_p$ for some p if and only if $w \in RH_q$ for some q,
(*f*) if $w \in A_p$, then $w^\tau \in A_p$ for some $\tau > 1$,
(*g*) if $w \in A_p$, $p > 1$, then $w \in A_{p-\epsilon}$ for some $\epsilon > 0$.

An excellent reference is [4].

We will also use the close connection between A_p and *BMO*, i.e., the space of functions satisfying

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| \, dx < \infty,$$

where

$$f_Q = \frac{1}{|Q|} \int_Q f.$$

The sup is a semi-norm on *BMO* which gives constants norm 0. The connection between the two classes is the following. For any $w \in A_p$, $\log w \in BMO$, and for any $u \in BMO$ and $1 < p < \infty$, $e^{\lambda u} \in A_p$ for some $\lambda > 0$. This last result is not true for $p = 1$.

The classes RH_{p_0} and A_∞ are closely related as the following theorem [12] shows.

Theorem 1.1. *$w \in RH_{p_0}$ if and only if $w^{p_0} \in A_\infty$.*

We postpone the simple proof till Section 3 where we need a quantitative version of this result.

As a first general composition type theorem we have

Theorem 1.2. *Let h: $\mathbb{R} \to \mathbb{R}$ be arbitrary. Then the following statements are equivalent for $1 < p_0 < \infty$.*

(1) *$w \in A_\infty$ implies $w \circ h \in A_\infty$,*
(2) *$w \in RH_{p_0}$ implies $w \circ h \in RH_{p_0}$,*
(3) *$w \in A_1$ implies $w \circ h \in \bigcap_{p>1} A_p$.*

Either one of these statements implies, but is not implied by

(4) *$w \in A_{p_0}$ implies $w \circ h \in A_{p_0}$.*

PROOF. Theorem 1.1 implies (1) $\Leftrightarrow$ (2). The implication (1) $\Rightarrow$ (4) follows from the fact that $w \in A_{p_0}$ if and only if w and $w^{1-p_0'}$ are in A_∞, [4, p.408]. This also shows (1) $\Rightarrow$ (3). For the proof of (3) $\Rightarrow$ (1), let $w \in A_\infty$. Then there is $1 < p < \infty$, and there are $u_1, u_2 \in A_1$ with $w = u_1 \cdot u_2^{1-p}$. Hence $w \circ h = u_1 \circ h \cdot (u_2 \circ h)^{1-p}$. Since $u_1 \circ h \in A_2$, there is $\tau > 1$ so that $(u_1 \circ h)^\tau \in A_2$ and $\tau'/(p'-1) + 1 \geqslant 2$. We let $q = \tau'/(p'-1) + 1$ and note that $(u_1 \circ h)^\tau \in A_q$ and $u_2 \circ h \in A_{q'}$. We claim that $w \circ h \in A_q$. To prove this, we note that

$$\frac{1}{|I|}\int_I u_1 \circ h \cdot (u_2 \circ h)^{1-p} \leqslant \left(\frac{1}{|I|}\int_I u_1 \circ h^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)}\right)^{1/\tau'}$$

and

$$\frac{1}{|I|}\int_I (u_1 \circ h)^{1-q'}(u_2 \circ h)^{(1-p)(1-q')} \leqslant \left(\frac{1}{|I|}\int_I (u_1 \circ h)^{\tau(1-q')}\right)^{1/\tau} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)(1-q')}\right)^{1/\tau'}.$$

Consequently,

$$\frac{1}{|I|}\int_I w \circ h\left(\frac{1}{|I|}\int_I (w \circ h)^{1-q'}\right)^{q-1} \leqslant \left(\frac{1}{|I|}\int_I (u_1 \circ h)^\tau\right) \cdot \left(\frac{1}{|I|}\int_I (u_1 \circ h)^{\tau(1-q')}\right)^{(q-1)/\tau} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)}\right)^{1/\tau'} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)(1-q')}\right)^{(q-1)/\tau'}$$

Since $\tau'(1-p)(1-q') = 1$ or $\tau'(1-p) = 1-q$ we see that

$$A_q(w \circ h) \leqslant A_q(u_1 \circ h^\tau)^{1/\tau} A_{q'}(u_2 \circ h)^{(q-1)/\tau'}.$$

In order to complete the proof we need to give an example for (4) $\not\Rightarrow$ (1). We let $h(x) = 1/x$. Then, as observed in the introduction, (4) holds for $p_0 = 2$. However, h cannot preserve A_∞, since

$$w(x) = |x|^2 \in A_\infty, \quad \text{but} \quad w \circ h(x) = |x|^{-2} \notin A_\infty.$$

Remark. The main problem with which this paper is concerned is to find conditions on h so that (4) is equivalent with the other conditions of Theorem 1.2, and such a condition will have to be independent of p_0.

2. Homeomorphisms Preserving A_p

We begin with several preliminary lemmas some of which are known [3] and are included here for the sake of completeness.

Lemma 2.1. *Let $w \in A_\infty$ and assume that $w^\epsilon \in A_1$ for some $\epsilon > 0$. Then $w \in A_1$.*

PROOF. We may suppose that $\epsilon = p_0' - 1$, since we may decrease ϵ and increase p_0 by property (*b*) of Section 1. Then

$$\begin{aligned}\frac{1}{|I|}\int_I w\,dx &\leqslant A_{p_0}(w)\Big/\left(\frac{1}{|I|}\int_I w^{-\epsilon}\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)\left(\frac{1}{|I|}\int_I w^{\epsilon}\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)\left(A_1(w^\epsilon)\inf_I w^\epsilon\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)A_1(w^\epsilon)^{1/\epsilon}\inf_I w,\end{aligned}$$

where in the second inequality we used Hölder's inequality in the form

$$1 \leqslant \left(\frac{1}{|I|}\int_I \varphi\right)\left(\frac{1}{|I|}\int_I \frac{1}{\varphi}\right).$$

Our necessary and sufficient condition will involve the space $\bigcap_{p>1} A_p$. We will see that this space contains A_1 properly, and for this we need the simple

Lemma 2.2. *If $w, w^{-1} \in A_1$, then $w \simeq 1$, i.e., w is bounded above and below.*

PROOF.

$$\frac{1}{A_1\left(\dfrac{1}{w}\right)\inf\limits_I \dfrac{1}{w}} \leqslant \frac{1}{\dfrac{1}{|I|}\displaystyle\int_I \frac{1}{w}} \leqslant \frac{1}{|I|}\int_I w \leqslant A_1(w)\inf_I w.$$

Thus

$$\sup_I w \leqslant A_1\left(\frac{1}{w}\right)A_1(w)\inf_I w$$

and the result follows.

Lemma 2.3. $A_1 \subsetneq \bigcap_{p>1} A_p$.

PROOF. If equality held, then

$$\left\{w: w, w^{-1} \in \bigcap_{p>1} A_p\right\} = \{w: w, w^{-1} \in A_1\} = \{w: w \simeq 1\}.$$

However, by [4, p. 474],

$$\left\{w: w, w^{-1} \in \bigcap_{p>1} A_p\right\} = \{e^{\varphi}: \varphi \in \text{BMO-Closure } L^{\infty}\}$$

$$\supseteq \{e^{Hf}: f \text{ continuous of compact support}\}$$

by [10]. Then $e^{Hf} \simeq 1$, for each continuous f of compact support, and this is impossible.

