

The Dirichlet Problem for the Degenerate Monge-Ampère Equation

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Dedicated to Alberto Calderón on his 65th birthday

Let Ω be a bounded convex domain in R^n with smooth, strictly convex boundary $\partial\Omega$, i.e. the principal curvatures of $\partial\Omega$ are all positive. We study the problem of finding a convex function u in Ω such that

$$(1) \quad \det(u_{ij}) = 0 \quad \text{in } \Omega$$
$$(2) \quad u = \phi \quad \text{given on } \partial\Omega.$$

Here $u_i = \partial u / \partial x_i$, $u_{ij} = \partial^2 u / \partial x_i \partial x_j$ etc. The existence of a smooth solution in $\bar{\Omega}$ satisfying (2) of the corresponding elliptic problem

$$(1)' \quad \det(u_{ij}) = \psi > 0 \quad \text{in } \bar{\Omega}$$

has been shown recently by N. V. Krylov [5] and the authors [3] in case ψ and ϕ are sufficiently smooth. It is of interest to treat the degenerate problem (1), (2). The corresponding question for degenerate complex Monge-Ampère equations is also of interest: find a plurisubharmonic function w in a bounded strictly pseudoconvex domain Ω in \mathbb{C}^n satisfying

$$(3) \quad \det(w_{z_j \bar{z}_k}) = 0 \quad \text{in } \Omega$$

and (2). In fact in [4], with J. J. Kohn, we treated the equation

$$(3)' \quad \det(w_{z_j \bar{z}_k}) = \psi \geq 0 \quad \text{in } \Omega$$

and showed that there is a plurisubharmonic solution w belonging to $C^{1,1}(\bar{\Omega})$ provided $\psi \not\equiv 0$, ψ satisfies some other conditions, and ψ and ϕ are sufficiently

smooth. See [4] for further references on (3)', (2). E. Bedford and J. E. For-
naess [1] presented an example with ψ satisfying all our conditions and for
which the (unique) solution of (1)' is not in $C^2(\Omega)$.

Several authors have studied the Dirichlet problem (1), (2): J. Rauch and
B. A. Taylor [6], Bedford and Taylor [2] as well as [1], and, most recently,
N. S. Trudinger and J. I. F. Urbas [7]. The unique solution of (1), (2) is
given by

$$(4) \quad u(x) = \max \{ v(x) \mid v \in C(\bar{\Omega}), \quad v \text{ convex and } v \leq \phi \text{ on } \partial\Omega \},$$

and the papers cited study the regularity of u . The best regularity result is that
in [7] where it is shown that if $\partial\Omega \in C^{3,1}$ and $\phi \in C^{1,1}(\partial\Omega)$ then the function
 u belongs to $C^{0,1}(\bar{\Omega})$ and to $C^{1,1}$ in every compact subset of Ω . In this paper
we prove an extension up to the boundary of this regularity in case ϕ is suffi-
ciently smooth.

Theorem. *Assume $\partial\Omega$ is in $C^{3,1}$, and $\phi \in C^{3,1}(\partial\Omega)$; then the function u defined
by (4) is of class $C^{1,1}(\bar{\Omega})$.*

We do not know if a corresponding estimate holds for the solution w of (3),
(2) —which is characterized by

$$(4)' \quad w(z) = \max \{ v(z) \mid v \in C(\bar{\Omega}), \quad v \text{ plurisubharmonic in } \Omega \text{ and } v \leq \phi \text{ on } \partial\Omega \}.$$

The following examples show that the theorem is optimal; here Ω is the unit
disc centred at the origin in the (x, y) plane.

Ex. 1. The function

$$u = (1 + y)^{2-\epsilon}, \quad 0 < \epsilon \text{ small}$$

satisfies (1) and (2) with

$$\phi \in C^{3,1-2\epsilon}(\partial\Omega).$$

The function u is of class $C^{1,1-\epsilon}$ but not $C^{1,1}$ in $\bar{\Omega}$.

The same u shows that if we flatten $\partial\Omega$ at $(0, -1)$ to make the curvature of
 $\partial\Omega$ vanish here to high order, ϕ may be very smooth while u is not in $C^{1,\delta}(\bar{\Omega})$
for $\delta > 1 - \epsilon$.

Ex. 2. (This is due to John Urbas). Let

$$\phi(y) \equiv \left(y^2 - \frac{1}{4} \right)^2$$

this is convex in $y > \frac{1}{2}$. It is easily verified that the function u defined by (4) satisfies the conditions in Ω : $u \equiv 0$ in the inscribed rectangle in Ω bounded on top and bottom by $|y| = 1/2$, and $u \equiv \phi(y)$ above the rectangle.

Here $\phi \in C^\infty(\partial\Omega)$ but u is not in C^2 near $(0, \frac{1}{2})$.

