Revista Matemática Iberoamericana
Vol. 1, n.º 2, 1985

Stable Planar Polynomial Vector Fields

J. Sotomayor

1. Introduction

A vector field in R^2 of the form

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

where $P = \sum a_{ij}x^i y^j$ and $Q = \sum b_{ij}x^i y^j$, $0 \leqslant i + j \leqslant n$, is called a *planar polynomial vector field of degree* $\leqslant n$. The $N = (n+1)(n+2)$ real numbers a_{ij}, b_{ij} are called the *coefficients of* X. The space of these vector fields, endowed with the structure of affine R^N-space in which X is identified with the N-tuple $(a_{00}, a_{10}, \ldots, a_{0n}; b_{00}, \ldots, b_{0n})$ of its coefficients, is denoted by χ_n.

The Poincaré compactification of $X \in \chi_n$ is defined to be the unique analytic vector field $\mathcal{P}(X)$ tangent to the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and to the equator $S^1 = \{S^2, z = 0\}$, whose restriction to the northern hemisphere $S^2_+ = \{S^2, z > 0\}$ is given by $z^{n-1}p_*(X)$, where p is the central projection from R^2 to S^2_+, defined by $p(u, v) = (u, v, 1)/(u^2 + v^2 + 1)^{1/2}$. See 3 or [6] for a verification of the uniqueness and analyticity of $\mathcal{P}(X)$.

Definition 1.1. *a*) $X \in \chi_n$ is said to be topologically *stable* if there is a neighborhood V and a map h: $V \to \mathrm{Hom}(S^2, S^1)$ (homeomorphisms of S^2 which preserve S^1) such that $h_X = \mathrm{Id}$ and h_Y maps orbits of $\mathcal{P}(X)$ onto orbits of $\mathcal{P}(Y)$, for every $Y \in V$.

b) If furthermore, h can be chosen such that for each $x \in S^2$, $Y \to h_Y(x)$ is of class C^r, $r = 1, 2, \ldots, \infty, \omega$, then X is said to be *r-stable*.

Define by Σ_n (resp. Σ_n^r) the set of $X \in \chi_n$ defined in *a*) (resp. *b*)).

The *topological* (resp. *r*) *bifurcation set* in χ_n is defined by $\chi_n^1 = \chi_n - \Sigma_n$ (resp. $\chi_n^{r,1} = \chi_n - \Sigma_n^r$).

The point topological properties of Σ_n have been studied by Pugh [17] and dos Santos [19] Their works describe an open dense set of χ_n denoted here by $\mathcal{S}_n$, such that $\mathcal{S}_n \subset \Sigma_n$, which is defined by properly extending to elements of the form $\mathcal{P}(X)$ the conditions given by Andronov-Pontryagin [1] and Peixoto [14] for smooth vector fields on compact domains. These papers were preceded by the work of Gonzáles [6], devoted to the generic properties of elements of χ_n at infinity, i.e. on S^1.

Definition 1.2. Denote by $\mathcal{S}_n$ the set of $X \in \chi_n$ for which $\mathcal{P}(X)$ has *a*) all its singularities hyperbolic, *b*) all its periodic orbits hyperbolic and *c*) no saddle connection contained in $S^2 - S^1$.

The characterization of Σ_n depends on a delicate point, apparently overlooked in [19], which for future reference is formulated here as a problem.

Problem 1.1. Prove (or disprove) that the hyperbolicity of an attracting or repelling periodic orbit in $S^2 - S^1$ is necessary for topological stability in χ_n.

The main results of this paper, characterize Σ_n^r as $\mathcal{S}_n$ and establishes the simplest affine, analytical and measure theoretical meagerness properties of the bifurcation sets χ_n^1 and $\chi_n^{r,1}$ of χ_n. These meagerness properties have obvious thickness counterparts for Σ_n and Σ_n^r.

Theorem A. *a*) *The set of r-stable vector fields* Σ_n^r, $r = 1, 2, \ldots, \omega$, *coincides with* $\mathcal{S}_n$.

b) *Furthermore,* $\chi_n^{r,1} = \chi_n - \mathcal{S}_n$, $r = 1, 2, \ldots, \omega$, *is contained in the union of countably many one-to-one immersed analytic submanifolds of codimension* $\geqslant 1$ *in* χ_n.

Corollary 1.1. $\chi_n^{r,1}$ *and, therefore,* χ_n^1 *have null Lebesgue measure in* χ_n.

