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Cohomology Theories on Compact and Locally Compact Spaces

To Alberto Calderón on his retirement
from the University of Chicago

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1. Introduction

This paper is devoted to an exposition of cohomology theories on categories of spaces where the cohomology theories satisfy the type of axiom system considered in [1, 12, 16, 17, 18]. The categories considered are $\mathcal{C}_{\text{comp}}$, the category of all compact Hausdorff spaces and continuous functions between them, and $\mathcal{C}_{\text{loc comp}}$, the category of all locally compact Hausdorff spaces and proper continuous functions between them. The fundamental uniqueness theorem for cohomology theories on a finite dimensional space implies a corresponding uniqueness theorem for cohomology theories on either of these two categories. The proof involves an extension of the uniqueness theorem for finite dimensional spaces to compact spaces which contrasts with the usual type of proof which involves a uniqueness proof for polyhedra and an extension to compact spaces.

A spectrum of ANR defines a cohomology theory on $\mathcal{C}_{\text{loc comp}}$. Applications of the uniqueness theorem to such cohomology theories gives a proof of the known result that the Chern character is an isomorphism of $K(X) \otimes \mathbb{Q}$ with

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$\check{H}_{ev}(X; \mathbb{Q})$ for every locally compact X. The uniqueness theorem is also used to give another proof of the duality theorem in stable homotopy theory for compact subsets of $\mathbb{R}^n$.

The results of this paper are not new. The objective of the paper is to give a reasonably self-contained expository account of the uniqueness theorem for cohomology theories on $\mathcal{C}_{\text{loc comp}}$ and how it can be applied to give theorems such as the one concerning the Chern character and the duality theorem referred to above.

The rest of the paper is divided into six sections. Section 2 is devoted to preliminaries including a review of the definitions of cohomology theory on a space and of an ES theory on a space (the latter satisfying some of the Eilenberg-Steenrod axioms). In Section 3 we consider cohomology theories on $\mathcal{C}_{\text{comp}}$ and $\mathcal{C}_{\text{loc comp}}$ and a uniqueness theorem for each. We also show that cohomology theories on $\mathcal{C}_{\text{comp}}$ correspond injectively to compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$.

Section 4 is devoted to ES theories on $\mathcal{C}_{\text{comp}}$. There is a uniqueness theorem for these and a proof of the equivalence of cohomology theories and ES theories on $\mathcal{C}_{\text{comp}}$.

In Section 5 it is shown how a spectrum of ANR determines a compactly supported ES theory on $\mathcal{C}_{\text{loc comp}}$. As a consequence the theorem concerning the Chern character is deduced. Section 6 contains definitions of the functional spectrum and of a spectrum approximating the complement of a compact pair in $\mathbb{R}^n$. In Section 7 these two spectra are compared to obtain a proof of the duality theorem in stable homotopy theory.

2. Preliminaries

In this section we recall the definitions of a cohomology theory and of an ES theory on a space X [16, 17, 18] and some related concepts.

All topological spaces will be assumed to be normal Hausdorff spaces. If H is a contravariant functor from a category of subsets (and inclusion maps between them) of such a space to the category of graded abelian groups (and homomorphisms of degree zero between them) we use the following notation. If H is defined for an inclusion map i: $B \subset A$ and $u \in H(A)$ then $u|B \in H(B)$ is defined by $u|B = H(i)(u)$. The statement that H is a contravariant functor is equivalent to the two conditions:

(i) for $u \in H(A)$, $u|A = u$.
(ii) for $C \subset B \subset A$ and $u \in H(A)$ then $(u|B)|C = u|C$.

In general ρ will be used to denote a homomorphism induced by an inclusion map or a family of inclusion maps (i.e. $\rho = H(i)$: $H(A) \to H(B)$ for

$i\colon B \subset A$). Similar notation will be used for contravariant functors H from a category of pairs of X to the category of graded abelian groups. Given a topological space X let $\mathrm{cl}(X)$ denote the category of all closed subsets of X and all inclusion maps between them and let $\mathrm{cl}(X)^2$ be the category of all pairs of closed subsets of X and inclusion maps between them.

A *cohomology theory* H, δ on X consists of:

(i) a contravariant functor H from $\mathrm{cl}(X)$ to the category of graded abelian groups ($H(A) = \{H^q(A)\}_{q\in\mathbb{Z}}$ for A in $\mathrm{cl}(X)$) such that $H(\varnothing) = 0$, and
(ii) a natural transformation δ assigning to every two closed subsets $A, B \subset X$ a homomorphism of degree 1

$$\delta\colon H^q(A \cap B) \longrightarrow H^{q+1}(A \cup B)$$

such that the following are satisfied:

Continuity. For every closed $A \subset X$ there is an isomorphism

$$\rho\colon \varinjlim \{H^q(N) | N \text{ a closed neighborhood of } A \text{ in } X\} \approx H^q(A)$$

where $\rho\{u\} = u|A$ for $u \in H^q(N)$.

MV exactness. For every two closed sets A, B in X there is an exact sequence

$$\cdots \xrightarrow{\delta} H^q(A \cup B) \xrightarrow{\alpha} H^q(A) \oplus H^q(B) \xrightarrow{\beta} H^q(A \cap B) \xrightarrow{\delta} H^{q+1}(A \cup B) \xrightarrow{\alpha} \cdots$$

where $\alpha(u) = (u|A, u|B)$ for $u \in H^q(A \cup B)$ and $\beta(u, v) = u|A \cap B - v|A \cap B$ for $u \in H^q(A)$, $v \in H^q(B)$.

The cohomology theory is *nonnegative* if $H^q(A) = 0$ for $q < 0$ and all closed $A \subset X/C$. It is *compactly supported* (or has *compact supports*) if given $u \in H^q(A)$ there is a decomposition $A = B \cup C$ where B is closed, C is compact, and $u|B = 0$. It is *additive* if given a discrete* family $\{A_j\}_{j\in J}$ of closed sets there is an isomorphism

$$\sigma\colon H^q\Big(\bigcup_{j\in J} A_j\Big) \approx \prod_{j\in j} H^q(A_j)$$

where $\sigma(u) = \{u|A_j\}_{j\in J}$ for $u \in H^q\big(\bigcup_{j\in j} A_j\big)$.

* A family $\{A_j\}_{j\in J}$ of subsets of a topological space X is *discrete* if every point of X has a neighborhood meeting at most one member of the family. This implies the members of the family are pairwise disjoint and, since a discrete family is obviously locally finite, if each is closed in X, then $\bigcup_{j\in J} A_j$ is also closed in X.

If H, δ and H', δ' are cohomology theories on the same space X, a *homomorphism* $\varnothing$ from H, δ to H', δ' is a natural transformation from H to H' commuting up to sign with δ, δ'.

Cohomology theories on X frequently arise from contravariant functors on $\mathrm{cl}(X)^2$ satisfying some of the Eilenberg-Steenrod axioms [5] as in the following.

An ES *theory* H, δ^* on X consists of:

(i) a contravariant functor H from $\mathrm{cl}(X)^2$ to the category of graded abelian groups, and

(ii) a natural transformation of degree 1

$$\delta^*\colon\ H^q(B, \varnothing) \to H^{q+1}(A, B)$$

for every (A, B) in $\mathrm{cl}(X)^2$

such that the following are satisfied:

Continuity. For every closed A in X there is an isomorphism

$$\rho\colon\ \varinjlim \{H^q(N, \varnothing) | N \text{ a closed neighborhood of } A \text{ in } X\} \approx H^q(A, \varnothing)$$

where $\rho\{u\} = u|(A, \varnothing)$ for $u \in H^q(N, \varnothing)$,

Exactness. For every closed pair (A, B) in X the following sequence is exact

$$\cdots \xrightarrow{\delta^*} H^q(A, B) \xrightarrow{H(j)} H^q(A, \varnothing) \xrightarrow{H(i)} H^q(B, \varnothing) \xrightarrow{\delta^*} H^{q+1}(A, B) \longrightarrow \cdots$$

where $i\colon (B, \varnothing) \subset (A, \varnothing)$ and $j\colon (A, \varnothing) \subset (A, B)$.

Excision. For closed sets A, B in X there is an isomorphism

$$\rho\colon\ H^q(A \cup B, B) \approx H^q(A, A \cap B).$$

It is standard [5] that if H, δ^* is an ES theory on X there is a cohomology theory H', δ' on X such that $H'(A) = H(A, \varnothing)$ and $\delta'\colon\ H'^q(A \cap B) \to H'^{q+1}(A \cup B)$ is suitably defined. In general we do not know if there is a way of associating an ES theory to a cohomology theory on X. With suitable definitions of cohomology theories and ES theories on larger categories we will show in Section 4 that the two theories are equivalent on the category of all compact spaces.

The concepts of *nonnegativity, compactly supported*, and *additivity* are defined for ES theories to correspond to the same properties of the associated cohomology theories.

3. Cohomology theories on categories of spaces

In this section we consider cohomology theories defined on categories of topological spaces and continuous functions. These consist of contravariant functors and natural transformations on the category whose restriction to cl(X) is a cohomology theory on X for every object X in the category. The categories of interest are the category of all compact spaces and continuous functions and the category of all compact spaces and proper continuous functions. We prove uniqueness theorems for cohomology theories on these two categories. Finally, we show that cohomology theories on the category of compact spaces correspond injectively to compactly supported cohomology theories on the category of locally compact spaces.

We begin by considering cohomology theories on cubes. By a *cube* we mean a product space $\prod_{j\in J} I_j$ where I_j is a closed interval of $\mathbb{R}$ for each $j \in J$. Given a cube $C(J) = \prod_{j\in J} I_j$ and given a subset $J' \subset J$ let $C(J') = \prod_{j\in J'} I_j$. There is a canonical projection map $p_{J'}$: $C(J) \to C(J')$.