In [5], [3], a priori estimates are established for the C^2 norms of solutions of (1)', (2). It seems natural to try to establish such estimates (independent of ϵ for $0 < \epsilon \leq 1$) of convex solutions of, say,

$$(1)_\epsilon \quad \det(u_{,ij}) = \epsilon \quad \text{in } \Omega$$

satisfying (2). However we have not done this. Instead we work directly with the characterization (4), and this paper is quite independent of [3].

Section 1. Beginning of proof

To prove the theorem we will establish the following estimate: there is constant A depending on Ω and the $C^{3,1}$ norm of ϕ such that

$$(5) \quad |u(y) - u(x) - (y - x) \cdot \nabla u(x)| \leq A|x - y|^2$$

holds for every pair of points x, y in Ω . It is easy to see that (5) then yields the conclusion of the Theorem. We have, namely, to show that for some constant B depending only on Ω and on A ,

$$(6) \quad |\nabla u(x) - \nabla u(y)| \leq B|x - y| \quad \text{for all } x, y \in \Omega.$$

Observe first that there are fixed positive numbers ϵ, δ depending only on Ω such that for every $y \in \Omega$, Ω contains a truncated cone

$$K(y) = \{z \neq y \mid |y - z| < \epsilon, \text{ and the angle } (z - y) \text{ makes} \\ \text{with some unit vector } \xi(y) \text{ is less than } \delta\}.$$

It follows that there is a positive number α depending only on δ (and so only on Ω) such that for any vector $\eta \neq 0$ there is a point z in $K(y)$ such that

$$|(z - y) \cdot \eta| \geq \alpha|z - y| \cdot |\eta|.$$

Clearly there is such a z in $K(y)$ with $|z - y| =$ any positive number $t < \epsilon$.

To establish (6), since by [6] and [7] we have a bound $|\nabla u| \leq$ constant, we may suppose $|x - y| < \epsilon$ (the ϵ determined above). After subtraction of an affine function we may assume

$$u(x) = |\nabla u(x)| = 0,$$

and we have then to show

$$(6)' \quad |\nabla u(y)| \leq B|x - y| \quad \text{if } |x - y| < \epsilon.$$

By (5), we have for every $z \in \Omega$,

$$|u(z)| \leq A|z - x|^2$$

and

$$|u(z) - u(y) - (z - y) \cdot \nabla u(y)| \leq A|z - y|^2,$$

and of course

$$|u(y)| \leq A|y - x|^2.$$

Hence

$$(7) \quad |(z - y) \cdot \nabla u(y)| \leq A(|z - x|^2 + |z - y|^2 + |y - x|^2).$$

Using what we have asserted above, we may choose z in Ω with $|z - y| = |x - y| < \epsilon$ and such that

$$|(z - y) \cdot \nabla u(y)| \geq \alpha|z - y| \cdot |\nabla u(y)|.$$

(for we may suppose $\nabla u(y) \neq 0$). Combining this with (7) we obtain

$$\alpha|x - y| \cdot |\nabla u(y)| \leq 6A|x - y|^2,$$

i.e., (6)', with $B = 6A/\alpha$.

We will derive (5) from the following local form, the heart of our proof:

Proposition. *There is a constant C depending only on Ω and the $C^{3,1}$ norm of ϕ , such that for every point x^0 in Ω , $\exists \epsilon(x^0) > 0$ so that for every x in Ω with $|x - x^0| \leq \epsilon(x^0)$, we have*

$$(8) \quad |u(x) - u(x^0) - (x - x^0) \cdot \nabla u(x^0)| \leq C|x - x^0|^2.$$

We claim that the proposition yields (5) with any constant A greater than C (and hence in the limit for $A = C$). Fix a constant $A > C$. If (5) does not hold, there are points x, y in Ω such that

$$|u(y) - u(x) - (y - x) \cdot \nabla u(x)| > A|x - y|^2.$$

By (8) there is a closest point z in $\bar{\Omega}$ to x such that

$$|u(z) - u(x) - (z - x) \cdot \nabla u(x)| = A|z - x|^2.$$

On the closed segment L joining x to z consider the function

$$f(y) = A|y - x|^2 - [u(y) - u(x) - (y - x) \cdot \nabla u(x)].$$

By (8), this is positive for y near x , $y \neq x$, but it is zero at x and z . So f achieves a positive maximum at some point w in the interior of L . At w the second difference quotient

$$(9) \quad \Delta_h^2 f = f(w + h(z - x)) + f(w - h(z - x)) - 2f(w) \leq 0 \quad \text{for } |h| \text{ small.}$$

But

$$\Delta_h^2 f = 2Ah^2|z - x|^2 - [u(w + h(z - x)) + u(w - h(z - x)) - 2u(w)]$$

and by (8) with x^0 replaced by w , the expression in the square bracket satisfies

$$|[\]| \leq 2Ch^2|z - x|^2 \quad \text{for } |h| \text{ small.}$$

Thus

$$\Delta_h^2 f \geq 2(A - C)h^2|z - x|^2 \quad \text{for } |h| \text{ small,}$$

contradicting (9). \square

To complete the proof of the theorem we have to prove the Proposition. First some preliminary simple lemmas.