Remark. An explicit example of

$$w \in \bigcap_{p>1} A_p \backslash A_1$$

is

$$w(t) = \left(\log \frac{1}{|t|}\right)^{-1}, \quad \text{for} \quad |t| \quad \text{close to} \quad 0.$$

This example was communicated to us by Rubio de Francia.

As we shall see now, Lemma 2.1 remains valid for $\bigcap_{p>1} A_p$.

Lemma 2.4. *Let* $w \in A_{p_0}$ *for some* $1 < p_0 < \infty$ *and suppose that for some* $\epsilon > 0$, $w^{\epsilon} \in \bigcap_{p>1} A_p$. *Then* $w \in \bigcap_{p>1} A_p$.

PROOF. Since we may decrease ϵ and increase p_0, we may suppose $\epsilon = p_0' - 1$. Then

$$\frac{1}{|I|} \int_I w \leqslant A_{p_0}(w) \Big/ \left(\frac{1}{|I|} \int_I w^{-\epsilon}\right)^{1/\epsilon} \leqslant A_{p_0}(w)\left(\frac{1}{|I|} \int_I w^{\epsilon}\right)^{1/\epsilon}.$$

Fix $1 < p < \infty$, and let

$$r = \frac{\epsilon}{p' - 1} + 1.$$

Since $w^{\epsilon} \in A_r$,

$$\frac{1}{|I|} \int_I w^{\epsilon} \left(\frac{1}{|I|} \int_I w^{\epsilon(1-r')}\right)^{r-1} \leqslant A_r(w^{\epsilon}),$$

and hence we have

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{\epsilon(1-r')}\right)^{(r-1)/\epsilon} \leqslant A_r(w^\epsilon)^{1/\epsilon}A_{p_0}(w).$$

Since $(r-1)/\epsilon = p-1$ and $\epsilon(1-r') = 1-p'$, $w \in A_p$ and $A_p(w) \leqslant A_r(w^\epsilon)^{1/\epsilon} \cdot A_{p_0}(w)$.

As we shall discuss further below, the case $n = 1$ is the most involved. We shall assume throughout unless otherwise noted, that $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism onto such that h, h^{-1} are locally absolutely continuous and, without loss of generality, that $h' \geqslant 0$.

Our next result implies a quantitative version of a result known qualitatively by the result of [4, p. 402] on comparability of measures.

Lemma 2.5. *$(h^{-1})' \in RH_q$ if and only if $h' \in A_{q'}$ and $RH_q((h^{-1})') = A_{q'}(h')^{1/q'}$.*

PROOF. If $(h^{-1})' \in RH_{q'}$ then for every interval J,

$$\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q} \leqslant RH_q((h^{-1})')\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt,$$

where $h(I) = J$. Let

$$L = \frac{1}{|I|}\int_I h'\left(\frac{1}{|I|}\int_I h'^{(1-q)}\right)^{q'-1}.$$

The first term of L is $|J|/|I|$ and the second is, by the change of variables $t = h(x)$,

$$\begin{aligned}\frac{1}{|I|}\int_I h'^{(1-q)}(x)\,dx &= \frac{1}{|I|}\int_J \frac{1}{h'\circ h^{-1}(t)^q}\,dt \\ &= \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right) \leqslant \frac{|J|}{|I|}RH_q((h^{-1})')^q.\end{aligned}$$

$$\left(\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt\right)^q = RH_q((h^{-1})')^q\left(\frac{|J|}{|I|}\right)^{1-q}.$$

Consequently,

$$L \leqslant \left(\frac{|J|}{|I|}\right)RH_q((h^{-1})')^{q(q'-1)}\left(\frac{|J|}{|I|}\right)^{(1-q)(q'-1)}$$

and this gives

$$A_{q'}(h') \leqslant RH_q((h^{-1})')^{q'}.$$

Conversely, if $h' \in A_{q'}$, we write with the change of variables $t = h(x)$

$$\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q}}{\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt} = \left(\frac{1}{|J|}\int_I h'(x)^{1-q}\,dx\right)^{1/q}\left(\frac{|I|}{|J|}\right)^{-1}.$$

Raise both sides to the q'^{th} power and then

$$\left\{\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q}}{\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt}\right\}^{q'} \leqslant \frac{|J|}{|I|}\left(\frac{1}{|I|}\int_I h'(x)^{1-q}\,dx\right)^{q'-1},$$

and thus $RH_q((h^{-1})')^{q'} \leqslant A_{q'}(h')$.

An easy consequence is the following lemma.

Lemma 2.6. *$(h^{-1})' \in A_\infty$ if and only if $h' \in A_\infty$.*

PROOF. If $(h^{-1})' \in A_\infty$, then $(h^{-1})' \in RH_{p'}$, for some $1 < p' < \infty$, and hence $h' \in A_p \subset A_\infty$. The same argument gives the converse direction.

Below, and throughout the paper, we will use the notation $Q \prec P$ to mean that Q is majorized by and expresión depending on P only so that for $P \leqslant T$, $Q \prec T$.

Now we are ready for one of the main theorems of the paper.

Theorem 2.7. *Let $h: \mathbb{R} \to \mathbb{R}$ be a homeomorphism as above, and suppose $1 < p_0 < \infty$, $0 \leqslant \alpha \leqslant 1$. Then, for every $w \in A_{p_0}$, $w \circ h \cdot h'^{\alpha} \in A_{p_0}$ with $A_{p_0}(w \circ h \cdot h'^{\alpha}) \prec A_{p_0}(w)$ if and only if $h' \in \bigcap_{p>1} A_p$.*

PROOF. Suppose first that $h' \in \bigcap_{p>1} A_p$. If the result holds for $\alpha = 0$ and $\alpha = 1$, it holds for $0 < \alpha < 1$ by property (c) of A_p-weights. For $\alpha = 0$, we use Property (*f*) of A_p-weights. If $w \in A_{p_0}$, then there is $\tau > 1$ so that $w^\tau \in A_{p_0}$. Let

$$L = \frac{1}{|I|}\int_I w \circ h\left(\frac{1}{|I|}\int_I (w \circ h)^{1-p_0'}\right)^{p_0-1}$$

and make the change of variables $t = h(x)$, $h(I) = J$. The first term of the product can be estimated by

$$\frac{1}{|I|}\int_I w \circ h = \frac{1}{|I|}\int_J \frac{w(t)}{h' \circ h^{-1}(t)}\,dt$$
$$\leqslant \left(\frac{1}{|I|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_J \left(\frac{1}{h' \circ h^{-1}}\right)^{\tau'}\right)^{1/\tau'}.$$

We change back to the x-variable in the last integral and obtain

$$\frac{1}{|I|}\int_I w\circ h \leqslant \left(\frac{|J|}{|I|}\right)^{1/\tau}\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau')}\right)^{1/\tau'}$$
$$\leqslant \left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau} A_\tau(h')^{1/\tau}.$$

Similarly the second term in L is

$$\frac{1}{|I|}\int_I (w\circ h)^{1-p_0'} \leqslant \left(\frac{1}{|J|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau} A_\tau(h')^{1/\tau},$$

and hence

$$A_{p_0}(w\circ h)\leqslant A_{p_0}(w^\tau)^{1/\tau}A_\tau(h')^{p_0/\tau}.$$

It is well-known [4, p. 397-9] that τ and $A_{p_0}(w^\tau)$ depend only on $A_{p_0}(w)$; in fact, $A_{p_0}(w^\tau)\leqslant cA_{p_0}(w)$ once τ has been chosen to depend on $A_{p_0}(w)$.

