

A Harnack Inequality
Approach to the
Regularity of Free
Boundaries.
Part I: Lipschitz Free
Boundaries are $C^{1,\alpha}$

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Introduction

1. This is the first in a series of papers where we intend to show, in several steps, the existence of «classical» (or as classical as possible) solutions to a general two-phase free-boundary problem.

2. We plan to do so by

- (a) constructing rather weak generalized solutions of the free-boundary problems,
- (b) showing that the free boundary of such solutions have nice measure theoretical properties (i.e., finite $(n - 1)$ -dimensional Hausdorff measure and the associated differentiability properties),
- (c) showing that near a «flat» point free boundaries are Lipschitz graphs and
- (d) showing that Lipschitz free boundaries are really $C^{1,\alpha}$.

From then on, the theory of regularity developed by Kinderlehrer-Nirenberg and Spruck applies.

We start here with the last part of the project, that is, to show that Lipschitz free boundaries are $C^{1,\alpha}$, mainly for two reasons: the first because many of the ideas in this part reappear in a much more entangled way than in the others, and the second, because this part is of immediate interest, since the existence of solutions to which these theorems will apply has been obtained already in many cases by different means.

An heuristic discussion of this paper can be found in [C]. The ideas presented here originated in a joint work with J. Athanasopoulos (see [At-C]).

Notion of Weak Solution

We denote a point in \mathbb{R}^{n+1} as X or (x, y) , with $x = (x_1, \dots, x_n)$. To state the simplest version of our results, let us define what we mean by a weak solution of a free-boundary problem.

Definition 1. *In the unit cylinder $C_1 = B_1 \times [-1, 1]$ of \mathbb{R}^{n+1} , we are given a continuous function u satisfying*

- (i) $\Delta u = 0$ on $\Omega^+ = \{u > 0\}$,
- (ii) $\Delta u = 0$ on $\Omega^- = \{u \leq 0\}^0$,
- (iii) (The weak free-boundary condition). Along $F = \partial\{u > 0\}$ u satisfies the free-boundary condition

$$u_{\nu^+} = G(u_{\nu^-})$$

in the following sense.

If $X_0 \in F$ and F has a one-sided tangent ball at X_0 (i.e. $\exists B_p(Y)$ such that $X_0 \in \partial B_p(Y)$ and $B_p(Y)$ is contained either in Ω^+ or Ω^-) then

$$u(X) = \alpha \langle X - X_0, \nu \rangle^+ - \beta \langle X - X_0, \nu \rangle^- + o(|X - X_0|)$$

and $\alpha = G(\beta)$.

The basic requirements on G will be strict monotonicity and continuity in u_{ν} .

Theorem 1. *Let u be a continuous function in the unit ball. Assume that u satisfies*

- (i) $\Delta u = 0$ in $\Omega^+ = \{u > 0\}$ and $\Omega^- = \{u \leq 0\}^0$.
- (ii) $\Omega^+ = \{(x, y): y > f(x)\}$, with $f(x)$ a Lipschitz continuous function.
- (iii) $0 \in F = \partial\Omega^+$ and along F , the free-boundary condition $u_{\nu^+} = G(u_{\nu^-})$ is satisfied in the sense described above.

Assume further that $G(s)$ is strictly increasing and for some C large, $s^{-C}G(s)$ is decreasing. Then, on $B_{1/2}$, f is a $C^{1,\alpha}$ function.

1. Some Properties of Harmonic Functions in a Lipschitz Domain

In this section we recall some properties of nonnegative harmonic functions in a Lipschitz domain.

Lemma 1. (Dahlberg, see [D], see also [C-F-M-S]). *Let u_1, u_2 be two nonnegative harmonic functions in a (Lipschitz) domain D of \mathbb{R}^{n+1} of the form*

$$D = \{|x| < 1, |y| < M, y > f(x)\}$$

with f a Lipschitz function with constant less than M and $f(0) = 0$. Assume further that u_1 and u_2 take continuously the value $u_1 = u_2 = 0$ along the graph of f . Then, on the domain

$$D_{1/2} = \left\{ |x| < \frac{1}{2}, |y| < \frac{M}{2}, y < f(x) \right\},$$

we have

$$0 < C_1 \leq \frac{u_1(x, y)}{u_2(x, y)} \cdot \frac{u_2\left(0, \frac{M}{2}\right)}{u_1\left(0, \frac{M}{2}\right)} \leq C_2$$

with C_1, C_2 depending only on M . In particular, if

$$\frac{u_2(0, M/2)}{u_1(0, M/2)} = 1$$

we get

$$0 < C_1 \leq \frac{u_1(x, y)}{u_2(x, y)} \leq C_2.$$

Lemma 2 (Jerison and Kenig [J-K], see also [At-C]). *Let D , u_1 and u_2 be as in Lemma 1. Assume further that*

$$\frac{u_1(0, M/2)}{u_2(0, M/2)} = 1.$$

Then, $u_1(x, y)/u_2(x, y)$ is Hölder continuous in $\bar{D}_{1/2}$ (i.e. up to the graph of $f(x)$) for some coefficient α , both α and the C^α norm of u_1/u_2 depending only on M .

Lemma 3 (Dahlberg [D], see also [C-F-M-S]). *Let u be as u_1 (or u_2) above. Then, there exists a constant $\delta = \delta(M)$ such that for*

$$D_\delta = \{ |x| < \delta, |y| < \delta M, y > f(x) \}$$

we have

$$u|_{D_\delta} \leq \frac{1}{2} u\left(0, \frac{M}{2}\right).$$

Lemma 4. *Let u be as in Lemma 3. Assume further that $D_y u \geq 0$ on D . Then,*

$$0 < C_1 \leq \frac{D_y u\left(0, \frac{M}{2}\right)}{u\left(0, \frac{M}{2}\right)} \leq C_2.$$

As usual $C_i = C_i(M)$.

