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Forms Equivalent to Curvatures

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Abstract

The 2-forms, Ω and Ω' on a manifold M with values in vector bundles $\xi \to M$ and $\xi' \to M$ are *equivalent* if there exist smooth fibered-linear maps $U: \xi \to \xi'$ and $W: \xi' \to \xi$ with $\Omega' = U\Omega$ and $\Omega = W\Omega'$. It is shown that an ordinary 2-form equivalent to the curvature of a linear connection has locally a non-vanishing integrating factor at each point in the interior of the set rank $(\omega) = 2$ or in the set rank $(\omega) > 2$. Under favorable conditions the same holds at points where the rank of ω changes from $=2$ to >2. Global versions are also considered.

Forms equivalent to curvatures

The 2-forms Ω and Ω' on a manifold M with values in vector bundles $\xi \to M$ and $\xi' \to M$ are *equivalent*, $\Omega \sim \Omega'$, if there exist smooth fibered-linear maps $U: \xi \to \xi'$ and $W: \xi' \to \xi$ such that $\Omega' = U\Omega$ and $\Omega = W\Omega'$. Examples: *a*) If Ω is a symplectic structure on M, the Lagrangian submanifolds of M depend only on the equivalence class of Ω; *b*) If $\eta \to N$ is a vector bundle with a connection ∇, the notion of ∇-homotopy $\phi: M \times [0,1] \to N$ depends only on the equivalence class of the curvature of the induced connection $\phi * \nabla$ on $\phi * \eta \to M \times [0,1]$. For details see [PR].

The second example motivates this work where we consider an ordinary 2-form equivalent to the curvature of a linear connection. The conclusion is that locally it is also equivalent to a *closed* 2-form (i.e., the curvature of a connection on a 1-dimensional bundle; for related matters see [K], [T]; in other words, a *2-form equivalent to a curvature has an integrating factor locally.*

(1) **Theorem.** *Let ω be a 2-form on M equivalent to a curvature. For $x \in M$ suppose that one of the following holds*:

(1.*a*) rank $(\omega) = 2$ *near* x; *or*,
(1.*b*) rank $(\omega) > 2$ *at* (*hence also near*) x.

Then ω has an integrating factor near x, i.e., there exists a nonvanishing smooth function f satisfying $d(f\omega) = 0$ on a neighborhood of x.

We make the following basic assumptions throughout: $\theta \to M$ is a smooth vector bundle with a conection ∇ and ω is a 2-form on M not zero at all points. The curvature $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ of ∇ is considered as a 2-form on M with values in the bundle $\xi = \operatorname{End}(\theta)$ (of smooth fibered-linear self-maps of θ), and it is equivalent to ω, i.e., $UR = \omega$, $W\omega = R$ for appropriate U and W (from ξ into the trivial one-imensional bundle $M \times \mathbb{R}$ and back). Denote by A the image under W of the constant section $1/2$ on $M \times \mathbb{R}$. Thus, A is a global section of ξ and

$$R(X, Y)v = 2\omega(X, Y)Av \tag{2}$$

for $v \in \theta$ and $X, Y \in TM$. If $\theta = M \times V$ is trivial (V a vector space) and

$$\nabla_X \sigma = X(\sigma) + \Gamma(X)\sigma$$

with Γ and $\operatorname{End}(V)$-valued 1-form, then $R/2 = d\Gamma + \Gamma \wedge \Gamma$ and (2) reads:

$$\omega A = d\Gamma + \Gamma \wedge \Gamma. \tag{3}$$

(A is now a function from M into $\operatorname{End}(V)$.) In this and similar formulas we use the canonical bilinear maps $\xi \times \xi \to \xi$ (composition) and $\xi \times \theta \to \theta$ (evaluation) to extend the exterior calculus to forms with values in $\mathbb{R}$, ξ, and θ (as long as the mixing is meaningful). In particular

$$(\Gamma \wedge \Gamma)(X, Y) = (1/2)[\Gamma(X), \Gamma(Y)].$$

We write $\alpha \wedge \beta - \beta \wedge \alpha = [\alpha, \beta]$ for ξ-valued forms α, β of arbitrary degree, which includes $\phi\psi - \psi\phi = [\phi, \psi]$ for sections ϕ, ψ of ξ. The identity

$$\begin{aligned} d(\Gamma \wedge \Gamma) = d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma &= (\omega A - \Gamma \wedge \Gamma) \wedge \Gamma - \Gamma \wedge (\omega A - \Gamma \wedge \Gamma) \\ &= \omega \wedge [A, \Gamma] \end{aligned}$$

and differentiation of (3) give

$$(d\omega)A + \omega \wedge (dA + [\Gamma, A]) = 0 \tag{4}$$

In terms of a basis of V this translates into n^2 relations of the form

$$a_{ij}\, d\omega + \omega \wedge \alpha_{ij} = 0.$$

Since $A \neq 0$ (because $2UA = \omega \neq 0$) some quotient α_{ij}/a_{ij} is defined near each point, whence

(5) *locally there exist* 1-*forms* α *such that* $d\omega = \alpha \wedge \omega$.