Lemma 3.1. *Let φ: $H \to H'$ be a homomorphism of cohomology theories on a cube $C(J)$ and let $n \in \mathbb{Z}$ be such that φ_A: $H(A) \to H'(A)$ is an n-equivalence* for every A of the form $A = p_F^{-1}(x)$ where F is a finite subset of J and $x \in C(F)$. Then φ_A is an n-equivalence for every closed $A \subset C(J)$.*

PROOF. 1) Let F be an arbitrary finite subset of J and let H_F, H'_F be the cohomology theories on $C(F)$ which equal the direct images (in the sense of Remark 2.6 of [18]) of the cohomology theories H, H' on $C(J)$ under p_F: $C(J) \to C(F)$ (so $H_F(B) = H(p_F^{-1}(B))$ and $H'_F(B) = H'(p_F^{-1}(B))$ for every closed $B \subset C(F)$). The hypotheses on φ imply that φ induces a homomorphism φ_F: $H_F \to H'_F$ of cohomology theories on $C(F)$ which is an n-equivalence for every $x \in C(F)$. Since $C(F)$ is a finite dimensional compact metric space, it follows from Theorem 4.2 of [18] that φ_F is an n-equivalence for every closed $B \subset C(F)$.

2) Let A be a closed subset of $C(J)$ and for F a finite subset of J let $A_F = p_F^{-1} p_F(A)$. If $F \subset F'$ are finite subsets of J, there is a projection p: $C(F') \to C(F)$ such that $p_F = p \circ p_{F'}$. It follows that $p_F(A) = pp_{F'}(A)$ so $p_{F'}(A) \subset p^{-1} p_F(A)$ and

$$A_{F'} = p_{F'}^{-1}(p_{F'}(A)) \subset p_{F'}^{-1}(p^{-1} p_F(A)) = p_F^{-1} p_F(A) = A_F.$$

Hence, the collection $\{A_F | F \text{ finite} \subset J\}$ is a family of closed subsets of $C(J)$ directed downward by inclusion. Clearly $A \subset A_F$ for every F so $A \subset \bigcap_F A_F$.

We show $A = \bigcap_F A_F$. If $y \in C(J) - A$ there is a nbhd of y disjoint from A (because A is closed). Every nbhd of y contains a subset of the form $p_F^{-1}(N)$ where F is a finite subset of J and N is a closed nbhd of $p_F(y)$ in $C(F)$. Clearly $p_F^{-1}(N)$ is disjoint from A if and only if N is disjoint from $p_F(A)$. This implies $p_F(y) \notin p_F(A)$ and so $y \notin p_F^{-1} p_F(A) = A_F$. Therefore, $\bigcap_F A_F = A$.

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* A homomorphism φ: $G \to G'$ of degree 0 between graded abelian groups is an *n-equivalence* if φ: $G^q \to G'^q$ is an isomorphism for all $q < n$ and a monomorphism for $q = n$.

It follows from Proposition 2.7 part 2) of [18] that $\varinjlim \{H(A_F)|F \text{ finite} \subset \subset J\} \approx H(A)$ and $\varinjlim \{H'(A_F)|F \text{ finite} \subset J\} \approx H'(A)$. By 1) above, φ_J: $H_F(p_F(A)) \to H'_F(p_F(A))$ is an n-equivalence for every finite $F \subset J$. This is equivalent to the assertion that φ: $H(A_F) \to H'(A_F)$ is an n-equivalence for every finite $F \subset J$. Passing to the direct limit we see that φ: $H(A) \to H'(A)$ is an n-equivalence for an arbitrary closed $A \subset C(J)$. □

Let $\mathcal{C}$ be a category of topological spaces and continuous functions such that if X is an object of $\mathcal{C}$ then cl(X) is a subcategory of $\mathcal{C}$. A *cohomology theory H,* δ on consists of:

(i) A contravariant functor H from $\mathcal{C}$ to the category of graded abelian groups such that $H(\phi) = 0$, and
(ii) A natural transformation δ: $H^q(A \cap B) \to H^{q+1}(A \cup B)$ for every triad $(X; A, B)$ in $\mathcal{C}$ (by such a triad we mean X is an object of $\mathcal{C}$ and A, B are closed subsets of X).

such that for every object X in $\mathcal{C}$ the restriction of H, δ to cl(X) is a cohomology theory on X.

A cohomology theory on $\mathcal{C}$ is *nonnegative, compactly supported* or *additive*, respectively, if its restriction to cl(X) has the corresponding property for every object X in $\mathcal{C}$. A cohomology theory H, δ is *invariant under homotopy* if for every f_0, f_1: $X \to Y$ which are homotopic in $\mathcal{C}$ (i.e. there is a continuous map F: $X \times I \to Y$ in $\mathcal{C}$ such that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ for all $x \in X$) then $H(f_0) = H(f_1)$: $H(Y) \to H(X)$.

Of primary interest are the categories $\mathcal{C}_{\text{comp}}$ of all compact spaces and continuous functions and $\mathcal{C}_{\text{loc comp}}$ of all locally compact spaces and proper continuous functions.

Proposition 3.2. *Every cohomology theory on* $\mathcal{C}_{\text{comp}}$ *is invariant under homotopy.*

PROOF. It is shown in [10] that every contravariant functor H on $\mathcal{C}_{\text{comp}}$ whose restriction to cl(X) is continuous for every compact space X is invariant under homotopy. □

A homomorphism φ: $H, \delta \to H', \delta'$ between two cohomology theories on the same category $\mathcal{C}$ is a natural transformation of degree 0 from H to H' commuting up to sign with δ, δ' for every triad $(X; A, B)$ in $\mathcal{C}$. We have the following extension of the uniqueness theorem.

Theorem 3.3. *Let* φ: $H, \delta \to H', \delta'$ *be a homomorphism between cohomology theories on* $\mathcal{C}_{\text{comp}}$ *such that for some one-point space P,* φ_P: $H(P) \to H'(P)$ *is an n-equivalence for some* $n \in \mathbb{Z}$. *Then* φ_X: $H(X) \to H'(X)$ *is an n-equivalence for every compact space X.*

PROOF. Because H, H' are contravariant functors on $\mathcal{C}_{\text{comp}}$ it follows that for every one-point space Q, φ_Q: $H(Q) \to H'(Q)$ is an n-equivalence. Consider a cube $C(J) = \prod_{j \in J} I_j$ and let $y \in C(F)$ for F a finite subset of J. Since the projection map p_F: $p_F^{-1}(y) \to y$ is a homotopy equivalence, there is a commutative square whose vertical maps are isomorphisms by Proposition 3.2

$$\begin{array}{ccc} H(y) & \xrightarrow{\varphi_y} & H'(y) \\ {\scriptstyle H(p_F)}\downarrow{\scriptstyle \approx} & & {\scriptstyle \approx}\downarrow{\scriptstyle H'(p_F)} \\ H(p_F^{-1}(y)) & \xrightarrow{\varphi} & H(p_F^{-1}(y)) \end{array}$$

It follows that φ: $H(p_F^{-1}(y)) \to H'(p_F^{-1}(y))$ is an n-equivalence for every finite $F \subset J$ and every $y \in C(F)$. By Lemma 3.1, φ_A is an n-equivalence for every $A \subset C(J)$. Since every compact X is homomorphic to a closed subset of some cube, φ_X: $H(X) \to H'(X)$ is an n-equivalence for every X. □

Corollary 3.4. *Let φ: $H, \delta \to H', \delta'$ be a homomorphism between compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$ such that for some one-point space P, φ_P: $H(P) \to H'(P)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then φ_X: $H(X) \to H'(X)$ is an n-equivalence for every locally compact space X.*

PROOF. Since $\mathcal{C}_{\text{comp}}$ is a subcategory of $\mathcal{C}_{\text{loc comp}}$, we can apply Theorem 3.3 to deduce that φ_X: $H(X) \to H'(X)$ is an n-equivalence for every compact X. The Corollary follows from this and Proposition 2.8 of [18]. □

In the above Corollary we used the fact that $\mathcal{C}_{\text{comp}}$ is a subcategory of $\mathcal{C}_{\text{loc comp}}$. Therefore, every cohomology theory on $\mathcal{C}_{\text{loc comp}}$ defines by restriction a cohomology theory on $\mathcal{C}_{\text{comp}}$. We now present a way of obtaining a compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$ from a cohomology theory on $\mathcal{C}_{\text{comp}}$.

For a subset $A \subset X$ we say A is *cobounded* in X if $\overline{X - A}$ is compact. We need de following lemma.

Lemma 3.5. *Let H be a contravariant functor from* cl(X) *to graded abelian groups such that $H(\emptyset) = 0$ and such that for every $A \subset X$*

(*) ρ: $\varinjlim \{H(N) | N$ a closed cobounded nbhd of A in $X\} \approx H(A)$.

Then H is continuous and compactly supported.

PROOF. This follows immediately from Proposition 2.6 of [17] applied to the family of supports consisting of all compact subsets of X. □

Given an arbitrary locally compact space X let X^+ be the compact space consisting of X together with exactly one more point ∞ such that $X^+ - \{\infty\} = X$. In case X is compact, X^+ is the topological sum of X and $\{\infty\}$. In case X is non-compact, X^+ is the one-point compactification of X. Note that $A \subset X \Rightarrow A^+ \subset X^+$, $\emptyset^+ = \{\infty\}$, and $(A \cap B)^+ = A^+ \cap B^+$, $(A \cup B)^+ = A^+ \cup B^+$ for A, B closed in X.