PROOF. From Lemma 3,

$$\frac{1}{2} u\left(0, \frac{M}{2}\right) \leq \int_\delta^{M/2} D_y u(0, t) dt \leq u\left(0, \frac{M}{2}\right).$$

But D_y is positive and harmonic in Ω . Therefore, by Harnack's inequality, all the values along the segment of integration are comparable, and the formula with $d = M/2$ follows. For $0 < d < M/2$ we may use rescaling. \square

Lemma 5. *Let u be as in Lemma 3. Then, in D_δ , for some $\delta(M)$, $D_y u \geq 0$.*

PROOF. Let $u_1 = u$ and u_2 be the (bounded) auxiliary function

$$\begin{cases} u_2 = C > 0, & \text{on } \partial D \setminus \text{graph } f \\ u_2 = 0, & \text{on } \text{graph } f \\ \Delta u_2 = 0, & \text{on } D. \end{cases}$$

If we compare u_2 with vertical translations in their common domain of definition, we obtain

$$D_y u_2 > 0 \quad \text{on } D.$$

Let us adjust C so that

$$\frac{u_1\left(0, \frac{M}{2}\right)}{u_2\left(0, \frac{M}{2}\right)} = 1.$$

Then, from Lemma 2, on $D_{1/2}$

$$0 < C_1 \leq \frac{u_1(x, y)}{u_2(x, y)} \leq C_2$$

and further, from Lemma 3

$$\left| \frac{u_1(x, y)}{u_2(x, y)} - \frac{u_1(\bar{x}, \bar{y})}{u_2(\bar{x}, \bar{y})} \right| \leq C(|x - \bar{x}| + |y - \bar{y}|)^\alpha.$$

In particular, if we freeze (\bar{x}, \bar{y}) , at distance d from graph of f , and let (x, y) vary in a $d/2$ -neighborhood of (\bar{x}, \bar{y}) , we get

$$\begin{aligned} \left| u_1(x, y) - u_2(x, y) \left[\frac{u_1(\bar{x}, \bar{y})}{u_2(\bar{x}, \bar{y})} \right] \right| &< C u_2(x, y) (|x - \bar{x}| + |y - \bar{y}|)^\alpha \\ &\leq C u_2(\bar{x}, \bar{y}) d^\alpha \\ &\leq C D_y u_2(\bar{x}, \bar{y}) d^{\alpha+1} \end{aligned}$$

(we may substitute $u_2(x, y)$ by $u_2(\bar{x}, \bar{y})$ by Harnack's inequality, and $u_2(\bar{x}, \bar{y})$ by $d(D_y u_2(\bar{x}, \bar{y}))$, because of Lemma 4). Therefore, taking D_y derivative on the unfrozen variable y , and evaluating at \bar{y} , we get, from standard interior *a priori* estimates for $w = u_1 - u_2 k$, $k = u_1(\bar{x}, \bar{y})/u_2(\bar{x}, \bar{y})$

$$\left| D_y u_1(\bar{x}, \bar{y}) - \left[\frac{u_1(\bar{x}, \bar{y})}{u_2(\bar{x}, \bar{y})} \right] D_y u_2(\bar{x}, \bar{y}) \right| \leq C D_y u_2(\bar{x}, \bar{y}) \cdot d^\alpha$$

that is

$$D_y u_1(\bar{x}, \bar{y}) \geq \left\{ \left[\frac{u_1(\bar{x}, \bar{y})}{u_2(\bar{x}, \bar{y})} \right] - C d^\alpha \right\} \cdot D_y u_2(\bar{x}, \bar{y}).$$

And this last term is positive if d^α is small enough. \square

2. Subsolutions to Our Free-Boundary Problems and Comparison Principles

In this section we define weak subsolutions to our free-boundary problem, and discuss a comparison principle.

We start by defining the notion of a weak subsolution.

Definition 2. *The continuous function $v(X)$ is a subsolution to our free-boundary problem in Ω if*

- (i) $\Delta v \geq 0$ both in $\Omega^+ = \{v > 0\}$ and $\Omega^- = \{v \leq 0\}$ ⁰
- (ii) let $X_0 \in F = (\partial\Omega^+) \cap \Omega$,

assume that at X_0 , F has a tangent ball B_ϵ from the Ω^+ side (i.e. $B_\epsilon \subset \Omega^+$, $X_0 \in \partial B_\epsilon \cap F$). Then, for some $\alpha \geq 0$, $\beta = G(\alpha)$, ν the unit inner radial direction of ∂B_ϵ at X_0 ,

$$v(X) \geq \beta \langle X - X_0, \nu \rangle^+ - \alpha \langle X - X_0, \nu \rangle^- + o(|X - X_0|).$$

Definition 3. Given a subsolution v to our F.B. Problem, a point $X_0 \in F$, at which F has a tangent ball from Ω^+ (as in Definition 2(ii)) will be called a regular point.

We now state a strong comparison principle.

Lemma 6. Let $v \leq u$ be two continuous functions in Ω , $v < u$ in $\Omega^+(v)$, v a subsolution and u a solution. Let $X_0 \in F(v) \cap F(u)$ (the free boundaries of v and u). Then X_0 cannot be a regular point for $F(v)$.

PROOF. Since $\Omega^+(v) \subset \Omega^+(u)$, X_0 automatically will be a point for which u has the desired asymptotic development (Definition 1)

$$u(X) = \beta \langle X - X_0, \nu \rangle^+ - \alpha \langle X - X_0, \nu \rangle^- + o(|X - X_0|)$$

with $\beta = G(\alpha)$

$$v(X) \geq \bar{\beta} \langle X - X_0, \nu \rangle^+ - \bar{\alpha} \langle X - X_0, \nu \rangle^- + o(|X - X_0|)$$

with $\bar{\beta} = G(\bar{\alpha})$. This implies that $\beta \geq \bar{\beta}$ and $\alpha \leq \bar{\alpha}$.

Since G is assumed to be monotone $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. But $u - v$ is a positive superharmonic function in $\Omega^+(v)$. By Hopf principle, since X_0 is regular

$$(u - v)(X) \geq \epsilon |X - X_0|$$

radially into $\Omega^+(v)$, along ν from X_0 . \square

We refine the previous lemma to a continuous family of subsolutions.

Lemma 7. Let v_t , for $0 \leq t \leq 1$, be a continuous family of subsolutions in Ω (continuous in $\bar{\Omega} \times [0, 1]$). Let u be a solution in Ω , continuous in $\bar{\Omega}$. Assume that

- (i) $v_0 \leq u$ in Ω .
- (ii) $v_t \leq u$ on $\partial\Omega$ and $v_t < u$ in $[\bar{\Omega}^+(v_t) \cap \partial\Omega]$ for $0 \leq t \leq 1$.
- (iii) every point $X_0 \in F(v_t)$ is regular and
- (iv) the family $\Omega^+(v_t)$ is continuous, that is $\Omega^+(v_{t_1}) \subset N_\epsilon(\Omega^+(v_{t_2}))$ whenever $|t_1 - t_2| < \delta(\epsilon)$ (N_ϵ denotes the ϵ -neighborhood of the set).

