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On the Boundary Values of Harmonic Functions

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1. Introduction

Over the years many methods have been discovered to prove the existence of a solution of the Dirichlet problem for Laplace's equation. A fairly recent collection of proofs is based on representations of the Green's function in terms of the Bergman kernel function or some equivalent linear operator [3]. Perhaps the most fundamental of these aproaches involves the projection of an arbitrary function onto the class of harmonic functions in a Hilbert space whose norm is defined by the Dirichlet integral [5]. Here a problem has remained open concerning continuity at the boundary of the solution that is constructed by orthogonal projection. Past discussions of this question turned out to be successful in space of two or three dimensions, but failed for larger numbers of independent variables [2]. It is the purpose of the present note to remove any such restriction and simultaneously to give a concise treatment of the boundary condition that is applicable to other existence proofs.

Let D be a domain in n-dimensional space that has a smooth boundary ∂D. We introduce the Hilbert space H whose elements are the gradients of harmonic functions u with a finite Dirichlet integral

$$\|u\|^2 = (u, u) = \int_D |\nabla u|^2 \, d\tau.$$

That H is complete follows easily from the mean value theorem for the partial derivatives of u.

Let w stand for a continuous function on ∂D that can be extended inside D so that its Dirichlet integral there is finite. According to the Riesz representation theorem the bounded linear functional (u, w) can be expressed as the scalar product

$$(u, w) = (u, U)$$

of u with an element U of the Hilbert space H. In fact U may be viewed as the orthogonal projection of w onto H. It might be anticipated that U solves the Dirichlet problem for Laplace's equation in D with boundary values w assigned on ∂D. However, we shall not attempt to establish this directly.

Let us consider the auxiliary function

$$v = \left(w - U, \frac{1}{\omega_n r^{n-2}}\right),$$

where r stands for the distance from a point x in D to a variable point of integration ξ, and where $\omega_n/(n-2)$ is the surface area of a unit sphere. After a preliminary application of Green's theorem that resolves the singularity at $\xi = x$, differentiation under the sign of integration shows that v satisfies the partial differential equation

$$\Delta v = \Delta w$$

in D, since $1/(\omega_n r^{n-2})$ is a fundamental solution of Laplace's equation. To prove that $w - v$ solves the Dirichlet problem formulated above it therefore suffices to show that v vanishes continuously at the boundary ∂D. It is our intention to develop an elementary proof of this result in the next two sections of the paper.

2. Green's function of a nearly spherical domain

In Neumann's method of the arithmetic mean [4] the solution of the Dirichlet problem is sought as a double-layer potential

$$u = \frac{1}{\omega_n} \int_{\partial D} \mu \frac{\partial}{\partial \nu} \frac{1}{r^{n-2}} d\sigma,$$

where ν stands for the inner normal. A Fredholm integral equation

$$\frac{\mu}{2} + \frac{1}{\omega_n} \int_{\partial D} \mu \, d\omega = w$$

is found for the determination of the unknown density μ on ∂D, where

$$d\omega = \frac{\partial}{\partial \nu} \frac{1}{r^{n-2}} d\sigma$$

becomes, after division by $n - 2$, the solid angle subtended from the point $x = x_0$ on ∂D by the surface element $d\sigma$. An exact solution may be obtained when D is a half-space, since in that case $d\omega = 0$, so $\mu = 2w$.

More generally, following Neumann, one can try to determine μ as the limit of a sequence of successive approximations μ_j defined by the formula

$$\mu_j = 2w - \frac{2}{\omega_n} \int_{\partial D} \mu_{j-1} \, d\omega.$$

A proof of convergence hinges on estimating the difference

$$\mu_{j+1} - \mu_j = -\frac{2}{\omega_n} \int_{\partial D} [\mu_j - \mu_{j-1}] \, d\omega.$$

For the moment let us suppose that ∂D consists of two pieces, one being the infinite section S_1 of an $(n - 1)$-dimensional hyperplane that is cut out by a small cell S_2 of some convex surface, and the other being S_2 itself. Furthermore, let us assume that with reference to any point x_0 on S_1 or S_2 the solid angles subtended by either S_1 or S_2 are both less than $\epsilon/(n - 2)$.