Proposition 3.6. *Given a cohomology theory H, δ on $\mathcal{C}_{\mathrm{comp}}$ there is a compactly supported cohomology theory $\tilde{H}, \tilde{\delta}$ on $\mathcal{C}_{\mathrm{loc\,comp}}$ where $\tilde{H}(X) = \ker[H(X^+) \xrightarrow{\rho} H(\infty)]$ for a locally compact space X.*

PROOF. $\tilde{H}$ is defined by the above. To define $\tilde{\delta}$ note that if A is a closed subset of X and $c\colon A^+ \to \infty$ is the constant map, then the composite

$$H(\infty) \xrightarrow{H(c)} H(A^+) \xrightarrow{\rho} H(\infty)$$

is the identity. Therefore, in the following commutative diagram with exact rows the vertical maps are epimorphisms

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\beta} & H^{q-1}(A^+ \cap B^+) & \xrightarrow{\delta} & H^q(A^+ \cup B^+) & \xrightarrow{\alpha} & H^q(A^+) \oplus H^q(B^+) & \xrightarrow{\beta} & H^q(A^+ \cap B^+) & \xrightarrow{\delta} \cdots \\
& \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho & \\
\cdots \xrightarrow{\beta} & H^{q-1}(\infty) & \xrightarrow{0} & H^q(\infty) & \xrightarrow{\alpha} & H^q(\infty) \oplus H^q(\infty) & \xrightarrow{\beta} & H^q(\infty) & \xrightarrow{0} \cdots
\end{array}$$

It follows that there is an exact sequence of the kernels of ρ

$$\cdots \xrightarrow{\tilde{\beta}} \tilde{H}^{q-1}(A \cap B) \xrightarrow{\tilde{\delta}} \tilde{H}^q(A \cup B) \xrightarrow{\tilde{\alpha}} \tilde{H}^q(A) \oplus \tilde{H}^q(B) \xrightarrow{\tilde{\beta}} \tilde{H}^q(A \cap B) \xrightarrow{\tilde{\delta}} \cdots$$

This defines the natural transformation $\tilde{\delta}$ and shows that $\tilde{H}, \tilde{\delta}$ satisfy *MV* exactness.

Clearly $\tilde{H}(\emptyset) = \ker[H(\infty) \xrightarrow{\rho} H(\infty)] = 0$ and the closed nbhds of A^+ in X^+ are precisely the sets N^+ where N is a closed cobounded nbhd of A in X. Hence, the continuity of H on X^+ implies

$$\rho\colon \varinjlim \{\tilde{H}(N) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \tilde{H}(A)$$

It follows from Lemma 3.5 that $\tilde{H}$ is continuous and compactly supported on X. □

Theorem 3.7. *The map H, δ to $\tilde{H}, \tilde{\delta}$ is an injection from cohomology theories on $\mathcal{C}_{\mathrm{comp}}$ to compactly supported cohomology theories on $\mathcal{C}_{\mathrm{loc\,comp}}$.*

PROOF. Given H, δ on $\mathcal{C}_{\mathrm{comp}}$ let $\tilde{H}, \tilde{\delta}$ be the compactly supported cohomology theory on $\mathcal{C}_{\mathrm{loc\,comp}}$ defined by it as in Proposition 3.6. If X is any compact space, the exactness of $0 = H(\emptyset) \to H(X^+) \xrightarrow{\alpha} H(X) \oplus H(\infty) \to \to H(\emptyset) = 0$ implies that $\tilde{H}(X) = \ker[H(X^+) \xrightarrow{\beta} H(\infty)] \approx H(X)$. This shows that $\tilde{H}$ when restricted to $\mathcal{C}_{\text{comp}}$ is isomorphic to H. Similarly $\tilde{\delta}$ restricted to compact triads is isomorphic to δ. Thus, the restriction of $\tilde{H}, \tilde{\delta}$ to $\mathcal{C}_{\text{comp}}$ is isomorphic to H, δ. □

If H', δ' is a compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$, let H, δ be its restriction to $\mathcal{C}_{\text{comp}}$ and $\tilde{H}, \tilde{\delta}$ the compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$ determined by H, δ as in Proposition 3.6. In general we do not know what relation there is between H', δ' and $\tilde{H}, \tilde{\delta}$ both compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$.

4. ES theories on categories of spaces

We consider ES theories on a category of topological spaces and continuous mappings. Since ES theories define cohomology theories, the uniqueness theorem is valid for ES theories. We also show that there is an equivalence between ES theories and cohomology theories on the category of all compact spaces and continuous functions. Thus, cohomology theories are «single space» equivalents to ES theories. Other single space equivalents to ES theories have been given in [2, 3, 7, 11].

Let $\mathcal{C}$ be a category of topological spaces and continuous functions such that if X is an object of $\mathcal{C}$ then $\text{cl}(X)$ is a subcategory of $\mathcal{C}$. An ES *theory* H, δ^* on $\mathcal{C}$ consists of:

(i) A contravariant functor H from $\mathcal{C}^2$ (the category of closed pairs in $\mathcal{C}$) to the category of graded abelian groups, and
(ii) A natural transformation δ^*: $H^q(B, \emptyset) \to H^{q+1}(A, B)$ for every (A, B) in $\mathcal{C}^2$

such that for every object X in $\mathcal{C}$ the restriction of H, δ^* to $\text{cl}(X)^2$ is an ES theory on X.

Since ES theories are continuous, the result in [10] implies they are invariant under homotopy. Therefore, they are continuous extraordinary cohomology theories because they satisfy all of the Eilenberg-Steenrod axioms [5] except the dimension axiom and are continuous.

As in Section 2 every ES theory on $\mathcal{C}$ determines a cohomology theory on $\mathcal{C}$. The concepts of *nonnegativity, compactly supported*, and *additivity* for ES theories on $\mathcal{C}$ are defined to correspond to the same properties of the associated cohomology theories.

A homomorphism φ: $H, \delta^* \to H', \delta'^*$ between ES theories on $\mathcal{C}$ is a natural transformation of degree 0 from H to H' commuting up to sign with δ^*, δ'^*

for every pair (X, A) in $\mathcal{C}^2$. The following uniqueness theorem for ES theories is valid.

Theorem 4.1. *Let φ: $H, \delta^* \to H', \delta'^*$ be a homomorphism between two compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$ such that for some one-point space P, φ_P: $H(P, \emptyset) \to H'(P, \emptyset)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then φ: $H(X, A) \to H'(X, A)$ is an n-equivalence for every locally compact pair.*

PROOF. The homomorphism φ determines a homomorphism $\bar{\varphi}$: $\bar{H}, \bar{\delta} \to \bar{H}', \bar{\delta}'$ between the cohomology theories on $\mathcal{C}_{\text{loc comp}}$ defined by the ES theories H, δ^* and H', δ'^* respectively. Since these are compactly supported and $\bar{\varphi}$ is an n-equivalence for the one-point space P, it follows from Corollary 3.4 that $\bar{\varphi}$ is an n-equivalence for every locally compact space X. This is equivalent to the assertion that φ: $H(X, \emptyset) \to H'(X, \emptyset)$ is an n-equivalence for every locally compact space X. Then the «five-lemma» shows that $\varphi: H(X, A) \to$ $\to H'(X, A)$ is an n-equivalence for every locally compact pair. □

The next result asserts the equivalence between cohomology theories and ES theories on $\mathcal{C}_{\text{comp}}$.

Theorem 4.2. *The assigment of a cohomology theory to an ES theory is an equivalence on $\mathcal{C}_{\text{comp}}$.*

PROOF. We have already seen that for an ES theory H, δ^* on $\mathcal{C}_{\text{comp}}$ there is associated a cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$ with $H'(X) = H(X, \emptyset)$ for every compact space X.

For the converse we consider for every compact space X the cone CX over X with vertex v (so CX is the join of X with a point v not in X). Given a cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$ define a contravariant functor H on $\mathcal{C}^2_{\text{comp}}$ by

$$H(X, A) = \ker[\rho\colon\ H'(X \cup CA) \to H'(CA)]$$

(in case $A = \emptyset$, $CA = \{v\}$). Then

$$H(X, \emptyset) = \ker[\rho\colon\ H'(X \cup \{v\}) \to H'(v)]$$

and by the exactness of

$$H'(\emptyset) \xrightarrow{\delta'} H'(X \cup \{v\}) \xrightarrow{\alpha} H'(X) \oplus H'(v) \xrightarrow{\beta} H'(\emptyset),$$

there is an isomorphism α': $H(X, \emptyset) \approx H'(X)$. Therefore, continuity of H' implies continuity of H.