Then $v_t \leq u$ in Ω for any t .

PROOF. The set of t 's for which $v_t \leq u$ is obviously closed. Let us show that it is open: first, if $v_{t_0} \leq u$, it follows from (ii) and the strong maximum principle, that $v_{t_0} < u$ in $\Omega^+(v_{t_0})$. And since every point of $F(v_{t_0})$ is regular (assumption (iii)), it follows that $[\overline{\Omega^+(v_{t_0})}]$ is compactly contained in $\Omega^+(u)$ (up to $\partial\Omega$, from assumption (ii)). From assumption (iv), the openness follows. \square

Remark. Since u may be the solution of a one-phase problem, that is $u|_{\Omega^-(u)} \equiv 0$, assumption (iv) is necessary (an easy counterexample where $\Omega^+(v_t) = \Omega$ for $t > 0$, can be constructed).

3. Continuous Families of Subsolutions

In this section we construct particular families of subsolutions, starting from a given solution. The simplest family is the following:

Lemma 8. *Let u , a continuous function in Ω , be a weak solution of our F. B. Problem. Let*

$$v_t(X) = \sup_{B_t(X)} u(Y), \quad t > 0.$$

Then v_t is a subsolution of our F. B. Problem in its domain of definition. Furthermore, any point of $F(v_t)$ is regular.

PROOF. v_t is the supremum of a family of translations of u , and as such, v is subharmonic both in $\Omega^+(v_t)$ and $\Omega^-(v_t)$. Let now $X_0 \in F(v_t)$. That means that $B_t(X_0)$ is tangent from $\Omega^-(u)$ to $F(u)$ at a point Y_0 . Therefore

- (a) X_0 is regular since $B_t(Y_0) \subset \Omega^+(v)$ and is tangent to $F(v)$ at X_0 .
- (b) At Y_0 , u has the asymptotic behavior

$$u = \beta \langle X - Y_0, \nu \rangle^+ - \alpha \langle X - Y_0, \nu \rangle^- + o(|X - Y_0|),$$

with $\beta = G(\alpha)$, and ν the outer normal to $\partial B_t(X_0)$ at Y_0 , and hence

$$v \geq \beta \langle X - X_0, \nu \rangle^+ - \alpha \langle X - X_0, \nu \rangle^- + o(|X - X_0|). \quad \square$$

The family v_t on the previous lemma is an admissible family for the comparison lemma (Lemma 7) and as such it can be used for a comparison principle that says: «If u_1 and u_2 are two weak solutions, with $u_1 \leq u_2$ and near $\partial\Omega$, $\sup_{B_\epsilon(X)} u_1 \leq u_2(X)$, then also in the interior of Ω $\sup_{B_\epsilon(X)} u_1 \leq u_2(X)$ », keeping, in particular $F(u_2)$, ϵ -away from $F(u_1)$.

This family has the problem of being too rigid. If u_2 is, for instance, much larger than u_1 in some section of $\partial\Omega$, one cannot exploit that fact. Therefore,

we will now introduce a more delicate family of perturbations, where we make the radius of the ball $B_t(X_0)$ dependent on X_0 itself ($t = t(X_0)$).

The key lemma is the following.

Lemma 9. *Let $\varphi(x)$ be a C^2 -positive function satisfying*

$$\Delta\varphi \geq \frac{C|\nabla\varphi|^2}{|\varphi|}$$

(for C large enough) in $B_1(0)$ (the unit ball of \mathbb{R}^n). Let u be continuous, defined in a domain Ω large enough so that the following function be defined in $B_1(0)$

$$w(X) = \sup_{|\nu|=1} u(X + \varphi(x)\nu).$$

Then, if u is harmonic in $\{u > 0\}$, w is subharmonic in $w > 0$.

PROOF. Assume $w(0)$ to be positive. We will show that

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \left[\int_{B_1(0)} (w(X) - w(0)) dx \right] \geq 0.$$

For that purpose, we will estimate $w(x)$ by below near 0, choosing an appropriate value for $\nu = \nu(X)$: Choose the system of coordinates so

- (1) $w(0) = u(\varphi(0)e_n)$
- (2) $\nabla\varphi(0) = \alpha e_1 + \beta e_n$.

We evaluate w by below by choosing $\nu(X) = \nu^*/|\nu^*|$ with

$$(3) \nu^*(X) = e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)} \left(\sum_2^{n-1} x_i e_i \right).$$

Here γ is chosen so that

$$(4) (1 + \gamma)^2 = (1 + \beta)^2 + \alpha^2.$$

Let us examine the point Y obtained by such a choice.

$$Y = X + \left\{ \varphi(0) + \nabla\varphi(0)X + \frac{1}{2} (D_{ij}\varphi)x_i x_j + o(|X|^2) \right\} \\ \left\{ \left[e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)} \sum_2^{n-1} x_i e_i \right] \right. \\ \left. \left[1 - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi^2(0)} - \left(\frac{\gamma}{\varphi} \right)^2 \sum_2^{n-1} x_i^2 + o(|X|^4) \right] \right\}.$$

The above expression has a constant (translation) term $\varphi(0)e_n$. A first-order term

$$Y^* - \varphi(0)e_n = X + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum x_i e_i$$

than can be thought as a rotation followed by and expansion by $1 + \gamma$ since

$$\frac{1}{1 + \gamma} [Y^* - \varphi(0)e_n] = \begin{bmatrix} \frac{1 + \beta}{1 + \gamma} & \dots & \frac{-\alpha}{1 + \gamma} \\ \vdots & \ddots & \vdots \\ \alpha & \dots & 1 + \beta \\ \frac{1}{1 + \gamma} & \dots & \frac{1}{1 + \gamma} \end{bmatrix} X = MX.$$

where M is a rotation in the e_1, e_n plane (by the definition of γ) and a quadratic term

$$Y - Y^* = \left[\frac{1}{2} (D_{ij}\varphi)x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)} \sum_{i=1}^{n-1} x_i^2 \right] e_n + O\left(\frac{|\nabla\varphi|^2}{\varphi} |X|^2\right) \mu$$

with $\mu \perp e_n$ and $|\mu| = 1$. Hence

$$\begin{aligned} \int w(X) - w(0) &\geq \int u(Y(X)) - u(Y(0)) \\ &= \int u(Y(X)) - u(Y^*(X)) + \int u(Y^*(X)) - u(Y(0)) \\ &= \int u(Y(X)) - u(Y^*(X)). \end{aligned}$$