Lemma 1. *Let S be a straight segment with endpoints on $\partial\Omega$. Suppose one of these is the origin and that S makes an angle $\pi/2 - \alpha$ with the interior normal to $\partial\Omega$ there, $0 < \alpha$ small. Then length of $S = O(\alpha)$.*

PROOF. We may assume the positive x_n axis is interior normal to $\partial\Omega$ at the origin and that S is of the form

$$x_n = \beta x_1, \quad 0 \leq x_1 \leq t, \quad \beta = \tan \alpha, \quad x_2 = \cdots = x_{n-1} = 0.$$

At the other end point of S we have

$$x_n = c_0 t^2 + O(|t|^3) = \beta t,$$

$c \geq c_0$ a fixed positive constant. It follows that

$$t = O(\beta) = O(\alpha)$$

and also $x_n = O(\alpha^2)$. \square

In the following lemma, u is the function defined by (4).

Lemma 2. *Let x^0 be any point in Ω . Subtracting from u a linear plane of support there we may suppose*

$$u \geq 0, \quad u(x^0) = 0.$$

Then x^0 is a convex combination of $(n + 1)$ points x^1, \dots, x^{n+1} in $\partial\Omega$ with $u(x^i) = 0$ for $i = 1, \dots, n + 1$.

PROOF. By Caratheodory's theorem it suffices to show that x^0 is in the convex hull of

$$S = \{x \in \partial\Omega \mid u(x) = 0\}.$$

If not, there is a hyperplane separating them; i.e. every point x in S satisfies (after rotation and traslation of coordinates)

$$x_n < -\epsilon < \epsilon < x_n^0.$$

Thus at points on $\partial\Omega$ where $x_n \geq 0$ we have $u \geq a > 0$ for some positive constant a . But then the function

$$v = \delta x_n \quad \text{for } 0 < \delta \text{ small}$$

satisfies

$$v \leq u \quad \text{on } \partial\Omega.$$

Consequently, by (4), $u(x^0) \geq v(x^0) > \delta\epsilon$ —contradicting the fact that $u(x^0) = 0$. \square

Section 2. Proof of the Proposition

We will say that a constant is under control if it depends only on Ω and the $C^{3,1}$ norm of ϕ . Fix $x^0 \in \Omega$. After subtraction of an affine function we may suppose $u \geq 0$, $u(x^0) = 0$. According to Lemma 2, x^0 lies in a n -dimensional simplex S with vertices on $\partial\Omega$, and on which $u \equiv 0$. It may be that x^0 lies in a lower, say k , dimensional simplex with this property. Using induction on k we will prove (8).

(i) Postponing the case $k = 1$, suppose we have proved the result for all x^0 lying in any $(k - 1)$ -dimensional simplex with the stated properties, and with constant $C = C_{k-1}$ under control. We wish to prove it for x^0 in such a k -dimensional simplex S with some constant C_k also under control. Let x^1 be the closest point to x^0 on any $(k - 1)$ -dimensional face of S . Then $y^0 = 2x^0 -$

– x^1 lies in S . By induction, in the ball B (around x^1 with radius $\epsilon(x^1)$), we have

$$|u(x)| \leq C_{k-1}|x - x^1|^2.$$

It follows by convexity of u (recall $u(y^0) = 0$) that in the ball around x^0 with radius $\epsilon(x^0) = \frac{1}{2}\epsilon(x^1)$

$$|u(x)| \leq 2C_{k-1}|x - x^0|^2.$$

Thus we have established (8) with a constant $C_k = 2C_{k-1}$.

Consequently (8) holds for any x^0 in Ω with

$$C = 2^{n-1}C_1$$

and the proof is finished —once we have treated the case $k = 1$.

(ii) Turning to that case, we suppose x^0 lies in a segment S with end points x^1, x^2 on $\partial\Omega$, on which $u = 0$, and that $u \geq 0$ in Ω . Of the two end points, suppose that x^2 is the closer to x^0 . We may suppose $x^2 = 0$ and that the positive x_n -axis is interior normal to $\partial\Omega$ there. In addition we may suppose $x^0 = (x_1^0, 0, \dots, 0, x_n^0)$ with $x_1^0 \geq 0$, $x_n^0 = x_1^0 \tan \alpha$, $0 < \alpha \leq \frac{\pi}{2}$, i.e., S makes angle $\frac{\pi}{2} - \alpha$ with the interior normal to $\partial\Omega$ at 0.