The case $\alpha = 1$ is similar but requires more complicated indices. We call again

$$L=\left(\frac{1}{|I|}\int_I w\circ hh'\right)\left(\frac{1}{|I|}\int_I (w\circ hh')^{1-p_0'}\right)^{p_0-1},$$

and proceeding as above, estimate the first term by

$$\frac{1}{|I|}\int_I w\circ hh' = \frac{1}{|I|}\int_J w(t)\,dt$$
$$= \frac{1}{|I|}\int_I h'\left(\frac{1}{|J|}\int_J w\right)$$
$$\leqslant \frac{1}{|I|}\int_I h'\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}.$$

The second term of the product is

$$\frac{1}{|I|}\int_I (w\circ hh')^{1-p_0'} \leqslant \left(\frac{1}{|I|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau}\left(\frac{1}{|I|}\int_J (h'\circ h^{-1})^{-\tau' p_0'}\right)^{1/\tau'}$$
$$= \left(\frac{|J|}{|I|}\right)^{1/\tau}\left(\frac{1}{|J|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau' p_0')}\right)^{1/\tau'}.$$

Hence

$$L\leqslant A_{p_0}(w^\tau)^{1/\tau}\left(\frac{1}{|I|}\int_I h'\right)^{1+(p_0-1)/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau' p_0')}\right)^{(p_0-1)/\tau'}.$$

Since $h' \in \bigcap_{p>1} A_p$, we have $h' \in A_{p'}$, $p' = \tau' p_0'$. We note that

$$p - 1 = \frac{1}{\tau' p_0' - 1} = \frac{(\tau - 1)(p_0 - 1)}{p_0 + \tau - 1}$$

and

$$\frac{p_0}{p} = \frac{p_0(p' - 1)}{p'} = \frac{\tau + p_0 - 1}{\tau},$$

and this allows us to write

$$L \leqslant A_{p_0}(w^\tau)^{1/\tau} \left\{ \frac{1}{|I|} \int_I h' \left(\frac{1}{|I|} \int_I h'^{(1-p')} \right)^{\frac{p_0 - \tau}{\tau'(\tau + p_0 - 1)}} \right\}^{(\tau + p_0 - 1)/\tau}$$
$$\leqslant A_{p_0}(w^\tau)^{1/\tau} \cdot A_{p_0}(h')^{p_0/p}$$

which completes the proof for $\alpha = 1$.

Now we suppose that for a fixed α and p_0, and any $w \in A_{p_0}$, $w \circ h \cdot h'^\alpha \in A_{p_0}$ with $A_{p_0}(w \circ h \cdot h'^\alpha) \prec A_{p_0}(w)$. We will first show, using extrapolation, that there is $\eta > 0$ so that $h'^\eta \in \bigcap_{p>1} A_p$. Since $w \circ h \cdot h'^\alpha \in A_{p_0}$, by [6]

$$\int M f^{p_0} w \circ h h'^\alpha \leqslant C \int f^{p_0} w \circ h h'^\alpha$$

where $C = C_{p_0} A_{p_0}(w \circ h h'^\alpha)^{p(p_0'+1)}$ [2]. The substitution $t = h(x)$ gives

$$\int M f^{p_0}(h^{-1}(t)) \frac{w(t)}{h' \circ h^{-1}(t)^{1-\alpha}} \, dt \leqslant C \int f(h^{-1}(t))^{p_0} \frac{w(t)}{h' \circ h^{-1}(t)^{1-\alpha}} \, dt.$$

Let $g(t) = f \circ h^{-1}(t) / h' \circ h^{-1}(t)^{(1-\alpha)/p_0}$. Then $f(x) = h'(x)^{(1-\alpha)/p_0} g \circ h(x)$. The sublinear operator

$$Tg(t) = \frac{M(h'^{(1-\alpha)/p_0} g \circ h)(h^{-1}(t))}{h' \circ h^{-1}(t)^{(1-\alpha)/p_0}}$$

satisfies

$$\int T g^{p_0} w \leqslant C \int g^{p_0} w, \qquad w \in A_{p_0} \quad \text{and} \quad C \prec A_{p_0}(w).$$

We can now apply the extrapolation theorem [4, p. 448] and obtain

$$\int T g^p w \leqslant C \int g^p w,$$

$w \in A_p$, $1 < p < \infty$ with $C \prec A_p(w)$. We undo the change of variables and get

$$\int M(h'^{(1-\alpha)/p_0} g \circ h)^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \leqslant C \int g \circ h^p w \circ hh'$$
$$= C \int (g \circ h \cdot h'^{(1-\alpha)/p_0})^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0},$$

which in terms of $f = g \circ hh'^{(1-\alpha)/p_0}$ says

$$\int Mf^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \leqslant C \int f^p w \circ hh'^{1-(1-\alpha)p/p_0}$$

guaranteeing that $w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \in A_p$. We choose $w = 1$ and restrict $1 < p < p_0$, and then $h'^{\alpha} \in A_p$, $1 < p < p_0$, if $\alpha > 0$, while if $\alpha = 0$, we can restrict $1 < p < gp_0/2$ some $0 < g < 1$ and conclude $h'^{1-g/2} \in \bigcap_{p>1} A_p$, and this completes the proof of the claim that $h'^{\eta} \in \bigcap_{p>1} A_p$ for some $\eta > 0$.

To conclude that $h' \in \bigcap_{p>1} A_p$, we want to apply Lemma 2.4 which requires us to show that $h' \in A_\infty$. To do this, we first show that h preserves BMO. If $u \in \mathrm{BMO}$, then $e^{\lambda u} \in A_{p_0}$ for some $\lambda > 0$, and thus by hypothesis, $e^{\lambda u \circ h} h'^{\alpha} \in A_{p_0}$. Hence $\lambda u \circ h + \alpha \log h' \in \mathrm{BMO}$. Since $h'^{\eta} \in \bigcap_{p>1} A_p$, $\log h'$ $\in \mathrm{BMO}$ and thus $u \circ h \in \mathrm{BMO}$. By [5], $(h^{-1})' \in A_\infty$ and hence by Lemma 2.6, $h' \in A_\infty$.

The situation for $p_0 = 1$ is different depending on whether $\alpha = 0$ or $0 < \alpha < 1$.