To define δ^* and verify exactness for H, δ^* note that

$$\delta'\colon\ H'(A) \to H'(X \cup CA)$$

for the triad $(CX; X, CA)$ has image lying in

$$\ker [\rho\colon\ H'(X \cup CA) \to H'(CA)]$$

by exactness of the MV sequence of X, CA. Therefore,

$$\operatorname{im} [\delta'\colon\ H'(A) \to H'(X \cup CA)] \subset \ker [\rho\colon\ H'(X \cup CA) \to H'(CA)] = H(X, A).$$

Thus, there is a unique homomorphism $\delta'\colon H(A, \varnothing) \to H(X, A)$ such that there is a commutative square

$$\begin{array}{ccc} H(A, \varnothing) & \xrightarrow[\approx]{\alpha'} & H'(A) \\ {\scriptstyle \delta^*}\downarrow & & \downarrow{\scriptstyle \delta'} \\ H(X, A) & \subset & H(X \cup CA) \end{array}$$

Then δ^* is a natural transformation of degree 1 defined for every (X, A) in $\mathcal{C}^2_{\text{comp}}$. Furthermore, if $c\colon X \cup CA \to v$ is the constant map, the composite

$$H'(v) \xrightarrow{H'(c)} H'(X \cup CA) \xrightarrow{\rho} H'(CA)$$

is an isomorphism (by Proposition 3.2 since $c|CA\colon CA \to v$ is a homotopy equivalence). Therefore, in the commutative diagram with exact rows all vertical maps are epimorphisms

$$\begin{array}{ccccccccc} \cdots \to & H'(A) & \xrightarrow{\delta'} & H'(X \cup CA) & \to & H'(X) \oplus H'(CA) & \to & H'(A) & \xrightarrow{\delta'} \cdots \\ & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & & \\ \cdots \to & 0 & \to & H'(CA) & \approx & 0 \oplus H'(CA) & \to & 0 & \to \cdots \end{array}$$

hence, there is an exact sequence of kernels

$$\cdots \to H'(A) \to H(X, A) \to H'(X) \to H'(A) \to \cdots$$

Replacing $H'(A)$ by $H(A, \varnothing)$ and $H'(X)$ by $H(X, \varnothing)$ we obtain the exact sequence

$$\cdots \to H(A, \varnothing) \xrightarrow{\delta^*} H(X, A) \xrightarrow{H(j)} H(X, \varnothing) \xrightarrow{H(i)} H(A, \varnothing) \xrightarrow{\delta^*} \cdots$$

where $i\colon (A, \varnothing) \subset (X, \varnothing)$ and $j\colon (X, \varnothing) \subset (X, A)$. Therefore, H, δ^* satisfy exactness.

To prove the excision property let $(X; A, B)$ be a triad in $\mathcal{C}_{\text{comp}}$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H'(A) \oplus H'(CB) & \xrightarrow{\beta} & H'(A \cap B) & \xrightarrow{\delta'} & H'(A \cup CB) & \xrightarrow{\alpha} & H'(A) \oplus H'(CB) & \xrightarrow{\beta} & H'(A \cap B) \\
\rho \downarrow \approx & & \rho \downarrow \approx & & \rho \downarrow & & \approx \downarrow \rho & & \approx \downarrow \rho \\
H'(A) \oplus H'(C(A \cap B)) & \xrightarrow{\beta} & H'(A \cap B) & \xrightarrow{\delta'} & H'(A \cap C(A \cap B)) & \xrightarrow{\alpha} & H'(A) \oplus H'(C(A \cap B)) & \xrightarrow{\beta} & H'(A \cap B)
\end{array}$$

From the «five lemma» the middle vertical map is an isomorphism and there is a commutative square

$$\begin{array}{ccc}
H'(A \cup CB) & \xrightarrow{\rho} & H'(CB) \\
\rho \downarrow \approx & & \rho \downarrow \approx \\
H'(A \cup C(A \cap B)) & \xrightarrow{\rho} & H'(C(A \cap B))
\end{array}$$

Therefore, $H(A \cup B, B) = \ker [\rho\colon H'(A \cup CB) \to H'(CB)]$ is isomorphic to $H(A, A \cap B) = \ker [\rho\colon H'(A \cup C(A \cap B)) \to H'(C(A \cap B))]$ by the restriction map.

Thus, H, δ^* is an ES theory on $\mathcal{C}_{\text{comp}}$. It is clear that α' induces an isomorphism of $H(A, \emptyset)$ with $H'(A)$ so the cohomology theory induced on $\mathcal{C}_{\text{comp}}$ by the ES theory H, δ^* is isomorphic to the original cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$.

Conversely, if H, δ^* is an ES theory on $\mathcal{C}_{\text{comp}}$ and H', δ' is the corresponding cohomology theory on $\mathcal{C}_{\text{comp}}$ let H'', δ'' be the ES theory on $\mathcal{C}_{\text{comp}}$ constructed as above from H', δ'. Then

$$\begin{aligned}
H''(X, A) &= \ker [H'(X \cup CA) \longrightarrow H'(CA)] \\
&= \ker [H(X \cup CA, \emptyset) \longrightarrow H(CA, \emptyset)]
\end{aligned}$$

From the exact sequence

$$\xrightarrow{0} H(X \cup CA, CA) \longrightarrow H(X \cup CA, \emptyset) \longrightarrow H(CA, \emptyset) \xrightarrow{0}$$

we see that $\ker [H(X \cup CA, \emptyset) \to H(CA, \emptyset)] \approx H(X \cup CA, CA)$. Since there is an excision isomorphism

$$H(X \cup CA, CA) \approx H(X, A)$$

we finally obtain $H''(X, A) \approx H(X, A)$. This isomorphism carries δ'' to δ^* and completes the proof. □

5. Cohomology defined by spectra

In this section we show that a spectrum of ANR defines an ES theory on the category $\mathcal{C}_{\text{loc comp}}$. In particular, K-theory, which is defined by such a spectrum, is an ES theory. The uniqueness theorem of the preceeding section yields another proof of the fact that the Chern character is an isomorphism of $K \otimes \mathbb{Q}$ with $\check{H}_{ev}$ (rational Cěch cohomology with compact supports) on $\mathcal{C}_{\mathrm{loc\,comp}}$.

For a pointed space X we let C_0X denote the reduced cone over X (so $C_0X = I \wedge X$ where $0 \in I$ is the base point of I) and $\Sigma_0 X = S^1 \wedge X$ the reduced suspension of X. If (X, A) is a compact pair with base point $x_0 \in A$ and Y is a pointed space there is an exact sequence of based homotopy classes [15]

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A; Y] \xrightarrow{\bar{k}^\#} [X \cup C_0 A; Y] \xrightarrow{\bar{i}^\#} [X; Y] \xrightarrow{i^\#} [A; Y]$$

where $i\colon A \subset X$, $\bar{i}\colon X \subset X \cup C_0 A$ and $\bar{k}\colon X \cup C_0 A \to \Sigma_0 A$ is the map collapsing X to the base point. In case Y is an ANR (absolute neighborhood retract for normal spaces) the contractibility of C_0A implies that the quotient map $q\colon X \cup C_0 A \to X/A$ induces, for every $n \geqq 0$, a bijection $(\Sigma_0^n q)^\#$: $[\Sigma_0^n (X/A); Y] \approx [\Sigma_0^n (X \cup CA); Y]$. Since $q \circ \bar{i} = k\colon X \to X/A$ where k is the quotient map, there is an exact sequence

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A; Y] \xrightarrow{\delta} [X/A; Y] \xrightarrow{k^\#} [X; Y] \xrightarrow{i^\#} [A; Y]$$

where $\delta\colon [\Sigma_0^{n+1} A; Y] \longrightarrow [\Sigma_0^n (X/A); Y]$ is defined to equal the composite

$$(\Sigma_0^n q)^{\#-1} (\Sigma_0^n \bar{k})^\#\colon [\Sigma_0^{n+1} A; Y] \longrightarrow [\Sigma_0^n (X \cup C_0 A); Y] \underset{\approx}{\rightarrow} [\Sigma_0^n (X/A); Y]$$

Now suppose $\textcircled{Y} = \{Y_k, \epsilon_k\colon \Sigma_0 Y_k \to Y_{k+1}\}$ is a spectrum of pointed ANR's. Define

$$\{X, A; \textcircled{Y}\}^q = \varinjlim_n \{[\Sigma_0^n (X/A); Y_{n+q}]\}$$

where the direct limit is with respect to the maps

$$[\Sigma_0^n (X/A); Y_{n+q}] \xrightarrow{\Sigma_0} [\Sigma_0^{n+1} (X/A); \Sigma_0 Y_{n+q}] \xrightarrow{(\epsilon_{n+q})^\#} [\Sigma_0^{n+1} (X/A); Y_{n+q+1}]$$

Taking the direct limit of the exact sequences above we obtain an exact sequence

$$\cdots \xrightarrow{\delta^*} \{X, A; \textcircled{Y}\}^q \xrightarrow{j\#} \{X, x_0; \textcircled{Y}\}^q \xrightarrow{i\#} \{A, x_0; \textcircled{Y}\}^q \xrightarrow{\delta^*} \{X, A; \textcircled{Y}\}^{q+1} \to \cdots$$

where $i\colon (A, x_0) \subset (X, x_0)$, $j\colon (X, x_0) \subset (X, A)$.

We consider the category $\mathcal{C}^2_{\mathrm{loc\,comp}}$. As in Section 3 to every locally compact space X there is asociated a compact space X^+ with base point ∞ such that $X^+ - \{\infty\} = X$. Define a contravariant functor $H_{\textcircled{Y}}$ on $\mathcal{C}^2_{\mathrm{loc\,comp}}$ by $H^q_{\textcircled{Y}}(X, A) = \{X^+, A^+; \textcircled{Y}\}^q$ and define a natural transformation $\delta^*\colon H^q_{\textcircled{Y}}(A, \varnothing) \to H^{q+1}_{\textcircled{Y}}(X, A)$ to be the homomorphism δ^* in the exact sequence above for the pair of pointed spaces (X^+, A^+).

Theorem 5.1 *For every spectrum* Ⓨ *of ANR there is a compactly supported ES theory* $H_{Ⓨ}, \delta^*$ *on* $\mathcal{C}_{\text{loc comp}}$.