(Since the last term is zero, due to the fact that u is harmonic and Y^* is a rigid rotation plus a dilation of X). We now point out that, by the definition of w , ∇u must point in the direction of e_n at $Y(0)$. Hence

$$\begin{aligned} u(Y) - u(Y^*) &= \nabla u \circ (Y - Y^*) + O(|Y - Y^*|^2) \\ &= |\nabla u| \left[\frac{1}{2} D_{ij}\varphi x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)} \sum x_i^2 \right] + O(|X|^4) \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{r^2} \int u(Y) - u(Y^*) + O(|X|^2) &= \\ &= |\nabla u(Y(0))| \left\{ \frac{1}{n} \left(\Delta\varphi - [\beta^2 + \alpha^2 + (n-2)\gamma^2] \frac{1}{\varphi(0)} \right) \right\} \geq 0 \end{aligned}$$

if

$$\Delta\varphi > C \frac{|\Delta\varphi|^2}{\varphi}.$$

Remark.

$$\Delta\varphi \geq C \frac{|\nabla\varphi|^2}{\varphi}$$

if φ^{1-c} is superharmonic.

We now study a more flexible family of perturbations, namely, given a solution u of our F. B. Problem and a function φ satisfying the properties of Lemma 9, we want to consider $v = v_\varphi$ defined by

$$v(x) = \sup_{B_{\varphi(x)}(x)} u(y).$$

We start with the asymptotic behavior of v at the free boundary.

Lemma 10. *Let u be a continuous function and*

$$v(X) = \sup_{B_{\varphi(X)}(X)} u(Y).$$

with φ a positive C^2 function, and $|\nabla\varphi| < 1$. Assume that

$$X_0 \in \partial\Omega^+(v), \quad Y_0 \in \partial\Omega^+(u)$$

and that they are related by the fact that

$$Y_0 \in \partial B_{\varphi(X_0)}(X_0).$$

Then

- (a) X_0 is a regular point for $F(v)$.
- (b) If near Y_0 , u^+ (resp. u^-) has the asymptotic behavior

$$u^+ \text{ (resp. } u^-) = \alpha \langle Y - Y_0, \nu \rangle^+ + o(|Y - Y_0|)$$

then

$$v^+ \geq \alpha \langle X - X_0, \nu + \nabla\varphi \rangle^+ + o(|X - X_0|)$$

(resp. $v^- \leq \alpha \langle X - X_0, \nu + \nabla\varphi \rangle + o(|X - X_0|)$).

- (c) If $F(u)$ is a Lipschitz graph, and $|\nabla\varphi|$ is small enough (depending on the Lipschitz norm, λ , of $F(u)$), then $F(v)$ is a Lipschitz graph with Lipschitz norm

$$\lambda' \leq \lambda + C \sup |\nabla\varphi|.$$

PROOF. To prove (a), we notice that $\Omega^+(v)$ contains the set

$$\Theta = \{|X - Y_0|^2 < \varphi^2(X)\}.$$

The boundary of this set is a smooth (C^2) surface, since

$$\nabla(|X - Y_0|^2 - \varphi^2(X)) = 2(X - Y_0 - \varphi(X)\nabla\varphi(X)) \neq 0$$

along the surface. Since this surface goes through X_0 , (a) is proven.

To prove (b) we use the fact that near X_0 ,

$$\varphi(X) \geq \varphi(X_0) + \langle X - X_0, \nabla\varphi(X_0) \rangle + o(|X - X_0|^2).$$

Hence

$$v^+(X) \geq \alpha \langle X - X_0, \nu + \nabla\varphi(X_0) \rangle^+ + o(|X - X_0|)$$

and

$$v^-(X) \leq \alpha \langle X - X_0, \nu + \nabla\varphi(X_0) \rangle^- + o(|X - X_0|)$$

respectively.

To prove (c) it is enough to assume that $\Omega^+(u)$ is above the graph of a smooth convex cone $f(x)$, since the general case is a union of such sets. Then if X_0 and Y_0 are as before, $Y_0 - X_0$ is by definition parallel to the inner unit normal ν to a supporting plane to $F(u)$ at Y_0 . About ν we can say that it must lie in a cone of aperture $\arctan \lambda$ around e_{n+1} . On the other hand at X_0 , $F(v)$ has the upper and lower envelopes the implicit surfaces

$$S_1 = \{|X - Y_0|^2 - \varphi^2(X) = 0\}$$

and

$$S_2 = \{d(X, \pi)^2 - \varphi^2(X) = 0\}$$

where π is the support plane to $F(u)$ at Y_0 . Both surfaces are smooth with unit normal vector, $\bar{\nu}$, parallel to

$$Y_0 - X_0 + \varphi(X_0)\nabla\varphi(X_0)$$

or to

$$\nu + \nabla\varphi(X_0).$$

Therefore, the angle between $\bar{\nu}$ and e_{n+1} is less than

$$\arctan \lambda + |\nabla\varphi|.$$

If $|\nabla\varphi|$ is small enough depending on λ , more precisely $|\nabla\varphi|$, a small multiple of $1/(1 + \lambda)$, the angle between $\bar{\nu}$ and e_{n+1} is less than

$$\arctan(\lambda(1 + (c + \lambda)|\nabla\varphi|))$$

i.e., $F(v)$ is Lipschitz, with Lipschitz constant

$$\lambda' = \lambda(1 + (c + \lambda)|\nabla\varphi|).$$

An important corollary is our next lemma.