Theorem 2.8. *Let $h: \mathbb{R} \to \mathbb{R}$ be a homeomorphism as before and suppose that $0 < \alpha \leqslant 1$. Then for every $w \in A_1$, $w \circ h \cdot h'^{\alpha} \in A_1$, with $A_1(w \circ h \cdot h'^{\alpha}) \prec A_1(w)$ if and only if $h' \in A_1$.*

Theorem 2.9. *Let $h: \mathbb{R} \to \mathbb{R}$ be a homeomorphism as before. Then for every $w \in A_1$, $w \circ h \in A_1$ with $A_1(w \circ h) \prec A_1(w)$ if and only if $h' \in \bigcap_{p>1} A_p$.*

We begin with the proof of Theorem 2.9 since it requires the least change from the proof of Theorem 2.7. If $h' \in \bigcap_{p>1} A_p$ and $w \in A_1$, we first choose $\tau > 1$ so that $w^\tau \in A_1$. The argument gave

$$\frac{1}{|I|} \int_I w \circ h \leqslant \left(\frac{1}{|J|} \int_J w^\tau \right)^{1/\tau} A_\tau(h')^{1/\tau}.$$

Since $w^\tau \in A_1$, this can be estimated by

$$\frac{1}{|I|} \int_I w \circ h \leqslant \left(\inf_J w^\tau \right)^{1/\tau} A_\tau(h')^{1/\tau} = A_\tau(h')^{1/\tau} \inf_I w \circ h.$$

Conversely, if for every $w \in A_1$, $w \circ h \in A_1$, with $A_1(w \circ h) \prec A_1(w)$, we apply the factorization theorem [4, p. 434] to show that the conditions of Theorem 2.7 hold with, e.g., $p_0 = 2$. For $w \in A_2$, $w = u_1/u_2$, for $u_1, u_2 \in A_1$

with $A_1(u_j) \prec A_2(w)$. Since $w \circ h = u_1 \circ h / u_2 \circ h$, we see that $w \circ h \in A_2$ and $A_2(w \circ h) \prec A_2(w)$, and Theorem 2.7 gives the result.

The proof of Theorem 2.8 is easy with all the tools now available to us. We only need to show that, if $h' \in A_1$, then $w \circ h \cdot h' \in A_1$ (property (*c*) of A_p-weights and Theorem 2.9 show then $w \circ hh'^{\alpha} \in A_1$).

If $h' \in A_1$ and $h(I) = J$,

$$\frac{1}{|I|}\int_I w \circ hh' = \frac{1}{|I|}\int_J w = \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J w\right)$$
$$\leqslant A_1(h') \inf_I h' A_1(w) \inf_I w \circ h$$
$$\leqslant A_1(h') A_1(w) \inf_I w \circ hh'.$$

Conversely, suppose

$$A_1(w \circ h \cdot h'^{\alpha}) \prec A_1(w), \qquad 0 < \alpha \leqslant 1.$$

The choice $w = 1$ shows that $h'^{\alpha} \in A_1$. The factorization theorem again shows that $h' \in \bigcap_{p>1} A_p$ because

$$w \circ h = \frac{u_1 \circ h \cdot h'^{\alpha}}{u_2 \circ h \cdot h'^{\alpha}}.$$

By Lemma 2.1, $h' \in A_1$.

Remarks. **1.** The extension to n-dimensions presents no real difficulties. If a homeomorphism preserves A_p, it must also preserve BMO and then h and h^{-1} are quasi-conformal by [9]. We will therefore assume that h is smooth when stating the next result. We denote by J_h the Jacobian of h.

Theorem 2.10. *Let h be a smooth quasi-conformal homeomorphism,* $h: \mathbb{R}^n \to \mathbb{R}^n$.

(*a*) *If* $1 < p_0 < \infty$, $0 \leqslant \alpha \leqslant 1$, *then* $A_{p_0}(w \circ h |J_h|^{\alpha}) \prec A_{p_0}(w)$ *if and only if* $|J_h| \in \bigcap_{p>1} A_p$.

(*b*) $A_1(w \circ h) \prec A_1(w)$ *if and only if* $|J_h| \in \bigcap_{p>1} A_p$.

(*c*) $A_1(w \circ h |J_h|^{\alpha}) \prec A_1(w)$, $0 < \alpha \leqslant 1$, *if and only if* $|J_h| \in A_1$.

2. The fundamental estimate of Theorem 2.7 contains a sufficient condition for a homeomorphism to preserve a single weight $w \in A_{p_0}$ since it says

$$A_{p_0}(w \circ h) \leqslant C A_{p_0}(w^{\tau})^{1/\tau} A_{\tau}(h')^{p_0/\tau}.$$

It can also be applied to the local A_p-classes $A_{p,\Omega}$ [4, p. 438].

3. Theorems 2.7, 2.8, 2.9 contain generalizations of themselves allowing negative powers of h', because if $h' \in \bigcap_{p>1} A_p$ and $w \in A_{p_0}$, then $w^{1-p_0'} \in A_{p_0'}$, and hence $(w \circ h)^{1-p_0'} h'^{\alpha} \in A_{p_0'}$, $0 \leqslant \alpha \leqslant 1$. We use duality again and see that $w \circ h \cdot h'^{\alpha(1-p_0)} \in A_{p_0}$. Thus $w \circ h \cdot h'^{\beta} \in A_{p_0}$ for $1 - p_0 \leqslant \beta \leqslant 1$.

The mapping $T_h w(x) = w \circ h(x)$, where $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism as before, is not an onto map from $A_{p_0} \to A_{p_0}$ in general, but it is possible to determine precisely when it is onto.

Theorem 2.11. *Let* $1 \leqslant p_0 < \infty$. *Then* $T_h: A_{p_0} \to A_{p_0}$ *is onto with*

$$A_{p_0}(T_h w) \prec A_{p_0}(w), \qquad A_{p_0}(T_h^{-1} w) \prec A_{p_2}(w)$$

if and only if $\log h' \in \mathrm{BMO}$*-closure* L^∞.

PROOF. For the the necessity, if $T_h: A_{p_0} \to A_{p_0}$ is onto, T_h^{-1} is defined and $T_h^{-1} = T_{h^{-1}}$. It follows from Theorems 2.7 and 2.9 that h' and $(h^{-1})'$ are in $\bigcap_{p>1} A_p$. This implies that

$$\frac{1}{h'} \in \bigcap_{p>1} A_p,$$

for, if we fix $1 < p_1 < \infty$ and apply Theorem 2.7 to $h' \in \bigcap_{p>1} A_p$ and $(h^{-1})' \in A_{p_1}$ we obtain

$$(h^{-1})' \circ h(x) = \frac{1}{h'(x)} \in A_{p_1}.$$

By [4, p. 474] $\log h' \in \mathrm{BMO}$-closure L^∞.

For the sufficiency, if we suppose that $\log h' \in \mathrm{BMO}$-closure L^∞, then $h' \in \bigcap_{p>1} A_p$ and $h' \in \bigcap_{q<\infty} RH_q$. Then by Lemma 2.5, $(h^{-1})' \in \bigcap_{p>1} A_p$. By Theorem 2.7 $T_{h^{-1}}: A_{p_0} \to A_{p_0}$ and thus T_h is onto.