We can prove now the following proposition which contains Theorem 1 under hypothesis (1.*a*) (cf. Corollary, 3.6, of [BCG]).

(6) **Proposition.** *Let* ω *be a* 2-*form defined on a neighborhood* U *of the origin* 0 *of* $\mathbb{R}^n$ *satisfying on* U:

(6.*a*) $d\omega = \alpha \wedge \omega$ *for some* 1-*form* α;
(6.*b*) $\operatorname{rank}(\omega) = 2$.

Then there exist local coordinates $y = (y_1, \ldots, y_n)$ *and a smooth function* h *such that* $\omega = h(y) dy_1 \wedge dy_2$ *near* 0. *A fortiori* $h \neq 0$ *and therefore* $f = 1/h$ *is a local integrating factor for* ω.

PROOF. First, (6.*a*) implies that the kernel of ω

$$N = \{ Y; \omega(Y, Z) = 0 \quad \text{for all} \quad Z \}$$

is an integrable distribution of planes, for if X, Y are vector fields in N and Z is an arbitrary vector field,

$$\begin{aligned}
\omega([X, Y], Z) &= -X\omega(Y, Z) - Y\omega(Z, X) - Z\omega(X, Y) + \omega([X, Y], Z) \\
&\quad + \omega([Y, Z], X) + \omega([Z, X], Y) \\
&= -3\, d\omega(X, Y, Z) \\
&= -\alpha(X)\omega(Y, Z) - \alpha(Y)\omega(Z, X) - \alpha(Z)\omega(X, Y) = 0
\end{aligned}$$

so $[X, Y] \in N$ as claimed. Apply now Frobenius' theorem to get local coordinates $y = (y_1, y_2, \ldots, y_n)$ such that N is spanned by $\partial/\partial y_3, \partial/\partial y_4, \ldots, \partial/\partial y_n$ at each point near 0, and define $h = \omega(\partial/\partial y_1, \partial/\partial y_2)$. Clearly ω and $h\, dy_1 \wedge dy_2$ vanish on all pairs $(\partial/\partial y_i, \partial/\partial y_j)$ with $i = 1, 2$ and $3 \leqslant j \leqslant n$, and they coincide on $(\partial/\partial y_1, \partial/\partial y_2)$ so that $\omega = h\, dy_1 \wedge dy_2$ as claimed. This proves (6).

The proof of Theorem (1) under hypothesis (1.*b*) is as follows. We interpret $dA + [\Gamma, A]$ as the covariant differential of A for a connection with curvature zero to obtain a parallel local section of the type A/f and then use the fact:

(7) A/f *parallel implies* $d(f\omega) = 0$

which is proved below. Here are the details. Let $\tilde{\nabla}$ denote the conection on ξ defined by $(\tilde{\nabla}_X\phi)\sigma = \nabla_X(\phi\sigma) - \phi\nabla_X\sigma$ for ϕ a section of ξ and σ a section of θ. Direct calculations show that the curvature $\tilde{R}$ of $\tilde{\nabla}$ is given by

(8) $$\tilde{R}(X, Y)\phi = [R(X, Y), \phi] = 2\omega(X, Y)[A, \phi].$$

Also, if $\nabla\sigma = d\sigma + \Gamma\sigma$ in a trivialization then for a section ϕ of ξ, $\tilde{\nabla}\phi = d\phi + [\Gamma, \phi]$. In particular, (4) reads

(9) $$(d\omega)A + \omega \wedge \tilde{\nabla}A = 0.$$

Next, denoting by $\xi^0 \subset \xi$ the one-dimensional subbundle spanned by A, we show that A is «recurrent» in the sense of [S]. Precisely,

(10) **Proposition.** *On the open set where* rank$(\omega) > 2$ *the subbundle* ξ^0 *is invariant under* ∇ *and the curvature of the induced connection vanishes identically.*

PROOF. To show that for each $X \in TM_x$ the endomorphism $\tilde{\nabla}_X A$ of θ_x is a scalar multiple of A it suffices to use (2.3) and (2.4) to obtain

$$\omega \wedge (\tilde{\nabla}A + \alpha A) = \omega \wedge \tilde{\nabla}A + (\omega \wedge \alpha)A = \omega \wedge \tilde{\nabla}A + (d\omega)A = 0,$$

and then use the following cancellation lemma from linear algebra.