Lemma 11. *Let u be a solution of our F. B. Problem and both φ and $v = v_\varphi$ be the functions of Lemmas 9 and 10 (i.e. φ satisfies the hypothesis of both lemmas). Then*

- (a) v is subharmonic in $\Omega^+(v)$ and $\Omega^-(v)$.
- (b) Every point of $F(v)$ is regular.
- (c) At every point of $F(v)$, v satisfies the asymptotic inequality

$$v(X) \geq \beta \langle X - X_0, \nu \rangle^+ - \alpha \langle X - X_0, \nu \rangle^- + o(|X - X_0|)$$

with

$$\frac{\beta}{1 - |\nabla\varphi|} \geq G\left(\frac{\alpha}{1 + |\nabla\varphi|}\right).$$

4. Main Harnack

In this section we develop the basis of our iteration technique. First, two preliminary lemmas:

Lemma 12. *Let $0 \leq u_1 \leq u_2$ be harmonic functions in $B_\lambda(0)$. Let $\epsilon < \lambda/8$ and assume that on $B_{\lambda-\epsilon}(0)$*

$$v_\epsilon(X) = \sup_{B_\epsilon(X)} u_1(Y) \leq u_2(X)$$

and further

$$u_2(0) - v_\epsilon(0) \geq \sigma\epsilon u_2(0).$$

Then, for some $\bar{C} = \bar{C}(\lambda)$, $\mu = \mu(\lambda) > 0$, we have in $B_{(3/4)\lambda}$

$$u_2(X) - v_{(1+\mu\sigma)\epsilon}(X) \geq \bar{C}\sigma\epsilon u_2(0).$$

PROOF. For any $|\nu| < 1$

$$w(X) = u_2(X) - u_1(X + \epsilon\nu)$$

is harmonic and positive in $B_{\lambda-\epsilon}$. By Harnack's inequality in $B_{3\lambda/4}$

$$w(X) \geq Cw(0) \geq C\sigma\epsilon u_2(0).$$

Also, both

$$|\nabla u_i(X)| \leq \frac{C}{\lambda} u_i(0) \leq \frac{C}{\lambda} u_2(0)$$

on $B_{3\lambda/4}$. It follows that

$$\begin{aligned} u_2(X) - u_1(X + (1 + \sigma\mu)\epsilon\nu) &= w(X) + u_1(X + \epsilon\nu) - u_1(X + (1 + \sigma\mu)\epsilon\nu) \\ &\geq \sigma\epsilon u_2(0) - \frac{C\mu\sigma}{\lambda} \epsilon u_2(0) \\ &\geq \bar{C}\sigma\epsilon u_2(0) \end{aligned}$$

if μ is chosen small. \square

Lemma 13. *Let $0 < \lambda < 1/8$, then there exists a θ and a $\mu > 0$, $(\mu(\lambda), \theta(\lambda))$ and a C^2 family of functions φ_t ($0 \leq t \leq 1$) defined in $\bar{B}_1 \setminus B_{\lambda/2}(0, 3/4)$, such that*

- (i) $1 \leq \varphi_t \leq 1 + t\mu$
- (ii) $\varphi\Delta\varphi \geq C|\nabla\varphi|^2$
- (iii) $\varphi \equiv 1$ outside of $B_{7/8}$
- (iv) $\varphi|_{B_{1/2}} \geq 1 + \theta t\mu$
- (v) $|\nabla\varphi| < Ct\mu$.

PROOF. It is not hard to construct a smooth function ψ_0 in $B_1 \setminus B_{\lambda/2}(0, 3/4)$ such that

$$\begin{cases} 0 \leq \psi_0 \leq 1 \\ \psi_0 \equiv 0 \text{ outside } B_{7/8}(0) \\ |\nabla\psi_0| < C\Delta\psi_0, \text{ for some } C \text{ large} \\ \psi_0|_{B_{1/2}} \geq \gamma > 0. \end{cases}$$

Then $\varphi_t = 1 + t\mu\psi_0$ is our desired function, provided that μ is small enough. \square

Now, a comparison theorem:

Lemma 14. *Let $u_1 \leq u_2$ be two solutions of our free-boundary problem in $B_1 \subset \mathbb{R}^{n+1}$ with $F_2 = F(u_2)$ a Lipschitz free boundary through the origin. Assume further that*

$$v_\epsilon(x) = \sup_{B_\epsilon(x)} u_1(y) \leq u_2(x)$$

in $B_{1-\epsilon}$, that

$$v_\epsilon\left(0, \frac{3}{4}\right) \leq (1 - \sigma\epsilon)u_2\left(0, \frac{3}{4}\right)$$

and that

$$B_\lambda\left(0, \frac{3}{4}\right) \subset \Omega^+(u_1).$$

Then, for ϵ small enough, there exists a δ , depending only on λ and the various constants C , such that on $B_{1/2}$

$$v_{(1+\delta\sigma)\epsilon}(x) = \sup_{B_{(1+\delta\sigma)\epsilon}(x)} u_1(y) \leq u_2(x).$$

PROOF. We construct a continuous family of subsolutions \bar{v}_t , such that $\bar{v}_0 \leq u_2$, $\bar{v}_1|_{B_{1/2}} \geq v_{(1+\delta)\epsilon}$, and for which the comparison lemma (Lemma 7), applies. More precisely

$$\bar{v}_t(x) = \sup_{B_{\epsilon\varphi_\sigma(x)}} u_1(y) + C\sigma\epsilon w_t \equiv v_t(x) + C\sigma\epsilon w_t$$

for a small constant $C > 0$, with w_t a continuous function in

$$\Omega = B_{9/10} - B_{\lambda/2}(0, 3/4)$$

defined by

$$\begin{cases} \Delta w_t = 0 & \text{in } \Omega^+(v_t) \cap \Omega = \Omega_1 \\ w_t|_{\partial(\Omega^+(v_t) \cap B_{9/10})} = 0 \\ w_t|_{\partial B_{\lambda/2}(0, 3/4)} = u_2(0, 3/4). \end{cases}$$

Let us check that \bar{v}_t satisfies the hypothesis of Lemma 7 in Ω with respect to $u = u_2$:

- (i) comparison in $B_{9/10} - \Omega^+(v_0)$ is clear. In Ω_1 we compare the boundary values of \bar{v}_0 and u_2 thanks to Lemma 12
- (ii) follows from our hypothesis and Lemma 12, provided that $\mu = \mu(\lambda)$ is kept small (we should really replace ϵ by any smaller ϵ^1 , to ensure the validity of (ii) along ∂B_1 , but that is a minor detail)

- (iii) follows from part (a) of Lemma 10
(iv) is by construction.

It only remains to check the fact that \bar{v}_t are indeed subsolutions.