Remark. The situation for A_p contrasts with that for BMO, since Jones shows in [5] that whenever T_h is bounded on BMO, it is onto, because of the result we gave here as Lemma 2.6.

We can also characterize the pointwise multipliers of A_{p_0}.

Theorem 2.12. *Let* $1 < p_0 < \infty$, *and let* $\varphi: \mathbb{R} \to \mathbb{R}_+$. *Then for every* $w \in A_{p_0}$, $w \cdot \varphi \in A_{p_0}$ *with* $A_{p_0}(w\varphi) \prec A_{p_0}(w)$ *if and only if* $\log \varphi \in \mathrm{BMO}$*-closure* L^∞.

PROOF. For the necessity, note that $\varphi^n \in A_{p_0}$, $n = 1, 2, \ldots$, and thus $\varphi \in A_{(p_0-1)/n+1}$ by [4, p. 394] and so $\varphi \in \bigcap_{p>1} A_p$.

Next, we claim

$$\frac{1}{\varphi} \in \bigcap_{p>1} A_p.$$

If $w \in A_{p_0'}$, then $w^{1-p_0} \in A_{p_0}$ and hence $w^{1-p_0}\phi^n \in A_{p_0}$, $n = 1, 2, \ldots$. Thus $w \cdot \varphi^{n(1-p_0')} \in A_{p_0'}$ from which we get by taking $w = 1$, $\varphi^{-1} \in A_{1+1/n}$ or $\varphi^{-1} \in \bigcap_{p>1} A_p$. Again by [4, p. 474], $\log \varphi \in$ BMO-closure L^∞.

For the sufficiency, suppose $\log \varphi \in$ BMO-closure L^∞. We define

$$h(x) = \int_0^x \varphi.$$

Then $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $h' = \varphi$. Our first claim is that h is onto $\mathbb{R}$, i.e., $\int_0^x \varphi \to \infty$ as $x \to \infty$. The condition on φ implies that $\varphi \in A_2$ and hence, if

$$\varphi(E) = \int_E \varphi,$$

for any $E \subseteq I$,

$$\varphi(I) \leqslant c\left(\frac{|I|}{|E|}\right)^2 \varphi(E),$$

with c independent of E, I. Apply this to $I_n = [0, 2^n]$ and $J = [2^{n-1}, 2^n]$ and note that

$$\varphi(I_n) = \varphi(I_{n-1}) + \varphi(J).$$

Since

$$\varphi(I_{n-1}) \leqslant \varphi(I_n) \leqslant 4c\varphi(J),$$

we see that

$$\varphi(I_n) \geqslant \left(1 + \frac{1}{4c}\right)\varphi(I_{n-1}),$$

and iterating

$$\varphi(I_n) \geqslant \left(1 + \frac{1}{4c}\right)^n \varphi(I_0),$$

and our claim is proved.

Since our hypothesis is that $\log \varphi \in$ BMO-closure L^∞, by Theorem 2.12, T_h is onto A_{p_0} for any $1 \leqslant p_0 < \infty$, and thus by Theorem 2.7, $(h^{-1})' \in \bigcap_{p>1} A_p$. For a given $w \in A_{p_0}$, we write $w\varphi = [w \circ h^{-1} \circ h]h'$ which is in A_{p_0} because $w \circ h^{-1} \in A_{p_0}$. One checks that $A_{p_0}(w\varphi) \prec A_{p_0}(w)$.

Remark. The multiplier problem for BMO is known [11] but quite different.

The pointwise multipliers of A_1 require a slightly different approach. For $1 < r < \infty$, let $A_{1,r}$ be the class of all $f: \mathbb{R} \to \mathbb{R}_+$ for which $f^r \in A_1$.

Lemma 2.13. *$f \in \bigcap_{r<\infty} A_{1,r}$ if and only if $f \in A_1$ and $1/f \in \bigcap_{p>1} A_p$.*

PROOF. For the necessity simply observe that $f \in A_1$ and

$$\frac{1}{|I|}\int_I f^{-1}\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant \frac{C}{|I|}\int_I \frac{1}{f}\cdot \inf_I f \leqslant C \sup_I \frac{1}{f}\cdot \inf_I f \leqslant C.$$

For the converse,

$$\frac{1}{|I|}\int_I f \leqslant C \inf_I f,$$

and since

$$1 \leqslant \frac{1}{|I|}\int_I f\left(\frac{1}{|I|}\int_I \frac{1}{f}\right),$$

we see that

$$C \sup_I \frac{1}{f} \leqslant \frac{1}{|I|}\int_I \frac{1}{f}.$$

Since for $1 < p < \infty$,

$$\frac{1}{|I|}\int_I \frac{1}{f}\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant C_p < \infty,$$

we get

$$\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant C \inf_I f \quad \text{or} \quad f^{p'-1} \in A_1.$$

Theorem 2.14. *Let $\varphi: \mathbb{R} \to \mathbb{R}_+$. Then $w \in A_1$ implies $w \cdot \varphi \in A_1$ with $A_1(w\varphi) \prec A_1(w)$ if and only if $\varphi \in \bigcap_{r<\infty} A_{1,r}$.*

PROOF. If $w\varphi \in A_1$ for $w \in A_1$, then $\varphi \in A_1$ by taking $w = 1$. Let $1 < p_0 < 2$ and $w \in A_{p_0}$. Then $w = u \cdot v^{1-p_0}$, $u, v \in A_1$, and $A_1(u)$, $A_1(v) \prec A_{p_0}(w)$ as well as

$$A_{p_0}(w) \leqslant A_1(u)A_1(v)^{p_0-1}$$

[4, p. 434]. Thus

$$w \cdot \varphi^{2-p_0} = u \cdot \varphi \cdot (v \cdot \varphi)^{1-p_0}$$

and $w \cdot \varphi^{2-p_0} \in A_{p_0}$ with $A_{p_0}(w \cdot \varphi^{2-p_0}) \prec A_{p_0}(w)$. Therefore, by Theorem 2.12, $\log \varphi$ is in the closure of L^∞ in BMO which implies that $1/\varphi \in \bigcap_{p>1} A_p$. Hence $\varphi \in \bigcap_{r>\infty} A_{1,r}$ by Lemma 2.13.

For the converse we use the technique of Theorem 2.12 and let

$$h(x) = \int_0^x \varphi.$$

Then, as shown there, h: $\mathbb{R} \to \mathbb{R}$ is a homeomorphism onto and $(h^{-1})' \in \bigcap_{p>1} A_p$ $h' \in A_1$. For $w \in A_1$, write $w \cdot \varphi = w \circ h^{-1} \circ h \cdot h'$, and, since $v = w \circ h^{-1} \in A_1$, by Theorem 2.9, we see that $v \circ h \cdot h' \in A_1$ by applying Theorem 2.8.

3. Homeomorphisms Preserving A_∞ and RH_{p_0}

We shall see that the homeomorphisms preserving A_∞ and RH_{p_0} will be the same as those preserving A_{p_0} under the condition $h' \in \bigcap_{p>1} A_p$, but the pointwise multiplier condition is different. We begin with some preliminary results, including a proof of Theorem 1.1 that gives the constants we need.