Lemma. *Let T, E denote vector spaces, ω a (real) 2-form on T with* rank$(\omega) > 2$ *and B: $T \to E$ a linear map. If $\omega \wedge B = 0$ then $B = 0$.*

Assuming now hypothesis (1.*b*) we can use (10) to conclude that ξ^0 has locally a parallel section ϕ with $\phi = A$ at x, i.e. $\phi = A/f$ for some f with $f(x) = 1$ (and then $f \neq 0$ near x) which in view of (7) implies $d(f\omega) = 0$. To prove (7) simply go back to (3) and observe that for any $f \neq 0$,

$$(f\omega)(A/f) = d\Gamma + \Gamma \wedge \Gamma$$

implies the following analogue of (9):

(12) $$d(f\omega)A + \omega \wedge \tilde{\nabla}(A/f) = 0.$$

This concludes the proof of Theorem 1.

The following improves (7):

(13) **Proposition.** *For $f \neq 0$ an arbitrary C^1 function defined near $x \in M$ suppose that*

(13.*a*) rank $(\omega) = 2$ *at* x; *then* $d(f\omega) = 0$ *at* x *if and only if* $\tilde{\nabla}_X(A/f) = 0$ *for all* $X \in N_x$.

(13.*b*) rank $(\omega) > 2$ *at* x; *then* $d(f\omega) = 0$ *at* x *if and only if* $\tilde{\nabla}(a/f) = 0$ *at* x.

PROOF. From (12) follows that $d(f\omega) = 0$ is equivalent to $\omega \wedge \tilde{\nabla}(A/f) = 0$ and so when rank $(\omega) > 2$ it is also equivalent to $\tilde{\nabla}(A/f) = 0$ by the lemma above. Suppose rank $(\omega) > 2$ at x and let $X_1, \dots, X_n$ be a basis for TM_x with N_x spanned by $X_3, X_4, \dots, X_n$. If i, j, k are distinct then one of them, say i, is 3 or larger. Hence, by (12) again

$$Ad(f\omega)(X_i, X_j, X_k) = -\omega(X_j, X_k)\tilde{\nabla}_{X_i}(A/f)$$

and so $d(f\omega) = 0$ at x if and only if $\tilde{\nabla}_{X_i}(A/f) = 0$ for $i = 3, 4, \dots, n$.

A global version of (1.*b*) follows.

(14) **Theorem.** *Let* ω *be a* 2*-form on* M *with* rank $(\omega) > 2$ *everywhere. Then*

(14.*a*) ω *has a local non-vanishing integrating factor if and only if* $d\omega = \alpha \wedge \omega$ *for a closed globally defined* 1*-form* α.

(14.*b*) ω *has a global never vanishing integrating factor if and only if* $\alpha = dg$ *for a smooth globally defined function* g. *In particular, if* $H^1(M) = 0$ (*real cohomology*), ω *has a global never vanishing integrating factor if and only if* $d\omega = \alpha \wedge \omega$ *for a closed globally defined* 1*-form* α.

PROOF. Suppose $d(f\omega) = 0$ on an open set V. Then $d\omega = \alpha \wedge \omega$ with $\alpha = d(-\ln|f|)$, which is clearly closed on V. Using the cancellation lemma proved above, we conclude that α is unique, hence one implication in (14.*a*) follows. Conversely if $d\omega = du \wedge \omega$ locally then $d(f\omega) = 0$ for $f = \exp(-u)$, and (14.*a*) follows. This last remark also proves one implication in (14.*b*). To complete the proof suppose $d(f\omega) = 0$ globally. Then $d\omega = (-df/f) \wedge \omega$ and by the cancellation lemma again we get $\alpha = d(-\ln|f|)$. This finishes the proof.

The global integrability of α is necessary for let $M = S^1 \times \mathbb{R}^3$ (S^1 = circle) and let φ be the angle variable on S^1, (x, y, z) cartesian coordinates on $\mathbb{R}^3$. Also let $X = \partial/\partial\psi$, $\alpha = d\varphi$ (which are smooth and globally defined on M), and define $\omega = (-y\,dx + dz) \wedge \alpha + dx \wedge dy$. Then, $d\omega = \alpha \wedge \omega$, rank $(\omega) = 4$, $d\alpha = 0$, and α *is not globally integrable.*

In addition to the basic assumptions we suppose that local coordinates $(p, \dots)$ exist near $x_0 \in M$ such that the resulting situation in $\mathbb{R}^n$ is as follows:

(15.*a*) rank $(\omega) = 2$ for $p \leqslant 0$,

(15.*b*) rank $(\omega) > 2$ for $p > 0$.

As above, N_x (in particular N_0) denotes the kernel of ω at $x \in \mathbb{R}^n$ near $x_0 = 0$.