The subharmonicity in Ω^+ and Ω^- follows from Lemma 9. About the asymptotic behavior, we write

$$\bar{v}_t = v_t + C\sigma\epsilon w_t.$$

From Lemma 11, v_t satisfies the asymptotic inequality (c) with

$$\frac{\beta}{1 - \epsilon|\nabla\varphi_{\sigma t}|} \geq G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right).$$

Since outside $B_{7/8}$, $|\nabla\varphi| \equiv 0$ the right inequality is satisfied by v_t and hence by \bar{v}_t since w_t is positive. Inside $B_{7/8} \cap \Omega^+(v_t)$, we notice that by Dahlberg's theorem (Lemma 1) $(w_t/v_t) \geq C$, provided that $\epsilon\mu$ and hence $\epsilon|\nabla\varphi|$, is kept small to make sure that the $F(v_t)$ are uniformly Lipschitz domains (see Lemma 10(c)). Therefore, from the asymptotic development of Lemma 11(c), we may say that

$$(v_t + C\sigma\epsilon w_t)^+ \geq \bar{\beta}\langle X - X_0, \nu \rangle^+ + o(|X - X_0|)$$

with $\bar{\beta} \geq (1 + C\sigma\epsilon)\beta$. Therefore, to complete the proof of the theorem, we must prove that, for μ in the definition of $\varphi_{\sigma t}$ small enough,

$$\bar{\beta} \geq G(\alpha).$$

From the properties of $G(s)$, $s^{-c}G(s)$ is decreasing. Hence

$$\alpha^{-c}G(\alpha) \leq \left[\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right]^{-c} G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right)$$

or

$$\begin{aligned} G(\alpha) &\leq (1 + C\epsilon|\nabla\varphi_{\sigma t}|)G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right) \leq \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|}\beta \\ &\leq \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|} \frac{\bar{\beta}}{1 + C\epsilon}. \end{aligned}$$

Since $|\nabla\varphi_{\sigma t}| \leq C\mu t$, the proof is complete for μ small. \square

5. Intermediate Cones

In this section we state an auxiliary lemma about cones in \mathbb{R}^n .

We denote by $\alpha(e, f)$ the angle between the vectors e and f , and by $\Gamma(\theta, e)$ the cone of axis e and aperture θ , i.e.

$$\Gamma(\theta, e) = \{\tau: \alpha(\tau, e) < \theta\}.$$

Lemma 16. *Let $0 < \theta_0 < \theta < \pi/2$ and let*

$$\Gamma(\theta, e) \subset \Gamma\left(\frac{\pi}{2}, \nu\right) = H(\nu).$$

For $\tau \in \Gamma(\theta/2, e)$, let

$$E(\tau) = \frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)$$

and for μ small, define

$$\rho(\tau) = |\tau| \sin\left(\frac{\theta}{2} + \mu E(\tau)\right).$$

Finally, let

$$S_\mu = \bigcup_{\tau \in \Gamma(\theta/2, e)} B_{\rho(\tau)}(\tau).$$

Then, $\exists \bar{\theta}, \bar{e}$ such that

$$\Gamma(\theta, e) \subset \Gamma(\bar{\theta}, \bar{e}) \subset S_\mu$$

and

$$\frac{\bar{\theta} - \theta}{\pi/2 - \theta} \geq Q(\theta_0, \mu) > 0.$$

PROOF. We reduce it to a problem in the plane through stereographic projection. We first restrict ourselves to the sphere, and then project using ν as the north pole. By symmetry, the lemma reduces to the following question in the plane (changing slightly θ, θ_0, μ)

Let $D_\theta(e)$ be a disc in \mathbb{R}^2 of radius $\theta > \theta_0 > 0$. Assume that $D_\theta \subset D_1$, the unit disc. For $0 < \lambda_0 < \lambda < \lambda_1 < 1$, for any $\tau \in D_{\lambda_0}(e)$, define

$$E(\tau) = (1 - [|\tau| + (1 - \lambda)\theta])$$

(note that $E(\tau) > 0$, since $D_\theta \subset D_1$) and $\rho(\tau) = (1 - \lambda)\theta + \mu E(\tau)$ ($0 < \mu < 1$). Then

$$S_\mu = \bigcup_{\tau \in D_{\lambda_0}(e)} B_{\rho(\tau)}(\tau) \supset D_{\bar{\theta}}(\bar{e}) \supset D_\theta(e)$$

with

$$\frac{\bar{\theta} - \theta}{1 - \theta} \geq Q(\mu, \theta_0, \lambda_0, \lambda_1) > 0.$$

The proof is an elementary computation. \square

6. The Basic Iteration

We are now ready to prove our basic iterative lemmas.

Lemma 17. *Let u be a weak solution of our F. B. Problem on B_1 . Assume that, for some $0 < \theta_0 < \theta \leq \pi/2$, u is monotonically increasing for any direction $\tau \in \Gamma(\theta, e_n)$. Then, $\exists \mu < 1$, $(\mu(\theta_0))$ and e a unit vector such that, for*

$$\bar{\theta} - \pi/2 = \mu(\theta - \pi/2),$$

the cone

$$\Gamma(\bar{\theta}, e) \supset \Gamma(\theta, e_n)$$

and, on $B_{1/2}$, u is monotonically increasing for any direction $\tau \in \Gamma(\bar{\theta}, e)$.

PROOF. We first point out that $B_{1/4 \sin \theta_0}(\frac{3}{4}e_n)$ is all contained in Ω^+ by the monotonicity of u . Let ν be the direction of ∇u at $\frac{3}{4}e_n$. Then for any $\tau \in \Gamma(\theta, e_n)$, we have that on $B_{1/4 \sin \theta_0}(\frac{3}{4}e_n)$, $D_\tau u$ is harmonic and nonnegative, and

$$D_\tau u \left(\frac{3}{4}e_n \right) = \langle \nabla u, \tau \rangle = |\nabla u| \langle \nu, \tau \rangle.$$

From Lemma 4 and Harnack's inequality applied to both $D_\tau u$ and u in $B_{1/4 \sin \theta_0}(\frac{3}{4}e_n)$, we get

$$D_\tau u|_{B_{1/4 \sin \theta_0}(3e_n/4)} \geq C(\sup |\nabla u|) \langle \tau, \nu \rangle \geq \sup D_{\epsilon_n} u \geq C \left(\sup_{B_{1/8 \sin \theta_0}(3e_n/4)} u \right) \langle \nu, \tau \rangle.$$

Let τ be a small vector in $\Gamma(\theta/2, e_n)$, and let $\bar{u}(x) = u(x - \tau)$. We now apply the main Harnack-type Lemma 14 with

$$u_1(x) = \bar{u}(x)$$

$$u_2(x) = u(x)$$

$$\epsilon = |\tau| \sin \frac{\theta}{2}$$

and σ defined as

$$\sigma = C \left(\frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2} \right) \right) \sim C \cos \left(\alpha(\tau, \nu) + \frac{\theta}{2} \right)$$