Lemma 3.1. *Let* $1 < p_0 < \infty$. *Then*

$$\frac{A_\infty(w^{p_0})^{1/p_0}}{A_\infty(w)} \leqslant RH_{p_0}(w) \leqslant A_\infty(w^{p_0})^{1/p_0}.$$

PROOF. For the first inequality simply note that

$$\frac{1}{|I|}\int_I w^{p_0} \leqslant RH_{p_0}(w)^{p_0}\left(\frac{1}{|I|}\int_I w\right)^{p_0},$$

and hence for $q < \infty$,

$$\frac{1}{|I|}\int_I w^{p_0}\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{q-1}$$
$$\leqslant RH_{p_0}(w)^{p_0}\left\{\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{(q-1)/p_0}\right\}^{p_0}.$$

Thus

$$A_q(w^{p_0}) \leqslant RH_{p_0}(w)^{p_0} \cdot A_{\frac{p_0+q-1}{p_0}}(w)^{p_0}$$

Let $q \uparrow \infty$.

The second inequality can be obtained as follows

$$\frac{\left(\frac{1}{|I|}\int_I w^{p_0}\right)^{1/p_0}}{\frac{1}{|I|}\int_I w} = \frac{\left\{\frac{1}{|I|}\int_I w^{p_0}\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{q-1}\right\}^{1/p_0}}{\frac{1}{|I|}\int_I w\left\{\frac{1}{|I|}\int_I w^{p_0(1-q')}\right\}^{(q-1)/p_0}} \leqslant A_q(w^{p_0})^{1/p_0}$$

since the denominator is $\geqslant 1$. Hence

$$RH_{p_0}(w) \leqslant A_q(w^{p_0})^{1/p_0}.$$

Let $q \uparrow \infty$.

Remark. We shall see below in Theorem 3.4 that the quantity $RH_{p_0}(w)$ is not as convenient as $RH_{p_0}(w) \cdot A_\infty(w) \equiv \overline{RH}_{p_0}(w)$.

Lemma 3.2. *Let $1 < p_0 < \infty$. Then*

$$\max\{A_\infty(w), A_\infty(w^{1-p_0'})^{p_0-1}\} \leqslant A_{p_0}(w) \leqslant A_\infty(w)A_\infty(w^{1-p_0'})^{p_0-1}.$$

PROOF. The first inequality is easy since both terms on the left are $\leqslant A_{p_0}(w)$. For the second inequality simply note that

$$\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{1-p_0'}\right)^{p_0-1} = \frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}$$
$$\cdot\left\{\frac{1}{|I|}\int_I w^{1-p_0'}\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{(1-q)/(p_0-1)}\right\}^{p_0-1}$$

Using

$$1 \leqslant \left(\frac{1}{|I|}\int_I \varphi\right)\left(\frac{1}{|I|}\int_I \frac{1}{\varphi}\right),$$

the expression in { } is

$$\leqslant \frac{1}{|I|}\int_I w^{1-p_0'}\left(\frac{1}{|I|}\int_I w^{q'-1}\right)^{(q-1)/(p_0-1)}$$
$$= \left(\frac{1}{|I|}\int_I w^{(1-p_0')(q'-1)/(1-p_0')}\right)^{(q-1)/(p_0-1)}\left(\frac{1}{|I|}\int_I w^{1-p_0'}\right).$$

If

$$r - 1 = \frac{q-1}{p_0-1},$$

one checks that

$$r' - 1 = \frac{p_0 - 1}{q - 1} = \frac{q' - 1}{p_0' - 1}.$$

Hence for $q < \infty$,

$$A_{p_0}(w) \leqslant A_q(w)\{A_{1+(q-1)/(p_0-1)}(w^{1-p_0'})\}^{p_0-1}.$$

Let $q \uparrow \infty$.

Lemma 3.3.

(i) $A_\infty(w^{1-q'})^{q-1} \searrow \sigma(w)$ *as* $q \nearrow \infty$ *and*
(ii) $\sigma(w) \leqslant A_\infty(w)$.

PROOF. (i) For $p < \infty$ and $q_1 > q_2$ we have

$$\left(\frac{1}{|I|}\int_I w^{1-q_1'}\right)^{q_1-1} \leqslant \left(\frac{1}{|I|}\int_I w^{1-q_2'}\right)^{q_2-1},$$

so that

$$A_p(w^{1-q_1'})^{q_1-1} \leqslant A_p(w^{1-q_2'})^{q_2-1}.$$

Let $p \to \infty$ and (i) follows. To prove (ii), let $\lambda < \sigma(w)$. We claim that for $q < \infty$, $\lambda \leqslant A_q(w)$. Fix q, and note that

$$\lambda < A_\infty(w^{1-q'})^{q-1} \leqslant A_p(w^{1-q'})^{q-1},$$

for any $p < \infty$. Since

$$\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}\left(\frac{1}{|I|}\int_I w^{(1-q')(1-p')}\right)^{(p-1)(q-1)}$$
$$\leqslant \left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}\left(\frac{1}{|I|}\int_I w\right),$$

if, $(p-1)(q-1) > 1$, the claim and hence (ii) follows.

Theorem 3.4. *Let* $1 < p_0 < \infty$. *Then the following statements are equivalent for a homeomorphism h as in section* 2.

(1) $h' \in \bigcap_{p>1} A_p$.
(2) $w \in A_\infty$ *implies* $w \circ h \in A_\infty$ *with* $A_\infty(w \circ h) \prec A_\infty(w)$.
(3) $w \in RH_{p_0}$ *implies* $w \circ h \in RH_{p_0}$ *with* $\overline{RH}_{p_0}(w \circ h) \prec \overline{RH}_{p_0}(w)$.
(4) $w \in A_{p_0}$ *implies* $w \circ h \in A_{p_0}$ *with* $A_{p_0}(w \circ h) \prec A_{p_0}(w)$.

PROOF. We will be brief. First (1) $\Rightarrow$ (4) is Theorem 2.7. The equivalence (2) $\Rightarrow$ (3) is Lemma 3.1. The implication (2) $\Rightarrow$ (4) follows from Lemma 3.2. There remains (4) $\Rightarrow$ (2). Since (4) $\Rightarrow$ (1) by Theorem 2.7. $A_p(w \circ h) \prec A_p(w), p < \infty$. Let $w \in A_\infty$. By Lemma 3.3 we can choose p_0 so that $A_\infty(w^{1-p_0'})^{p_0-1} \leqslant 2A_\infty(w)$. Then $A_\infty(w \circ h) \leqslant A_{p_0}(w \circ h) \prec A_{p_0}(w)$, and $A_{p_0}(w) \leqslant A_\infty(w)A_\infty(w^{1-p_0'})^{p_0-1}$ by Lemma 3.2. Hence $A_\infty(w \circ h) \prec A_\infty(w)$.

We come now to the pointwise multipliers of A_∞ and RH_{p_0}.

Theorem 3.5. *Let $1 < p_0 < \infty$. Then the following statements are equivalent for $\varphi: \mathbb{R} \to \mathbb{R}_+$.*

(1) $\varphi^n \in A_\infty$, $n = 1, 2, \ldots$.