From the hypotheses we have formulated one can derive the estimate

$$|\mu_{j+1} - \mu_j| \leqslant \frac{4\epsilon}{\omega_n} \max |\mu_j - \mu_{j-1}|.$$

This follows because any line intersects the surfaces S_1 and S_2 in at most three points, so that the solid angle of integration $d\omega/(n - 2)$ does not become multiplied by more than two. Therefore μ_j converges to a solution μ of the Fredholm equation provided that $\epsilon < \omega_n/4$. We conclude that the Dirichlet problem can be solved and the Green's function

$$G = \frac{1}{\omega_n r^{n-2}} + \cdots$$

for Laplace's equation exists in either of the two infinite domains bounded by S_1 and S_2.

Let us return to the case of a smooth surface ∂D bounding a finite domain D. To assess boundary values we require that each point x_0 of ∂D can be touched by a closed sphere located enterely in the exterior of D. An inversion mapping this sphere onto a half-space transforms ∂D into a surface that becomes convex in the neighborhood of the boundary point x_0. Consequently a convex surface element S_2 of ∂D enclosing x_0 can be cut out by a hyperplane whose outer section S_1 combines with S_2 to meet the hypotheses announced above. Thus we are assured that a Green's function G exists in the infinite region bounded by S_1 and S_2 that contains D.

In the next section we shall use G as a parametrix to estimate the boundary values of the auxiliary function v. To complete our discussion of Neumann's method here we observe that, coupled to the Kelvin transformation [4], it provides a convenient construction of the Green's function for a nearly spherical domain. Moreover, convergence of the Neumann series can be proved without the assumption of partial convexity that we have introduced as a matter of convenience.

3. Continuity at the boundary

We proceed to establish that the auxiliary function v defined in Section 1 approaches zero as its argument x approaches any point x_0 on the boundary surface ∂D. The analysis of Section 2 shows that it suffices to consider the case where x_0 lies on a convex surface element S_2 of ∂D which, together with the outer section S_1 of a corresponding hyperplane, bounds a domain containing D and possessing a well defined Green's function G.

Let us recall that $w - U$ is orthogonal to every harmonic functon u of the Hilbert space H in the sense that

$$(w - U, u) = 0.$$

Since the difference between G and the fundamental solution $1/(\omega_n r^{n-2})$ of Laplace's equation lies in H, it follows that v has the representation

$$v = (w - U, G).$$

Because G vanishes on S_2 we wish to draw a similar conclusion about v.

Given any number $\delta > 0$, the locus of points ξ where

$$G = G(x, \xi) \geqslant \delta$$

is seen to lie inside D when x is chosen sufficiently close to the boundary point x_0. The estimate of G required for the proof follows from a comparison with the Green's function of a half-space enclosing S_1 and S_2. In this situation Green's theorem yields the identity

$$v = -\int_{G>\delta} (G - \delta)\,\Delta w\,d\tau + \int_{G<\delta} (\nabla w - \nabla U)\cdot\nabla G\,d\tau.$$

As $x \to x_0$ the first integral on the right tends to zero with δ provided that w

has bounded second derivatives. The second integral, which is evaluated over the part of D where $G < \delta$, has the same property in view of the Schwarz inequality

$$\left[\int_{G<\delta} (\nabla w - \nabla U) \cdot \nabla G \, d\tau\right]^2 \leqslant \| w - U \|^2 \int_{G=\delta} G \frac{\partial G}{\partial \nu} d\sigma = \delta \| w - U \|^2 .$$

This completes the proof that v vanishes on ∂D. Thus $w - v$ solves the Dirichlet problem posed in Section 1, and our existence theorem is established.

A similar treatment of the boundary condition can be given for the solution of the Dirichlet problema constructed from Dirichlet's principle rather than from projection onto the Hilbert space of harmonic functions. The method succeeds for more general linear partial differential equations of the elliptic type, too [1]. The main requirement is that a fundamental solution can be found in the large. The proof can also be based on other principles of functional analysis, such as the Hahn-Banach theorem [2]. One advantage of the present approach, as we have already indicated, is that it applies in space of arbitrary dimension. On the other hand, a disadvantage is the restriction to domains with a smooth boundary.

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