(16) **Theorem.** *The form ω has a non-vanishing local integrating factor near $x_0 = 0$ in any of the following cases*:

(16.*a*) *N_0 is transversal to the hyperplane $\{p = 0\}$*;
(16.*b*) *$N_x \subset \{p = 0\}$ for each x near 0 belonging to the hyperplane $\{p = 0\}$*.

PROOF. It suffices to consider the case where $\tilde{\nabla}A = 0$ for $p \geqslant 0$. In fact, the restriction of ξ^0 to $\{p \geqslant 0\}$ has curvature zero (by (10) for $p > 0$ and by continuity at $p = 0$) and therefore there is a parallel section A/f with $f = 1$ at $0 \in \mathbb{R}^n$, and f of class C^∞ on $\{p \geqslant 0\}$; after extending f to a smooth function on a neighborhood of 0 we replace A by A/f (and ω by $f\omega$) to obtain $\tilde{\nabla}A = 0$ for the new A on $\{p \geqslant 0\}$. Using now the proof of (6) and Frobenius' theorem we obtain a foliation $\mathfrak{F} = \{F\}$ of the manifold with boundary $M_1 \cap \{p \leqslant 0\}$, where M_1 is a small neighborhood of the origin. For each leaf F the restriction $\xi^0|_F \to F$ is stable under $\tilde{\nabla}$ because if $x \in F \subset M_1 \cap \{p \leqslant 0\}$, choosing $Y, Z \in T(M_1)_x$ with $\omega(Y, Z) = 1$ and $X \in TF_x = N_x$, from (9) follows that $\tilde{\nabla}_X A = -d\omega(X, Y, Z)A$. Also the curvature of $\tilde{\nabla}$ vanishes on $\xi^0|_F$ (8).

Consider the case (16.*a*). It is clear that each leaf intersects the boundary $M_1 \cap \{p = 0\}$ transversally. Using $\tilde{R} = 0$ on $\xi^0|F$ we can find parallel sections in each $\xi^0|F$ extending the values of A on $M_1 \cap \{p = 0\}$ (which are parallel on $F \cap \{p = 0\}$ since $\tilde{\nabla}A = 0$ throughout $\{p \geqslant 0\}$). Thus a smooth section A/f of ξ^0 is defined on $M_1 \cap \{p \leqslant 0\}$ with $f = 1$ on $M^1 \cap \{p = 0\}$ and A/f «parallel on each leaf»:

$$(17) \qquad \tilde{\nabla}_X(A/f) = 0 \quad \text{for} \quad x \in N_x, \qquad x \in M_1 \cap \{p \leqslant 0\}.$$

Setting $f = 1$ on $M_1 \cap \{p \geqslant 0\}$ gives a C^1 extension of f satisfying

$$(18) \qquad \tilde{\nabla}_X(A/f) = 0 \quad \text{for} \quad X \in T(M_1)_x, \qquad x \in M_1 \cap \{p \geqslant 0\}.$$

Now (13) aplies to give $d(f\omega) = 0$ on M_1 (and this in turn forces f to be C^∞).

The proof assuming (16.*b*) is similar. First, each leaf is fully contained in $\{p = 0\}$ or disjoint from $\{p = 0\}$. Denote by Σ the set of points in $M_1 \cap \{p \leqslant 0\}$ orthogonal to N_0. Then if M_1 is small each leaf intersects F in exactly one point. Consider A extended smoothly to Σ and find parallel extensions along the leaves using the initial values of A at the point in $F \cap \Sigma$, in each case. As above we get A/f satisfying (17) and (18), and $d(f\omega) = 0$ follows again.

The following results indicate how some properties of ω translate into properties of A, ∇ or $\tilde{\nabla}$. We omit the proofs.

(19) *Under hypothesis* (1.*a*) *of Theorem* 1, *there exist local coordinates $y = (y_1, y_2, \ldots, y_n)$ and a trivialization $\theta = M \times \mathbb{R}^d$ near x such that*

$$\omega = h(y)\, dy_1 \wedge dy_2$$

$$\Gamma = B_1(y_1, y_2)\, dy_1 + B_2(y_1, y_2)\, dy_2$$

$$hA = [B_1, B_2] - (\partial B_1/\partial y_2) + (\partial B_2/\partial y_1)$$

where B_1, B_2 *are* $d \times d$ *matrices depending on* y_1, y_2 *only.*

(20) *Under hypothesis* (1.*b*) *of Theorem* 1, *parallel trasport operators for* $\tilde{\nabla}$ *are obtained locally by conjugation with the parallel transport operators for* ∇. *In particular, if* $U: \theta_u \to \theta_v$ (u, v *near* x) *is parallel transport along any curve joining* u, v, *then* $A_v = UA_uU^{-1}$.

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