PROOF. From the exact sequence above for the pair (X^+, A^+) it is clear that $H_{Ⓨ}, \delta^*$ satisfies the exactness property for the pair (X, A). It also satisfies excision because if $(X; A, B)$ is a locally compact triad then $(X^+; A^+, B^+)$ is a compact pointed triad with $A^+ \cup B^+ = (A \cup B)^+$ and $A^+ \cap B^+ = (A \cap B)^+$. Therefore, there are isomorphisms

$$H^q_{Ⓨ}(A \cup B, B) = \{(A \cup B)^+, B^+; Ⓨ\}^q = \{A^+ \cup B^+, B^+; Ⓨ\}^q$$
$$\approx \{A^+, A^+ \cap B^+; Ⓨ\}^q = \{A^+, (A \cap B)^+; Ⓨ\}^q = H^q_{Ⓨ}(A, A \cap B).$$

If (X, A) is a locally compact pair, then (X^+, A^+) and $(\Sigma_0^n X^+, \Sigma_0^n A^+)$ are compact pairs for every n. Since Y_{n+q} is an ANR it follows that

$$\rho\colon \varinjlim \{[\Sigma_0^n B; Y_{n+q}] \mid B \text{ a closed nbhd of } A^+ \text{ in } X^+\} \approx [\Sigma_0^n A^+; Y_{n+q}]$$

Taking direct limits with respect to n we obtain an isomorphism

$$\rho\colon \varinjlim \{(B; Y)^q | B \text{ a closed nbhd of } A^+ \text{ in } X^+\} \approx \{A^+; Y\}^q$$

Since the closed nbhds B of A^+ in X^+ are exactly the sets of the form $B = N^+$ where N is a closed cobounded nbhd of A in X, it follows that

$$\rho\colon \varinjlim \{H^q_{Ⓨ}(N) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx H^q_{Ⓨ}(A).$$

Since $H_{Ⓨ}(\emptyset) = 0$ it follows from Lemma 3.5 that $H_{Ⓨ}$ is continuous and compactly supported.

There is a spectrum of ANR that defines an ES theory known as K-theory [9]. This ES theory is known to be periodic of order 2 (i.e. $K^{q+2}(X, A) \approx \approx K^q(X, A)$) and for a one-point space P,

$$K^q(P) \approx \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

Let $\check{H}_{ev}$ be the FS theory defined in terms of rational Čech cohomology with compact supports by

$$\check{H}^q_{ev}(X, A) = \begin{cases} \bigoplus_{i \text{ even}} \check{H}^i_c(X, A; \mathbb{Q}), & q \text{ even} \\ \bigoplus_{i \text{ odd}} \check{H}^i_c(X, A; \mathbb{Q}), & q \text{ odd.} \end{cases}$$

This is an ES Theory on $\mathcal{C}_{\text{loc comp}}$ (by analogues of Remarks 2.3 and 2.4 of [18] for ES theories). By an analogue of Remark 2.5 of [18] for ES theories there is also an ES theory $K \otimes \mathbb{Q}$ on $\mathcal{C}_{\text{loc comp}}$.

The Chern character [9] Ch: $K \otimes \mathbb{Q} \to \check{H}_{ev}$ is a homomorphism of ES theories on $\mathcal{C}_{\text{loc comp}}$.

Theorem 5.2. *For every locally compact space X,* Ch: $K^q(X) \otimes \mathbb{Q} \approx \check{H}^q_{ev}(X)$.

PROOF. Since $K \otimes \mathbb{Q}$ and $\check{H}_{ev}$ are both compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$ and Ch is an isomorphism for a one-point space [9], the result follows from Theorem 4.1. □

If $\textcircled{Y} = \{Y_k, \epsilon_k\}$ is a spectrum its *homotopy groups* $\pi_q(\textcircled{Y})$ are defined by $\pi_q(\textcircled{Y}) = \varinjlim_n \{\pi_{n+q}(Y_n)\}$. If $\textcircled{Y}$ is an ANR spectrum, it is clear that if P is a one-point space, then $H^q_{\textcircled{Y}}(P, \varnothing) \approx \pi_{-q}(\textcircled{Y})$ for all q.

If $\textcircled{Y}$, $\textcircled{Y}'$ are spectra, a *map g*: $\textcircled{Y} \to \textcircled{Y}'$ between them is defined to be a sequence of pointed continuous functions g_k: $Y_k \to Y'_k$ for each k such that the following square is homotopy commutative for every k

$$\begin{array}{ccc} \Sigma_0 Y_k & \xrightarrow{\epsilon_k} & Y_{k+1} \\ {\scriptstyle \Sigma_0 g_k}\downarrow & & \downarrow{\scriptstyle g_{k+1}} \\ \Sigma_0 Y'_k & \xrightarrow{\epsilon'_k} & Y'_{k+1} \end{array}$$

Such a map g induces a homomorphism $g_\#$: $\pi_q(\textcircled{Y}) \to \pi_q(\textcircled{Y}')$ and, in case $\textcircled{Y}$, $\textcircled{Y}'$ are ANR spectra, it induces a homomorphism g_*: $H_{\textcircled{Y}} \to H_{\textcircled{Y}'}$ of the corresponding ES theories.

Theorem 5.3. *Let g*: $\textcircled{Y} \to \textcircled{Y}'$ *be a map between ANR spectra which induces an isomorphism* $g_\#$: $\pi_q(\textcircled{Y}) \approx \pi_q(\textcircled{Y}')$ *for all q. Then for every locally compact pair* (X, A),

$$g_*: H_{\textcircled{Y}}(X, A) \approx H_{\textcircled{Y}'}(X, A).$$

PROOF. Since $g_\#$ is an isomorphism, it follows that if P is a one-point space, then g_*: $H_{\textcircled{Y}}(P, \varnothing) \approx H_{\textcircled{Y}'}(P, \varnothing)$. Since $H_{\textcircled{Y}}, H_{\textcircled{Y}'}$ are compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$, the result follows from Theorem 4.1. □

6. The functional spectrum and the complementary spectrum of a pair in $\mathbb{R}^n$

First we introduce the functional spectrum $\mathbb{F}(A, B)$ whose k^{th} term consists of the space of continuous functions $(A^+, B^+) \to (S^k, \infty)$ topologized with the compact-open topology. The homotopy groups $\mathbb{F}^*$ of $\mathbb{F}(A, B)$ are part of an ES theory $\mathbb{F}^*, \delta^*$ on $\mathcal{C}_{\text{loc comp}}$.

We regard S^k as $(\mathbb{R}^k)^+ = \mathbb{R}^k \cup \{\infty\}$ with ∞ as base point. Given a locally compact pair (A, B) let $F(A, B; S^k)$ be the space of all continuous functions $(A^+, B^+) \to (S^k, \infty)$ in the compact-open topology with the constant map $A^+ \to \infty$ as base point. Clearly $F(A, B; S^k)$ can also be regarded as the space of pointed continuous functions $A^+/B^+ \to S^k$ in the compact-open topology. It follows from the exponential theorem [4], the compactness of A^+ and the fact that S^k is an ANR that $F(A, B; S^k)$ is an ANR (the proof of Theorem 4 on p. 38 of [14] applies to our case as well).

It is known [6] that if (A, B) is a locally compact pair the restriction map $F(A, \emptyset, S^k) \to F(B, \emptyset; S^k)$ is a fibration with fiber $F(A, B; S^k)$. Therefore, there is an exact sequence of homotopy groups

$$(*) \cdots \longrightarrow \pi_{q+k}(F(A, B; S^k)) \longrightarrow \pi_{q+k}(F(A; \emptyset; S^k)) \longrightarrow \pi_{q+k}(F(B, \emptyset; S^k) \longrightarrow \\ \longrightarrow \pi_{q+k-1}(F(A, B; S^{k-1})) \longrightarrow \cdots$$

Define ϵ_k: $\sum_0 F(A, B; S^k) \to F(A, B; S^{k+1})$ by $(\epsilon_k(t \wedge f))(a) = t \wedge f(a)$ for $t \wedge f \in \sum_0 F(A, B; S^k) = S^1 \wedge F(A, B; S^k)$, $a \in A$ and $\sum_0 S^k = S^1 \wedge S^k \approx S^{k+1}$. Let $\mathbb{F}(A, B)$ be the spectrum $\{F(A, B; S^k), \epsilon_k\}$. The direct limit over k of the exact sequence (*) using the maps ϵ_k is an exact sequence

$$\cdots \longrightarrow \pi_q(\mathbb{F}(A, B)) \longrightarrow \pi_q(\mathbb{F}(A, \emptyset)) \longrightarrow \pi_q(\mathbb{F}(B, \emptyset)) \longrightarrow \pi_{q-1}(\mathbb{F}(A, B)) \longrightarrow \cdot$$

extending indefinitely on both ends.

If we define a contravariant functor $\mathbb{F}^*$ on $\mathcal{C}^2_{\text{loc comp}}$ by $\mathbb{F}^q(A, B) = \pi_{-q}(\mathbb{F}(A, B))$ and a natural transformation δ^*: $\mathbb{F}^q(B, \emptyset) \to \mathbb{F}^{q+1}(A, B)$ to correspond to ∂ in the exact sequence above, we see that $\mathbb{F}^*, \delta^*$ satisfy the exactness property of ES theories. We shall prove that $\mathbb{F}^*, \delta^*$ is an ES theory on $\mathcal{C}_{\text{loc comp}}$, but first we establish the following.