By Lemma 1,

$$(10) \quad |x^1| = O(\alpha).$$

We will distinguish two cases

$$\begin{aligned} \alpha &\leq \alpha_0, \quad \text{a positive small constant to be chosen,} \\ \alpha &> \alpha_0. \end{aligned}$$

Consider first the case $\alpha \leq \alpha_0$ small. We have $x^1 = (x_1^1, 0, \dots, 0, x_1^1 \tan \alpha)$. The orthogonal projection on the plane $x_n = 0$ of the segment joining the origin to x^1 is the segment L on the x_1 axis from the origin to $(x_1^1, 0, \dots, 0)$. If we think of the boundary values ϕ near 0 as a nonnegative function of (x_1, \dots, x_{n-1}) , then at the end points of the segment L we have $\phi = 0$ and hence $\phi_1 = 0$, and $\phi_{11} \geq 0$ there. It follows that

$$(11) \quad \int_L \phi_{11} dx_1 = 0.$$

Hence ϕ_{11} has an interior minimum in L , where necessarily $\phi_{111} = 0$. Consequently on L , $|\phi_{111}| \leq \text{constant} \cdot x_1^{\frac{1}{2}}$; *here, at last, is where we use the fact that the $C^{3,1}$ norm of ϕ is finite*. By (10), we have $|\phi_{111}| \leq \text{constant} \cdot \alpha$, and since ϕ_{11} necessarily vanishes somewhere on L , see (11), it follows that

$$(12) \quad |\phi_{11}(0)| \leq A\alpha^2$$

with A under control.

For other second derivatives of ϕ at the origin we have

$$(13) \quad |\phi_{ij}(0)| \leq A$$

while from (12) it follows that

$$(14) \quad |\phi_{1\beta}(0)| \leq A\alpha, \quad 1 < \beta \leq n-1,$$

since $\phi \geq 0$ and, hence, $\sum \phi_{ij}(0)x_i x_j \geq 0$.

Consider now a ball B in Ω with centre x^0 and radius $\epsilon \leq \alpha^3$. Let Γ be the cone with vertex x^1 generated by B and set $K = \partial\Omega \cap \Gamma$; recall that $|x^1 - x^0| \geq |x^0|$. For $\alpha \leq \alpha_0$ sufficiently small, depending only on Ω , it is not difficult to verify that for ϵ sufficiently small (depending possibly on the point x^0), the orthogonal projection of K onto the plane $x_n = 0$ is contained in an ellipsoid with axes

$$C\frac{\epsilon}{\alpha} \text{ in the } x_1\text{-direction, } C\epsilon \text{ in the other directions,}$$

for a fixed constant C depending only on Ω . Let us now fix such α_0 .

For any point x in the ϵ -ball B let \bar{x} represent the point on $\partial\Omega$ where the ray from x^1 to x strikes $\partial\Omega$. Since $u(x^1) = 0$ it follows from convexity that

$$(15) \quad u(x) \leq \phi(\bar{x}).$$

Set $\delta = |x - x^0|$. As described above, $(\bar{x}_1, \dots, \bar{x}_{n-1})$ lies in the ellipsoid

$$(16) \quad \alpha^2 x_1^2 + x_2^2 + \dots + x_{n-1}^2 \leq C^2 \delta^2.$$

Consequently

$$\phi(\bar{x}) = \sum_{i,j=1}^{n-1} \phi_{ij}(0) \bar{x}_i \bar{x}_j + O(|\bar{x}|^3).$$

Using (12), (13), (14), and (16) it follows that

$$\phi(\bar{x}) \leq CA\delta^2 + C\frac{\delta^3}{\alpha^3}$$

with C a (different) constant under control. By (15), and the relations $|x - x^0| = \delta \leq \epsilon \leq \alpha^3$, we find

$$u(x) \leq C_1 |x - x^0|^2.$$

We have proved (8) for $\alpha \leq \alpha_0$.

The other case to consider is $\alpha \geq \alpha_0$; this case is simple. There is a positive ϵ such that if $|x - x^0| < \epsilon$, then the point \bar{x} on $\partial\Omega$ where the ray from x^1 to

x strikes $\partial\Omega$ satisfies

$$|\bar{x}| \leq C|x - x^0|$$

with C depending only on Ω . Here we use also the fact that $|x^1 - x^0| \geq |x^0|$. Since $\phi \geq 0$ and $\phi(0) = 0$ we have $\phi(\bar{x}) \leq A|\bar{x}|^2$ where A depends on the C^2 norm of ϕ . By convexity of u , and the fact that $u(x^1) = 0$ we conclude that

$$u(x) \leq \phi(\bar{x}) \leq A|\bar{x}|^2 \leq AC^2|x - x^0|^2$$

i.e. (8) holds again in this case.

The proofs of the Proposition and of the Theorem are complete.

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