(2) $\varphi \in \bigcap_{r<\infty} RH_r$.

(3) $w \in A_\infty$ *implies* $w \cdot \varphi \in A_\infty$.

(4) $w \in RH_{p_0}$ *implies* $w \cdot \varphi \in RH_{p_0}$.

PROOF. From Lemma 3.1 we get (1) $\Leftrightarrow$ (2). It is clear that (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (3) can be seen as follows. Let $w \in A_\infty$. Then for some $p_0 < \infty$, $w \in A_{p_0}$. Hence there is $\tau > 1$ so that $w^\tau \in A_{p_0} \subset A_p, p \geqslant p_0$. Since by (1) $\varphi^{\tau'} \in A_\infty$, we can choose $p \geqslant p_0$ so that $\varphi^{\tau'} \in A_p$. An easy application of Hölder's inequality shows that

$$A_p(w\varphi) \leqslant A_p(w^\tau)^{1/\tau} A_p(\varphi^{\tau'})^{1/\tau'}.$$

Hence $w \cdot \varphi \in A_\infty$. Since (4) implies $\varphi^n \in RH_{p_0} \subset A_\infty$, $n = 1, 2, \ldots$, we get (4) $\Rightarrow$ (1). We complete the proof by showing (3) $\Rightarrow$ (4). If $\alpha > 0$, then from (3), $w \in A_\infty$ implies $w \cdot \varphi^\alpha \in A_\infty$, since (3) $\Rightarrow$ (1). Let $w \in RH_{p_0}$. Then, by Lemma 3.1,

$$RH_{p_0}(w \cdot \varphi)^{p_0} \leqslant A_\infty(w^{p_0}\varphi^{p_0}) < \infty$$

sience $w^{p_0} \in A_\infty$.

Remark. If $\varphi = P(x)$ is a polynomial, then (1) is satisfied, and hence a polynomial is a pointwise multiplier of A_∞ and RH_{p_0}. However, it is clear that this is not the case for A_{p_0}.

4. Homeomorphisms Preserving Double Weights

The argument presented in Theorem 2.7 for single weights is not applicable to double weights $(u, v) \in A_p$ since it is no longer true that there is $\tau > 1$ with $(u^\tau, v^\tau) \in A_p$. In fact the existence of such a $\tau > 1$ takes us back to the single weight case [7]. For the study of the homeomorphisms which preserve $(u, v) \in A_p$ we need the following extrapolation theorem.

Theorem 4.1. *Let $1 < p_0 < \infty$, and let $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing. Assume that T is a sublinear operator so that for every $(u, v) \in A_{p_0}$,*

$$u\{x: |Tf(x)| > y\} \leqslant \frac{\gamma(A_{p_0}(u, v))^{p_0}}{y^{p_0}} \|f\|_{p_0, v}^{p_0}.$$

Then, if $1 < p < p_0$, and $(u, v) \in A_p$,

$$u\{x: |Tf(x)| > y\} \leqslant \frac{C^p}{y^p} \|f\|_{p, v}^p.$$

where

$$C \leqslant cp_0 A_p(u, v)[\gamma(A_p(u, v)^{(p'-1)(p_0-1)})]^{1/p_0}.$$

We use the notation

$$A_p(u, v) = \max\left\{1, \sup_Q \left(\frac{1}{|Q|}\int_Q u\right)\left(\frac{1}{|Q|}\int_Q v^{1-p'}\right)^{p-1}\right\}.$$

We will not prove this theorem since the proof is the same as in [8] by keeping track of the constants involved. In the applications that follow, $\gamma(t) = ct^{1/p_0}$.

As before we shall assume that $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism onto so that h, h^{-1} are locally absolutely continuous and $h' \geqslant 0$.

Theorem 4.2. $1 < p < \infty$, $1 - p \leqslant \alpha \leqslant 1$. *If $h' \in A_1$, then for every $(u, v) \in A_p$, $(u \circ h \cdot h'^{\alpha}, v \circ h \cdot h'^{\alpha}) \in A_p$ with $A_p(u \circ h \cdot h'^{\alpha}, v \circ h \cdot h'^{\alpha}) \leqslant CA_p(u, v)$, C independent of α.*

PROOF. Let

$$L = \frac{1}{|I|}\int_I u \circ h \cdot h'^{\alpha}\left(\frac{1}{|I|}\int_I (v \circ h)^{1-p'} \cdot h'^{\alpha(1-p')}\right)^{p-1}.$$

Let $t = h(x)$, $h(I) = J$. Then

$$\frac{1}{|I|}\int_I u \circ h \cdot h'^{\alpha} = \frac{|J|}{|I|}\frac{1}{|J|}\int_J \frac{u(t)}{h'^{(1-\alpha)} \circ h^{-1}(t)},$$

and

$$\frac{1}{|I|}\int_I \frac{1}{v \circ h'^{p'-1} \cdot h'^{\alpha(p'-1)}} = \frac{|J|}{|I|}\frac{1}{|J|}\int_J \frac{1}{v(t)^{p'-1} \cdot (h' \circ h^{-1}(t))^{\alpha(p'-1)+1}}.$$

Hence

$$L \leqslant \left(\frac{|J|}{|I|}\right)^p \frac{1}{\inf h'^{(1-\alpha)} \cdot \inf h'^{(\alpha+p-1)}} A_p(u, v),$$

since $1 - \alpha \geqslant 0$, $\alpha + p - 1 \geqslant 0$. Since $h' \in A_1$, i.e.,

$$\frac{1}{|I|} \int_I h' = \frac{|J|}{|I|} \leqslant C \inf_I h',$$

the proof is complete.

Remark. We do not know whether or not the converse of Theorem 4.2 is true (except for $\alpha = 1$ as we will see). From Theorem 2.7, however, the A_p-norm inequality does imply that $h' \in \bigcap_{p>1} A_p$, if $0 \leqslant \alpha \leqslant 1$. If we change the problem somewhat, we obtain a necessary and sufficient condition.

Theorem 4.3. *Let* $1 < p_0 < \infty$, $1 - p_0 \leqslant \alpha$. *Then for every* $(u, v) \in A_{p_0}$, $(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \in A_{p_0}$ *with* $A_{p_0}(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \leqslant CA_{p_0}(u, v)$ *if and only if*

$$\frac{1}{|I|} \int_I h' \leqslant C \inf_I (h')^{\alpha/p_0 + 1/p_0'},$$

C independent of I.

PROOF. The sufficiency can be handled as in Theorem 5.

$$\frac{1}{|I|} \int_I u \circ h \cdot h' = \frac{|J|}{|I|} \frac{1}{|J|} \int_J u(t)\, dt$$

$$\frac{1}{|I|} \int_I \frac{1}{(v \circ h)^{p_0'-1} \cdot h'^{\alpha(p_0'-1)}} = \frac{|J|}{|I|} \frac{1}{|J|} \int_J \frac{1}{v(t)^{p_0'-1}(h' \circ h^{-1}(t))^{\alpha(p_0'-1)+1}}.$$

Hence

$$A_{p_0}(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \leqslant A_p(u, v) \cdot \left(\frac{|J|}{|I|}\right)^{p_0} \frac{1}{\inf_I (h')^{\alpha+p_0-1}},$$

since $\alpha + p_0 - 1 \geqslant 0$.