(C to be chosen). Then, the only nontrivial hypothesis is that

$$v_\epsilon \left(0, \frac{3}{4} \right) \leq (1 - \sigma\epsilon) u_2 \left(0, \frac{3}{4} \right).$$

Let $Y \in B_\epsilon(X)$, $u_1(Y) = u(Y - \tau) = u(X - \tau - (X - Y)) = u(X - \bar{\tau})$ with

$$\alpha(\bar{\tau}, \tau) \leq \theta/2$$

(since $|\bar{\tau} - \tau| = |X - Y| \leq |\tau| \sin \theta/2$). Also

$$|\bar{\tau}| \geq |\tau| - |\tau| \sin \frac{\theta}{2} \geq \frac{1}{2} |\tau|.$$

since, $\bar{\tau} \in \theta/2 < \pi/4$. It follows that

$$\begin{aligned} \inf_{B_{1/8}(3/4 e_n)} D_{\bar{\tau}} u &\geq C \left[\sup_{B_{1/8}(3/4 e_n)} u \right] \langle \nu, \bar{\tau} \rangle \\ &= C(\sup u) |\bar{\tau}| \cos \alpha(\nu, \bar{\tau}) \\ &\stackrel{\text{def}}{\geq} \sigma\epsilon(\sup u). \end{aligned}$$

(Here we chose C in the definition of σ). Hence

$$u(X - \bar{\tau}) \leq u(X) - D_{\bar{\tau}} u(\bar{X}) \geq (1 - \sigma\epsilon)u(X)$$

and the hypotheses of the Harnack lemma (Lemma 14) are satisfied.

It follows that on $B_{1/2}$

$$\sup_{B_{(1+\delta\tau)\epsilon}(x)} u(y - \tau) \leq u(x).$$

Recalling that $\epsilon = |\tau| \sin \theta/2$, $\sigma = C(\pi/2 - (\alpha(\tau, \nu) + \theta/2))$, we get, for any τ in $\Gamma(\theta/2, e_n)$, that

$$(1 + \delta\sigma)\epsilon = |\tau| \left(\sin \frac{\theta}{2} \right) (1 + \delta C E(\tau))$$

(in the notation of Lemma 16) and for $\theta_0 < \theta < \pi/2$ we get

$$(1 + \delta\sigma)\epsilon \geq |\tau| \sin \left(\frac{\theta}{2} + \mu E(\tau) \right), \quad \mu = \delta C \frac{\theta_0}{2}.$$

The statement above then translates into saying, for any Z of the form $Z = Y - \tau$ (for $Y \in B_{|\tau| \sin(\theta/2 + \mu E(\tau))}(X) = X - (Y - X) - \tau = X - \tau$, with τ in S_μ (of Lemma 16), we have

$$u(Z) \leq u(X).$$

That is, u is monotone for any direction τ in S_μ , and in particular in the intermediate cone $\Gamma(\bar{\theta}, \bar{e})$.

The proof of the lemma is now complete.

PROOF OF THEOREM 1. To prove Theorem 1, we repeat inductively Lemma 17, (notice that if u is a solution of our F. B. Problem; $u(\lambda X)/\lambda$ is also a solution in the corresponding domain). We get that if u is a weak solution as in Theorem 1, then on $B_{2^{-k}}$, u is monotone in a cone of directions

$$\Gamma(\theta_k, e_k)$$

with

$$\Gamma(\theta_{k+1}, e_{k+1}) \supset \Gamma(\theta_k, e_k)$$

and

$$\frac{\pi/2 - \theta_{k+1}}{\pi/2 - \theta_k} = \mu < 1.$$

It follows that $\pi/2 - \theta_k \leq b^k$ and hence the fact that the free boundary is $C^{1,\alpha}$ at the origin for some $\alpha(b) > 0$. (Note: the first step in the inductive process, i.e. the free boundary being Lipschitz implies u to be monotone in a cone of directions, follows from Lemma 5). \square

7. A Generalization

In this last section, we show how to treat the case in which X and ν dependence is introduced in the free-boundary relation and how the restriction on G at infinity are unnecessary. That is, we now consider weak solutions to the free-boundary problem

$$u_\nu^+ = G(u_\nu^+, X, \nu)$$

in the same sense as before, i.e. whenever X_0 has a tangent ball from Ω^+ or Ω^-

$$u = \beta \langle X - X_0, \nu \rangle^+ - \alpha \langle X - X_0, \nu \rangle^-$$

with

$$\beta = G(\alpha, X_0, \nu)$$

(ν given by the radial direction of the tangent ball at X_0) and assume that

- (a) $\log G$ is Lipschitz continuous on X and ν for bounded values of u_ν^- ,
- (b) for u_ν^- in a compact interval $[0, M]$, G is strictly monotone in u_ν^- and $s^{-C}G(s, X, \nu)$ is decreasing in s , ($C = C(M)$).

Then we have

Theorem 2. *Same geometric situation as in Theorem 1, u and G satisfying now the conditions above, the same conclusion as in Theorem 1 holds.*

In order to prove Theorem 2, we must do two things. First, to show that u is Lipschitz continuous, eliminating the need to impose conditions at infinity on G . Second, to verify that the dependence in X and ν introduce controllable perturbations in our argument. The first step is achieved by the following monotonicity formula, due to Alt, Friedman and myself.

Lemma 18. (See [A-C-F]). *Let u be a continuous function in B_1 , $u(0) = 0$. Assume that on $\{u > 0\}$, $\Delta u \geq 0$ and on $\{u < 0\}$, $\Delta u \leq 0$. Then, (ρ, σ) are radial and spherical coordinates in \mathbb{R}^n)*

$$g(r) = \frac{\int_{B_r} (\nabla u^+)^2 \rho \, d\rho \, d\sigma \int_{B_r} (\nabla u^-)^2 \rho \, d\rho \, d\sigma}{r^4}$$

is an increasing function of r .

Remark. g is shown to be finite from the continuity of u by an approximation of say, u^+ , by a smooth function and the fact that

$$(\nabla u^+)^2 \leq (\nabla u^+)^2 + u^+ \Delta u^+ = \frac{1}{2} \Delta (u^+)^2$$

and

$$\rho \, d\rho \, d\sigma = \frac{1}{|X|^{n-2}} \, dx.$$

By integrating by parts, this allows us to control $g(r)$ for say, $r < 1/2$, by $(\sup_{B_1} |u|)^4$.