Lemma 6.1. *Let (K', K) be a compact pair, N a closed cobounded nbhd of A in X, and λ: $K \to F(N, \emptyset; S^k)$, μ: $K' \to F(A, \emptyset; S^k)$ continuous functions such that $\mu|K$ is the composite $K \xrightarrow{\lambda} F(N, \emptyset; S^k) \xrightarrow{\rho} F(A, \emptyset; S^k)$. Then there is a closed cobounded nbhd N' of A in N and a map λ': $K' \to F(N', \emptyset; S^k)$ such that $\lambda'|K$ is the composite $K \xrightarrow{\lambda} F(N', \emptyset; S^k) \xrightarrow{\rho'} F(N, \emptyset; S^k)$ and μ is the composite $K' \xrightarrow{\lambda'} F(N', \emptyset; S^k) \xrightarrow{\rho''} F(A, \emptyset; S^k)$.*

PROOF. By the exponential theorem the functions λ and μ correspond to continuous functions $\bar{\lambda}$: $(K \times N^+, K \times \infty) \to (S^k, \infty)$ and $\bar{\mu}$: $(K' \times A^+, K' \times \infty) \to (S^k, \infty)$ such that $\bar{\lambda}|K \times A^+ = \bar{\mu}|K \times A^+$. Therefore, there is a continuous function

$$\bar{f}: (K \times N^+ \cup K' \times A^+, K' \times \infty) \to (S^k, \infty)$$

such that $\bar{f}|K \times N^+ = \bar{\lambda}$ and $\bar{f}|K' \times A^+ = \bar{\mu}$. Since $K' \times N^+$ is compact, $K \times N^+ \cup K' \times A^+$ is closed in $K' \times N^+$ and S^k is an ANR there is a mbhd U of $K \times N^+ \cup K' \times A^+$ in $K' \times N^+$ and an extension $\bar{f}'$: $U \to S^k$ of $\bar{f}$. U contains a subset of the form $K' \times N'^+$ where N'^+ is a closed cobounded nbhd of A in N and $\bar{f}'|K' \times N'^+$ corresponds by the exponential theorem to a map λ': $K' \to F(N', \emptyset; S^k)$ having all the requisite properties. □

Theorem 6.2. *The pair* $\mathbb{F}^*, \delta^*$ *is a compactly supported ES theory on* $\mathcal{C}_{\text{loc comp}}$.

PROOF. We have already seen that exactness is satisfied. The excision property is also satisfied because if $(X; A, B)$ is a locally compact triad then there are homeomorphisms for each k

$$F(A \cup B, B; S^k) \approx F((A^+ \cup B^+)/B^+; S^k) \approx$$
$$\approx F(A^+/(A^+ \cap B^+); S^k) \approx F(A, A \cap B; S^k)$$

so the map of spectra $\mathbb{F}(A \cup B, B) \to \mathbb{F}(A, A \cap B)$ induces an isomorphism of homotopy groups. Therefore, $\mathbb{F}^q(A \cup B, B) = \pi_{-q}(\mathbb{F}(A \cup B, B)) \approx \approx \pi_{-q}(\mathbb{F}(A, A \cap B)) \approx \mathbb{F}^q(A, A \cap B)$ for all q. To complete the proof it suffices to show that for every locally compact space X the restriction of the functor $\mathbb{F}^*(\cdot, \emptyset)$ to $\text{cl}(X)$ satisfies Lemma 3.5. Clearly $\mathbb{F}^*(\emptyset, \emptyset) = 0$ so we need only verify that the following homomorphism is an isomorphism

$$\rho\colon \varinjlim \{\mathbb{F}^*(N, \emptyset) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \mathbb{F}^*(A, \emptyset)$$

To show ρ is an epimorphism let μ: $S^{k+q} \to F(A, \emptyset; S^k)$ represent an element $\{[u]\} \in \pi_q(\mathbb{F}(A, \emptyset)) = \mathbb{F}^{-q}(A, \emptyset)$ where $A \subset X$. By Lemma 6.1 with $K' = S^{k+q}$, $K = \infty$, $N = X$, λ: $\infty \to F(X, \emptyset; S^m)$ the unique pointed map, μ: $S^{k+q} \to F(A, \emptyset; S^k)$ we obtain a closed cobounded nbhd N' of A in X and a map λ': $S^{k+q} \to F(N', \emptyset; S^k)$ representing an element $\{[\lambda]\} \in \pi_q(\mathbb{F}(N', \emptyset))$ whose restriction to $\mathbb{F}(A, \emptyset)$ equals $\{[\mu]\}$. This implies ρ is an epimorphism.

To show ρ is a monomorphism let λ: $S^{k'+q} \to F(N, \emptyset; S^{k'})$ represent an element $\{[\lambda]\} \in \pi_q(\mathbb{F}(N, \emptyset))$ whose restriction to $\mathbb{F}(A, \emptyset)$ is 0 (where N is a closed cobounded nbhd of A in X). Then there is a map μ: $C_0 S^{k+q} \to \to F(A, \emptyset; S^k)$ for some $k \geqslant k'$ such that $\mu | S^{k+q} = \rho \circ (\Sigma_0^{k-k'} \lambda)$. By Lemma 6.1 with $K' = C_0 S^{k+q}, K = \Sigma_0^{k-k'} S^{k'+q} = S^{k+q}$ and the maps $\Sigma_0^{k-k'}\lambda$: $S^{k+q} \to F(N, \emptyset; S^k)$, μ: $C_0 S^{k+q} \to F(A, \emptyset; S^k)$ there is a closed cobounded nbhd N' of A in N and a map λ': $C_0 S^{k+q} \to F(N', \emptyset; S^k)$ such that $\lambda' | S^{k+q} = = \rho' \circ \Sigma_0^{k-k'} \lambda$. Therefore, $\{[\lambda]\}$ maps to 0 in $\mathbb{F}(N', \emptyset)$ proving ρ is a monomorphism. □

Next, for (A, B) a closed pair in $\mathbb{R}^n$ we define a spectrum $\mathbb{C}(A, B)$ which stably aproximates $(\mathbb{R}^n - B, \mathbb{R}^n - A)$. The homotopy groups $\mathbb{C}^*$ of this spectrum are part of an ET theory $\mathbb{C}^*, \delta^*$ on $\mathrm{cl}(\mathbb{R}^n)$.

Let T be an arbitrary but fixed triangulation of S^n with ∞ as vertex and let $T^{(k)}$ be the k^{th} barycentric subdivision of T for $k \geqslant 0$. For a closed subset $A \subset \mathbb{R}^n = S^n - \{\infty\}$ let $T_k(A)$ be the compact polyhedron equal to the union of all closed simplexes of $T^{(k)}$ disjoint from $A^+ = A \cup \{\infty\}$. Clearly $T_k(A) \subset \subset T_{k+1}(A)$ for all $k \geqslant 0$. We also have:

Lemma 6.3. (1) $A \subset B \subset \mathbb{R}^n \Rightarrow T_k(B) \subset T_k(A)$ *for all* $k \geqslant 0$.

(2) $A, B \subset \mathbb{R}^n \Rightarrow T_k(A \cup B) = T_k(A) \cap T_k(B)$ *for all* $k \geqslant 0$.

(3) $A, B \subset \mathbb{R}^n \Rightarrow$ *for every* $k \geqslant 0$ *there is* N_k *such that if* $k' \geqslant N_k$ *then* $T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$.

(4) $A \subset N \subset \mathbb{R}^n$ *where* N *is a closed cobounded nbhd of* A *in* $\mathbb{R}^n \Rightarrow$ *for every* $k \geqslant 0$ *there is a closed cobounded nbhd* N' *of* A *in* $\mathbb{R}^n$ *such that* $N' \subset N$ *and* $T_k(N') = T_k(N)$.

PROOF. (1) If $A \subset B$ and s is a closed simplex of $T^{(k)}$ disjoint from B, s is disjoint from A. Hence, $T_k(B) \subset T_k(A)$. □

(2) By (1) $T_k(A \cup B) \subset T_k(A)$ and $T_k(A \cup B) \subset T_k(B)$ so $T_k(A \cup B) \subset \subset T_k(A) \cap T_k(B)$. Given x let s be the unique closed simplex of $T^{(k)}$ containing x in its interior. Then $x \in T_k(A) \cap T_k(B)$ if and only if $s \subset T_k(A) \cap T_k(B)$, but this implies s is disjoint from A and from B so is disjoint from $A \cup B$. Therefore, $T_k(A) \cap T_k(B) \subset T_k(A \cup B)$. □

(3) $T_k(A \cap B)$ is the union of a finite number of closed simplexes, say $T_k(A \cap B) = s_1 \cup \cdots \cup s_r$. For each j, s_j is disjoint from $A \cap B$ so $s_j \cap A$ and $s_j \cap B$ are disjoint compact sets. Let $d_j > 0$ be the distance between them (in some metric on S^n) and let $d = \min\{d_1, \ldots, d_r\}$, choose N_k so that $k' \geqslant N_k$ implies that the diameter of every closed simplex of $T^{(k')}$ in less than d. If s' is any closed simplex of $T^{(k')}$ contained in s_j for some $1 \leqslant j \leqslant r$, then $\operatorname{diam} s' < < d \leqslant d_j$ so s' cannot meet both A and B. Therefore, either $s' \in T_{k'}(A)$ or $s' \in T_{k'}(B)$. Hence, for $k' \geqslant N_{k'} T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$. □

(4) For $A \subset \mathbb{R}^n, S^n - T_k(A)$ is an open nbhd of A^+. Let M be a closed nbhd of A^+ contained in $S^n - T_k(A)$. Then $M - \{\infty\}$ is a closed cobounded nbhd of A in $\mathbb{R}^n$. Since $T_k(A) \subset S^n - M = S^n - (M - \{\infty\})^+$, it follows that $T_k(A) \subset T_k(M - \{\infty\})$. If N is any closed cobounded nbhd of A in $\mathbb{R}^n$, then $N' = N \cap (M - \{\infty\})$ is a closed cobounded nbhd of A in $\mathbb{R}^n$ such that $N' \subset N$ and since $N' \subset M - \{\infty\}$, it follows from (1) that

$$T_k(A) \subset T_k(M - \{\infty\}) \subset T_k(N') \subset T_k(A)$$

so that $T_k(N') = T_k(A)$. □

Given a closed pair (A, B) in $\mathbb{R}^n$ we define a spectrum $\mathbb{C}(A, B)$ whose k^{th} term is $\Sigma_0^k (T_k(B)^+/T_k(A)^+)$ with continuous maps

$$\Sigma_0 (\Sigma_0^k (T_k(B)^+/T_k(A)^+)) \\ = \Sigma_0^{k+1} (T_k(B)^+/T_k(A)^+) \to \Sigma_0^{k+1} (T_{k+1}(B)^+/T_{k+1}(A)^+)$$

where the last map is the $(k+1)st$ reduced suspension of the map

$$(T_k(B)^+/T_k(A)^+) \subset (T_{k+1}(B)^+/T_{k+1}(A)^+)$$

induced by the inclusion $(T_k(B)^+, T_k(A)^+) \subset (T_{k+1}(B)^+, T_{k+1}(A)^+)$.