For the necessity we note first that [6]

$$\int_{\{Mf>y\}} u \circ h \cdot h' \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int f^{p_0}(x) v \circ h(x) \cdot h'^{\alpha}(x)\, dx.$$

We change variables $t = h(x)$ and get

$$\int_{\{t:\, Mf[h^{-1}(t)] > y\}} u(t)\,dt \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int f[h^{-1}(t)]^{p_0} \frac{v(t)}{[h' \circ h^{-1}(t)]^{1-\alpha}}\,dt.$$

If we set

$$g(t) = \frac{f \circ h^{-1}(t)}{[h' \circ h^{-1}(t)]^{(1-\alpha)/p_0}},$$

then $f(x) = h'(x)^{(1-\alpha)/p_0} g \circ h(x)$, and

$$\int_{\{t:\, M(h'^{(1-\alpha)/p_0} g \circ h)[h^{-1}(t)] > y\}} u(t)\,dt \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int g^{p_0} v.$$

The hypothesis of Theorem 4.1 are satisfied with

$$\gamma(t) = ct^{1/p_0}$$

and

$$Tg(t) = M(h'^{(1-\alpha)/p_0} g \circ h)[h^{-1}(t)].$$

Hence for $1 < p < p_0$ and $(u, v) \in A_p$,

$$u\{x: |Tg(x)| > y\} \leqslant \frac{C^p}{y^p} \|g\|_{p,v}^p$$

with

$$C = cp_0 A_p(u, v)[\gamma(A_p(u, v)^{(p'-1)(p_0-1)})]^{1/p_0}.$$

We let now $u = v = 1$ so that $A_p(u, v) = 1$. With $t = h(x)$ we get

$$\int_{\{x:\, M(h'^{(1-\alpha)/p_0} g \circ h)(x) > y\}} h'(x)\,dt \leqslant \frac{C^p}{y^p} \int (g \circ h)^p(x) \cdot h'(x)\,dx,$$

where C is an absolute constant independent of p. Since

$$g \circ h(x) = \frac{f(x)}{h'^{(1-\alpha)/p_0}(x)},$$

the above is

$$\int_{\substack{I \\ \{x:\, Mf(x) > y\}}} h'(x)\,dx \leqslant \frac{C^p}{y^p} \int f^p(x) h'(x)^{1-(1-\alpha)p/p_0}\,dx.$$

We test this inequality with $f = \chi_I h'^{\gamma}$, where γ will be chosen later. Since

$$I \subset \left\{x: Mf(x) \geqslant \frac{1}{|I|} \int_I h'^{\gamma} \equiv y\right\}$$

we obtain

$$\int_I h'(x)\,dx \leqslant C^p \frac{|I|^p}{\left(\int_I h'^{\gamma}\right)^p} \int_I h'^{\gamma p + 1 - (1-\alpha)p/p_0}.$$

We choose γ so that

$$\gamma = \gamma p + 1 - \frac{1-\alpha}{p_0} p, \quad (p-1)\gamma = \frac{1-\alpha}{p_0} \cdot p - 1 \text{ or } \gamma = p'\left[\frac{1-\alpha}{p_0} - \frac{1}{p}\right].$$

Hence

$$\frac{1}{|I|} \int_I h' \leqslant C^p |I|^{p-1} \left(\int_I h'^{\gamma}\right)^{1-p}.$$

We let now $p \downarrow 1$. Then

$$\left(\int_I h'^{[(1-\alpha)/p_0 - 1/p]p'}\right)^{p/p'} \to \sup_I h'^{(1-\alpha)/p_0 - 1} = \frac{1}{\inf_I h'^{1-(1-\alpha)/p_0}}.$$

Hence

$$\frac{1}{|I|} \int_I h' \leqslant C \inf_I h'^{\alpha/p_0 + 1/p_0'}.$$

Corollary 4.4. *Let $1 < p_0 < \infty$. Then for every $(u, v) \in A_{p_0}$, $(u \circ h \cdot h', v \circ h \cdot h') \in A_{p_0}$ with $A_{p_0}(u \circ h \cdot h, v \circ h \cdot h') \leqslant C A_{p_0}(u, v)$ if and only if $h' \in A_1$.*

As in the single weight case we ask when $T_h(u, v) = (u \circ h \cdot h', v \circ h \cdot h'^{\alpha})$ takes A_{p_0} *onto* A_{p_0}.

Theorem 4.5. *Let $1 < p_0 < \infty$ and $\alpha > 1 - p_0$. Then $T_h: A_{p_0} \to A_{p_0}$ is onto with $A_{p_0}(T_h(u, v))$, and $A_{p_0}(T_h^{-1}(u, v))$ both $\leqslant C A_{p_0}(u, v)$ if and only if $h' \simeq 1$.*

PROOF. First assume that T_h is onto. Then, if $(u, v) \in A_{p_0}$, there is $(\bar{u}, \bar{v}) \in A_{p_0}$ with $(\bar{u} \circ h \cdot h', \bar{v} \circ h \cdot h'^{\alpha}) = (u, v)$. Hence

$$\bar{u}(t) = u \circ h^{-1}(t) \cdot \frac{1}{h' \circ h^{-1}(t)} = u \circ h^{-1}(t) \cdot (h^{-1})'(t)$$

and

$$\bar{v}(t) = v \circ h^{-1}(t) \cdot (h^{-1})'^{\alpha}(t).$$

Hence $T_h^{-1} = T_{h^{-1}}$, and Theorem 4.3 gives the two inequalities

$$\frac{1}{|I|} \int_I h' \leqslant C \inf_I h'^{\alpha/p_0 + 1/p_0'},$$

$$\frac{1}{|J|} \int_J (h^{-1})' \leqslant C \inf_J (h^{-1})'^{\alpha/p_0 + 1/p_0'}.$$

If $h(I) = J$, the second inequality can be rewritten as

$$C \sup_I h'^{\alpha/p_0 + 1/p_0'} \leqslant \frac{1}{|I|} \int_I h'.$$

This together with the first inequality shows that

$$C \sup_I h' \leqslant \inf_I h'$$

(note $\alpha/p_0 + 1/p_0'$ since $\alpha > 1 - p_0$), and $h' \simeq 1$.

Conversely, if $h' \simeq 1$, then there are constants $0 < c_1 \leqslant h' \leqslant c_2 < \infty$, and so

$$\frac{1}{|I|} \int_I h' \leqslant \frac{C_2}{C_1^{\gamma}} C_1^{\gamma} \leqslant C \inf_I h'^{\gamma}, \qquad \gamma = \frac{\alpha}{p_0} + \frac{1}{p_0'}.$$

Similarly

$$\frac{1}{|I|} \int_I (h^{-1})' \leqslant C \inf_I (h^{-1})'^{\alpha/p_0 + 1/p_0'}.$$

Hence T_h is onto.

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