Lemma 19. (Corollary to Lemma 18). *Let u be a weak solution as in Theorem 2. Then u is Lipschitz continuous in (say) $B_{1/4}$.*

PROOF. It is enough to prove that $|u(X)| < Cd(X, F)$.

From Lemma 18 and from the remark following it,

$$g(r) \leq C \left(\sup_{B_1} |u| \right)^4$$

for any $r < 1/2$ and taking as origin any point X_0 on $F \cap B_{1/4}$. We consider two cases:

- (a) $u|_{\Omega^-} \equiv 0$ or,
- (b) $u|_{\Omega^-}$ is never zero.

In Case (a) let $X \in \Omega^+$, $u(X) = \sigma$, $d(X, F) = \rho$, and $X_0 \in \partial B_\rho(X) \cap F \cap B_{1/2}$. Then by Harnack's inequality, $u|_{B_{\rho/2}(X)} \geq C\sigma$ and hence

$$u|_{B_\rho(X)} \geq h$$

where h is the auxiliary radially symmetric harmonic function on $B_\rho(X) - B_{\rho/2}(X)$ with values $h|_{\partial B_\rho(X)} = 0$ and $h|_{\partial B_{\rho/2}(X)} = C\sigma$. Since h has linear behaviour

$$h = C \frac{\sigma}{\rho} \langle X - X_0, \nu \rangle$$

near X_0 and

$$\begin{aligned} u &= \alpha \langle X - X_0, \nu \rangle^+ - \beta \langle X - X_0, \nu \rangle^- + \sigma(|X - X_0|) \\ &= G(0, X_0, \nu) \langle X - X_0, \nu \rangle^+ + \sigma(|X - X_0|) \end{aligned}$$

we get

$$C \frac{\sigma}{\rho} \leq G(0, X_0, \nu) \leq C,$$

or

$$\sigma \leq C\rho$$

and Case (a) is complete.

Case (b) (we only prove it for u^+). We proceed as in Case (a) and we obtain at X_0 the estimate

$$u(X) = \alpha \langle X - X_0, \nu \rangle^+ + \beta \langle X - X_0, \nu \rangle^- + \sigma(|X - X_0|)$$

with

$$C \frac{\sigma}{\rho} \leq \alpha$$

and

$$\alpha = G(\beta, X_0, \nu).$$

We now bring into play the monotonicity formula by pointing out that

$$g(O^+) \geq C\alpha^2\beta^2.$$

(Indeed, in any non-tangential domain, $|\langle X - X_0, \nu \rangle| > \delta$, ∇u converges to $\alpha\nu$ (resp. $\beta\nu$)). Therefore,

$$\alpha^2\beta^2 \leq C\|u\|_{L^\infty(B_1)}^4$$

and

$$\alpha = G(\beta, X_0, \nu).$$

Since G is monotone in β , and

$$\begin{aligned} G(1, X_0, \nu) &\geq \mu_0 > 0 \\ \beta G(\beta, X_0, \nu) &\geq \mu_0\beta. \end{aligned}$$

Therefore,

$$\beta \leq C\|u\|_{L^\infty(B_1)}^2 \leq C$$

and hence

$$\alpha \leq C.$$

It follows that $\sigma/\rho \leq C\alpha \leq C$ and Case (b) is also proven.

To complete the proof of the theorem, we only need to prove

Lemma 20. *Let \bar{v}_t be the one parameter family of functions constructed in the proof of Lemma 14. There exists a $\theta > 0$, depending only on λ and the various constants C such that if*

$$|\log(\alpha, X, \nu) - \log(\alpha, Y, \nu)| \leq \theta\sigma|X - Y|$$

for any $\alpha \leq \|\nabla u\|_{L^\infty}$ for any $\nu \in S_1$, then \bar{v}_t is still a subsolution of our generalized free boundary problem.

PROOF. We estimate once more the coefficients in the asymptotic inequality (c) of Lemma 11, satisfied by v_t at X_0 in $F(v)$. For that, we go back to Lemma 10, and with the notation there employed, we now have that v satisfies there the asymptotic inequality

$$v(X) \geq \alpha\langle X - X_0, \nu \rangle^+ - \beta\langle X - X_0, \nu \rangle^- + \sigma(|X - X_0|)$$

with

$$\frac{\beta}{1 - C\epsilon|\nabla\varphi_{\sigma t}|} \geq G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_{\sigma t}|}, Y_0, \nu_0\right)$$

where

(a) $Y_0 \in \partial B_{\epsilon\varphi_t}(X_0)$

(b) $\nu_0 = \frac{Y_0 - X_0}{|Y_0 - X_0|}$ and

(c) ν is parallel to $\nu + \epsilon\nabla\varphi_{\sigma t}$.

It follows that

$$|Y_0 - X_0| \leq C\epsilon$$

and

$$|\nu - \nu_0| \leq \epsilon|\nabla\varphi_t|.$$

Therefore

$$\begin{aligned} \log \beta - \log 1 + C\epsilon|\nabla\varphi_{\sigma t}| &\geq \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, Y_0, \nu\right) \\ &\geq \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, X_0, \nu\right) - \theta\sigma\epsilon - C\epsilon|\nabla\varphi_{\sigma t}| \\ &\geq \log G(\alpha, X_0, \nu) - C\epsilon|\nabla\varphi_{\sigma t}| - C\theta\sigma\epsilon. \end{aligned}$$

But $\log \bar{\beta} \geq \log \beta + C\sigma\epsilon$ (β and $\bar{\beta}$ being bounded). The proof of the lemma is complete.

PROOF OF THEOREM 2. To prove Theorem 2, we now want to apply the equivalent of Lemma 17 inductively. We want, therefore, to make sure that the hypothesis of the Harnack type Lemma 14 (now Lemma 20) holds. This follows from the fact that after a first Lipschitz expansion,

$$\tilde{u}(X) = \frac{1}{\lambda} u(\lambda X),$$

the Lipschitz norm of $\log G$ in X becomes as small as we wish, ($= \theta$) and that after a k^{th} expansion, $\|\log\|_{\Lambda^k(X)} \leq \theta 2^{-nk}$ and σ_k can be chosen $\geq 2^{-nk}$.

Remark. Only a Hölder condition in X and ν is necessary, but this requires a more careful argument.

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