We define a contravariant functor $\mathbb{C}^*$ on $\text{cl}(\mathbb{R}^n)^2$ by

$$\mathbb{C}^q(A, B) = \pi_{-q}(\mathbb{C}(A, B)) = \varinjlim_k \{\pi_{-q-k}(\Sigma_0^k (T_k(B)^+/T_k(A)^+))\}.$$

To define the natural transformation $\delta^*\colon \mathbb{C}^q(B, \phi) \to \mathbb{C}^{q+1}(A, B)$ recall [15, Corollary 9.3.6 on p. 487] that the collapsing map

$$(\Sigma_0^k T_k(B)^+, \Sigma_0^k T_k(A)^+) \to (\Sigma_0^k (T_k(B)^+/T_k(A)^+), \infty)$$

induces isomorphisms

$$\pi_i(\Sigma_0^k T_k(B)^+, \Sigma_0^k T_k(A)^+) \approx \pi_i(\Sigma_0^k (T_k(B)^+/T_k(A)^+))$$

for $i \leqslant 2k-2$ (because the k^{th} reduced suspension of a space, or pair, is $(k-1)$-connected). Hence, the connecting homomorphisms

$$\partial\colon \pi_{-q+k}(\Sigma_0^k T_k(\varnothing)^+, \Sigma_0^k T_k(B)^+) \to \pi_{-q+k-1}(\Sigma_0^k T_k(B)^+, \Sigma_0^k T_k(A)^+)$$

for various k correspond to a homomorphism

$$\delta^*\colon \pi_{-q}(\mathbb{C}(B, \varnothing)) \to \pi_{-q-1}(\mathbb{C}(A, B))$$

This is a natural transformation of degree 1 from $\mathbb{C}^*(B, \varnothing)$ to $\mathbb{C}^*(A, B)$ for $(A, B) \in \text{cl}(\mathbb{R}^n)^2$ such that $\mathbb{C}^*, \delta^*$ satisfy the exactness property of ES theories.

Theorem 6.4. *$\mathbb{C}^*, \delta^*$ is a compactly supported ES theory on $\mathbb{R}^n$.*

PROOF. We have seen above that $\mathbb{C}^*, \delta^*$ satisfy exactness. To verify excision assume A, B are closed subsets of $\mathbb{R}^n$. By (2) of Lemma 6.3

$$\varinjlim_k \{\pi_{k-q}(\Sigma_0^k (T_k(B)^+/(T_k(A)^+ \cap T_k(B^+))))\} \\ \approx \varinjlim_k \{\pi_{k-q}(\Sigma_0^k (T_k(B)^+/T_k(A \cup B^+)))\} = \mathbb{C}^q(A \cup B, B).$$

From (3) of Lemma (6.3) for given k if $k' \geqslant N_k$ there are homomorphisms induced by inclusion

$$\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+)) \to \pi_{k-q}(\Sigma_0^k(T_k(A\cap B)^+/T_k(A)^+))$$
$$\to \pi_{k-q}(\Sigma_0^k((T_{k'}(A)^+ \cup T_{k'}(B)^+)/T_{k'}(A)^+))$$

implying that

$$\varinjlim_k \{\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B^+)/T_k(A)^+))\}$$
$$\approx \varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(A\cap B)^+/T_k(A)^+))\} = \mathbb{C}^q(A, A\cap B).$$

Since

$$T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+) = (T_k(A)^+ \cup T_k(B^+))/T_k(A)^+,$$

it follows that

$$\varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+)))\}$$
$$\approx \varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+))\}$$

so that $\mathbb{C}^q(A\cup B, B) \approx \mathbb{C}^q(A, A\cap B)$ and excision is satisfied.

To complete the proof we show that the functor $\mathbb{C}^*(\cdot, \varnothing)$ satisfies Lemma 3.5. Clearly $\mathbb{C}^*(\varnothing, \varnothing) = 0$. Hence, we only need verify that

$$\rho\colon\ \varinjlim \{\mathbb{C}^*(N, \varnothing) | N \text{ a closed cobounded nbhd of } A \text{ in } \mathbb{R}^n\} \approx \mathbb{C}^*(A, \varnothing)$$

To show ρ is an epimorphism let

$$\mu\colon\ S^{k-q} \to \Sigma_0^k(T_k(\varnothing)^+/T_k(A)^+)$$

represent an element $\omega \in \pi_{-q}(\mathbb{C}(A, \varnothing)) = \mathbb{C}^q(A, \varnothing)$ for some $A \subset \mathbb{R}^n$. By (4) of Lemma 6.3 with $N = \mathbb{R}^n$ there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $T_k(N') = T_k(A)$. Then μ is also a map from S^{k-q} into $\Sigma_0^k(T_k(\varnothing)^+/T_k(N')^+)$ so determines an element $\omega' \in \pi_{-q}(\mathbb{C}(N', \varnothing))$ whose restriction to $\mathbb{C}(A, \varnothing)$ equals ω. Thus, ρ is an epimorphism.

To show ρ is a monomorphism let $\lambda\colon S^{k-q} \to \Sigma_0^k(T_k(\varnothing)^+/T_k(N)^+)$ represent an element $\omega \in \pi_{-q}(\mathbb{C}(N, \varnothing))$ whose restriction to $\mathbb{C}(A, \varnothing) = 0$ (where N is a closed cobounded nbhd of A in $\mathbb{R}^n$). Then there is a map $\mu\colon C_0S^{k'-q} \to \to \Sigma_0^{k'}(T_{k'}(\varnothing)^+/T_{k'}(A)^+)$ for some $k' \geqslant k$ such that $\mu|S^{k'-q} = \rho' \circ \Sigma_0^{k'-k}(\lambda)$ where $\rho'\colon \Sigma_0^{k'}(T_k(\varnothing)^+/T_k(N)^+) \to \Sigma_0^{k'}(T_{k'}(\varnothing)^+/T_{k'}(A)^+)$. By (4) of Lemma 6.3 there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $N' \subset N$ and $T_{k'}(N') = T_{k'}(A)$. Then μ is also a map from $C_0S^{k'-q}$ to $\Sigma_0^{k'}(T_{k'}(\varnothing)^+/ /T_{k'}(N')^+)$ implying that ω restricts to 0 in $\mathbb{C}(N', \varnothing)$. Thus, ρ is a monomorphism. □

7. Duality in $\mathbb{R}^n$

In the last section $\mathbb{C}^*$, δ^* was defined as a compactly supported ES theory on $\mathbb{R}^n$. By Theorem 6.2 the restriction of $\mathbb{F}^*$, δ^* to $\mathbb{R}^n$ is also a compactly supported ES theory on $\mathbb{R}^n$. This implies that $\sigma^n\mathbb{F}^*$, δ^* is also an ES theory on $\mathbb{R}^n$ (where σ^n is defined as in Remark 2.3 of [18] so that

$$(\sigma^n\mathbb{F}^*)^q(A,B) = \mathbb{F}^{q+n}(A,B)).$$

Since $\mathbb{F}^*(A,B)$ is the ES theory defined by the spectrum $\mathbb{F}(A,B)$, $\sigma^n\mathbb{F}^*(A,B)$ is the ES theory defined by the spectrum $\Sigma_0^n\mathbb{F}^*(A,B)$ whose m^{th} term is the $(n+m)^{\text{th}}$ term of $\mathbb{F}(A,B)$. We shall define a map of spectra

$$\mu\colon\ \mathbb{C}(A,B) \to \Sigma_0^n\mathbb{F}(A,B)$$

for $(A,B) \in \text{cl}(\mathbb{R}^n)^2$. This will induce a homomorphism

$$\begin{aligned}\mathbb{C}^q(A,B) &= \pi_{-q}(\mathbb{C}(A,B)) \to \pi_{-q}(\Sigma_0^n\mathbb{F}(A,B)) \approx \pi_{-n-q}(\mathbb{F}(A,B))\\ &= \mathbb{F}^{q+n}(A,B) = (\sigma^n\mathbb{F}^*)^q(A,B)\end{aligned}$$

and μ_* will be a homomorphism from $\mathbb{C}^*, \delta^*$ to $\sigma^n\mathbb{F}^*, \delta^*$.

Given a closed pair (A,B) in $\mathbb{R}^n$ for every $k \geqslant 0$ there is a continuous map

$$\lambda_k\colon\ \mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B) \to (\mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^{n+k}, \mathbb{R}^{n+k} - \{0\})$$

defined by $\lambda_k(x,y,z) = (x, y-z)$ for $x \in \mathbb{R}^k$, $y \in T_k(B)$, $z \in A$. Since $T_k(B)$ is a compact subset of $\mathbb{R}^n$, λ_k is a proper map so extends to a continuous map

$$\lambda_k^+\colon\ [\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B)]^+ \to (\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - \{0\})^+$$

There is a canonical homeomorphism

$$[\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B)]^+ \approx S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)$$

Therefore, λ_k^+ can also be regarded as a map

$$\lambda_k^+\colon\ S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+) \to (S^{n+k}, S^{n+k} - \{0\}).$$

This map λ_k^+ has the following two properties.

(1) If $(A', B') \subset (A,B)$, then $(T_k(B), T_k(A)) \subset (T_k(B'), T_k(A'))$ and $\lambda_k'^+ | [S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)] = \lambda_k^+ | [S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)]$.

(2) For any $k \geqslant 0$, $(T_k(B), T_k(A)) \subset (T_{k+1}(B), T_{k+1}(A))$ and $\lambda_{k+1}^+ | [S^{k+1} \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)] = \Sigma_0(\lambda_k^+)$.

Since ∞ is a deformation retract of $S^{n+k} - \{0\}$ if i: $(S_{n+k}, \infty) \subset \subset (S^{n+k}, S^{n+k} - \{0\})$ is the inclusion map, there is a map

$$\bar{\lambda}_k^+ : S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+) \to (S^{n+k}, \infty)$$

such that $\lambda_k^+ \simeq i \circ \bar{\lambda}_k^+$ and this condition determines $\bar{\lambda}_k^+$ up to homotopy. We may also consider $\bar{\lambda}_k^+$ as a pointed map $\bar{\lambda}_k^+$: $S^k \wedge (T_k(B)^+ / T_k(A)^+) \wedge (A^+ / /B^+) \to S^{n+k}$. By the exponential theorem [4] $\bar{\lambda}_k^+$ corresponds to a continuous function

$$\mu_k: \ \Sigma_0^k (T_k(B)^+ / T_k(A)^+) \to F(A, B; S^{n+k})$$

By (2) above the following square commutes up to homotopy

$$\begin{array}{ccc} \Sigma_0 \Sigma_0^k (T_k(B)^+ / T_k(A)^+) & \longrightarrow & \Sigma_0^{k+1} (T_{k+1}(B)^+, T_{k+1}(A)^+) \\ \Sigma_0 \mu_k \big\downarrow & & \big\downarrow \mu_{k+1} \\ \Sigma_0 F(A, B; S^{n+k}) & \xrightarrow{\epsilon_{n+k}} & F(A, B; S^{n+k+1}) \end{array}$$

Therefore, $\{\mu_k\}$ is a map of spectra

$$\mu: \ \mathbb{C}(A, B) \to \Sigma_0^n \mathbb{F}(A, B)$$

and it follows from (1) above that the map

$$\mu_*: \ \mathbb{C}^q(A, B) \to (\sigma^n \mathbb{F}^*)^q(A, B)$$

is a natural transformation from $\mathbb{C}^*$ to $\sigma^n \mathbb{F}^*$. It is easy to verify that μ_* commutes with δ^* for the two ES theories so it is a homomorphism of ES theories.

Theorem 7.1. *The homomorphism* μ_*: $\mathbb{C}^*, \delta^* \to \sigma^n \mathbb{F}^*, \delta^*$ *is an isomorphism of ES theories on* $\mathbb{R}^n$.

PROOF. Using the «five-lemma» it suffices to prove that μ_* is an isomorphism of the corresponding cohomology theories. Since each is compactly supported and $\mathbb{R}^n$ is finite dimensional, it suffices to prove μ_* is an isomorphism for $(x, \emptyset)$ for every $x \in \mathbb{R}^n$. But $F(x, \emptyset; S^k) \approx S^k$ so $\Sigma_0^n \mathbb{F}(x, \emptyset)$ is the spectrum $S^n, S^{n+1}, \ldots$. Also the pair $(T_k(\emptyset), T_k(x))$ for k large is an n-cell together with the complement of an open n-cell inside it and the map μ_k: $\Sigma_0^k (T_k(\emptyset)^+ / T_k(x)^+) \to S^{n+k}$ is of degree 1. Therefore,

$$\mu_*: \ \pi_{-q}(\mathbb{C}(x, \emptyset)) \approx \pi_{-q}(\Sigma_0^n \mathbb{F}(x, \emptyset)) \quad \text{for all} \quad q. \qquad \square$$

Corollary 7.2. *If (A,B), (C,D), are closed pairs in $\mathbb{R}^n$ there is a duality isomorphism*

$$\{A^+/B^+;\mathbb{C}(C,D)\}^q \approx \{C^+/D^+;\mathbb{C}(A,B)\}^q \quad \textit{for all} \quad q.$$

PROOF. If $(C,D)\in \mathrm{cl}(\mathbb{R}^n)^2$ then by Theorem 7.1 there is a map μ: $\mathbb{C}(C,D)\to$ $\to \Sigma_0^n \mathbb{F}(C,D)$ such that

$$\mu_*:\ \pi_q(\mathbb{C}(C,D)) \approx \pi_q(\Sigma_0^k \mathbb{F}(C,D)) \quad \text{for all} \quad q.$$

Since both $\mathbb{C}(C,D)$ and $\Sigma_0^n \mathbb{F}(C,D)$ are ANR spectra, it follows from Theorem 5.3 that for every locally compact pair (A,B)

$$\mu_*:\ H^q_{\mathbb{C}(C,D)}(A,B) \approx H^q_{\Sigma_0^n \mathbb{F}(C,D)}(A,B)$$

or equivalently,

$$\begin{aligned}\{A^+,B^+;\mathbb{C}(C,D)\}^q &\overset{\mu_*}{\approx} \{A^+,B^+;\Sigma_0^n \mathbb{F}(C,D)\}^q \\ &= \varinjlim_k \{[\Sigma_0^k(A^+/B^+);F(C,D;S^{n+k+q})]\}\end{aligned}$$

By the exponential theorem the last limit is isomorphic to

$$\varinjlim_k \{[\Sigma_0^k(A^+/B^+)\wedge(C^+/D^+);S^{n+k+q}]\}$$

There are canonical homomorphisms

$$\begin{aligned}\Sigma_0^k(A^+/B^+)\wedge(C^+/D^+) &\approx S^k\wedge(A^+/B^+)\wedge(C^+/D^+) \\ &\approx S^k\wedge(C^+/D^+)\wedge(A^+/B^+)\approx \Sigma_0^k(C^+/D^+)\wedge(A^+/B^+)\end{aligned}$$

so that

$$\begin{aligned}\{A^+,B^+,\mathbb{C}(C,D)\}^q &\approx \varinjlim_k \{[\Sigma_0^k(C^+/D^+)\wedge(A^+/B^+);S^{n+k+q}]\} \\ &\approx \varinjlim_k \{[\Sigma_0^k(C^+/D^+);F(A,B;S^{n+k+q})]\} \\ &\approx \{C^+,D^+;\Sigma_0^n \mathbb{F}(A,B)\}^q\end{aligned}$$

In case (A,B) is also a closed pair in $\mathbb{R}^n$ there is also an isomorphism

$$\{C^+,D^+;\mathbb{C}(A,B)\}^q \overset{\mu_*}{\approx} \{C^+,D^+;\Sigma_0^n \mathbb{F}(A,B)\}^q$$

Combining these isomorphisms gives the result. □

Our final result is due to Lima [13]. In the proof we essentially show that for a compact $A\subset S^n$, the spectrum of $S^n - A$ and $\Sigma_0^{n-1}\mathbb{F}(A,\varnothing)$ are equivalent (compare with [8, Theorem 4.5]).

Corollary 7.3. *Let A, B be nonempty proper compact subsets of S^n. Then*

$$\{A; S^n - B\}^q \approx \{B; S^n - A\}^q \quad \textit{for all} \quad q.$$

PROOF. If φ is a homeomorphism of S^n and the result is valid for A, B it is also valid for $\varphi(A), B$ and $A, \varphi(B)$ so, without loss of generality, we can assume $\infty \in A \cap B$. Then $A = (A')^+$, $B = (B')^+$ for closed subsets $A', B' \subset \mathbb{R}^n$. By Corollary 7.2 there is a duality isomorphism

$$\{A'^+/\varnothing; \mathbb{C}(B', \varnothing)\}^q \approx \{B'^+/\varnothing^+; \mathbb{C}(A', \varnothing)\}^q$$

For k large $T_k(\varnothing)$ is a closed n-cell containing $T_k(B')$ and, therefore, $T_k(\varnothing)^+/T_k(B')^+$ has the same homotopy type as $\sum_0 T_k(B')$, and we have

$$\begin{aligned}\{A'^+/\varnothing^+; \mathbb{C}(B', \varnothing)\}^q &= \varinjlim_k \{[\Sigma_0^k(A); \Sigma_0^{k+q}(T_{k+q}(\varnothing)^+/T_{k+q}(B')^+)]\} \\ &\approx \varinjlim_k \{[\Sigma_0^k(A); \Sigma_0^{k+q+1} T_{k+q}(B')]\}.\end{aligned}$$

(where in the above $T_k(B')$ is given an arbitrary base point for k large enough which is also the base point for $T_{k'}(B')$ for $k' > k$ and for $S^n - B$). Since $\{T_k(B')\}_k$ is an increasing sequence of subspaces of $S^n - B$ such that $\bigcup_k \operatorname{int} T_k(B') = S^n - B$, it follows that for the compact space A,

$$[A; \Sigma_0^{q+1}(S^n - B)] \approx \varinjlim_k \{[A; \Sigma_0^{q+1} T_k(B')]\},$$

and this implies that

$$\{A; S^n - B\}^{q+1} \approx \varinjlim_k \{[\Sigma_0^k(A); \Sigma_0^{k+q+1} T_k(B')]\} \approx \{A'^+/\varnothing^+, \mathbb{C}(B', \varnothing)\}^q.$$

Similarly $\{B; S^n - A\}^{q+1} \approx \{B'^+/\varnothing^+; \mathbb{C}(A', \varnothing)\}^q$. Combining these isomorphisms with the duality isomorphism gives the result. □

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