Corollary 1.2. *Let* $\xi: \mathbb{R} \to \chi_n$ *be a* C^1 *map. Call* $G(\xi)$ *the set of* $V \in \chi_n$ *such that* $\xi + V$ *meets* $\mathcal{S}_n$ *except at most in a countable set of points. Then* $G(\xi)$ *has total Lebesgue measure in* χ_n.

The null Lebesgue measure of the bifurcation set in compact plane regions was established by the author in [25].

Corolary 1.2 is a rather crude description of bifurcations of codimension one of χ_n. The actual geometry of the bifurcation phenomena of $\mathcal{P}(X)$ as the coefficients of X change along curves that meet $\chi_n^{r,1}$ transversally at regular points, has been studied in the simplest situations in [13].

2. Proof of Corollaries

Assume Theorem A, *b*).

Let $\chi_n - \mathcal{S}_n = US_j$, $j = 1, \ldots,$ where S_j are analytic submanifolds of codimension $\geqslant 1$.

The map $F(V, \cdot) = V + \xi(\cdot)$ is transversal of S_j if and only if V belongs to the set R_j of regular values of the projection of $F^{-1}(S_j)$ onto χ_n. Clearly $G(\xi) = R_j$. By Sard's Theorem [20], $G(\xi)$ has total Lebesgue measure. This argument applies to any map $\xi: R^k \to \chi_n$ of class C^k. It gives Corollary 1.1 if $k = 0$, and Corollary 1.2, if $k = 1$.

3. Proof of Theorem A

Take coordinates (θ, ρ), 2π-periodic in θ, defined by the covering map from $R \times (-1, 1)$ onto $S^2 - \{(0, 0, \pm 1)\}$, given by $(\theta, \rho) \to (x, y, z) = (1 + \rho^2)^{-1/2}$ $(\cos\theta, \sin\theta, \rho)$.

The expression for $z^{n-1} p_*(X)$, $X \in \chi_n$, in these coordinates is

$$(1 + \rho^2)^{(1-n)/2}\left[(\Sigma\rho^i A_{n-i}(\theta))\frac{\partial}{\partial\theta} - \rho(\Sigma\rho^i R_{n-i}(\theta))\frac{\partial}{\partial\rho}\right], \tag{3.1}$$

where $i = 0, 1, 2, \ldots, n$ and

$$\begin{aligned} A_k(\theta) = A_k(X, \theta) &= -P_k(\cos\theta, \sin\theta)\sin\theta + Q_k(\cos\theta, \sin\theta)\cos\theta \\ R_k(\theta) = R_k(X, \theta) &= P_k(\cos\theta, \sin\theta)\cos\theta + Q_k(\cos\theta, \sin\theta)\sin\theta, \end{aligned} \tag{3.2}$$

with $P_k = \sum a_{ij}x^i y^j$, $Q_k = \sum b_{ij}x^i y^j$, $i + j = k$.

This shows that $\mathcal{P}(X)$ must be given by (3.1), mod 2π, and is therefore analytic in S^2 and tangent to S^1.

Denote by $B(i)$ the set of $X \in \chi_n$ which to not satisfy condition $i = a$), *b*), *c*) of Definition 1.2. Theorem A, *b*) will follow from.

Proposition 3.1. *a*) *$B(a)$ is a semi-algebraic set in χ_n.*

b) *The set C of vector field of $\chi_n - B(a)$ with some graph of saddles and separatrices is closed in $\chi_n - B(a)$.*

c) *$B(b)$ is a closed semianalytic set in the open set $A = (\chi_n - B(a)) - C$.*

d) $B(c)$ is the union of finitely many one-to-one immersed analytic hypersurfaces in $\chi_n - B(a)$.

PROOF. *a*) Notice that $B(a)$ is the projection into χ_n of the union of the following semi-algebraic sets.

$$\{P = 0, Q = 0; \Delta = P_x Q_y - P_y Q_x = 0\},$$
$$\{P = 0, Q = 0; \Delta > 0, \sigma = P_x + Q_y = 0\},$$
$$\{A_n = 0, A'_n = 0\} \quad \text{and} \quad \{A_n = 0, R_n = 0\}.$$

The result follows from Tarski-Seindenberg Theorem [21].

b) If $X \to Y$ in $\chi_n - B(a)$, and Y does not have any graph, by continuation of all the saddle separatrices through saddle connections of Y one would arrive to separatrices whose limit sets are attractors or repellors.

By continuity, the same would hold for neighboring systems and, therefore, X could not have had graphs.

c) The following remaks will be needed.

Remark 3.1. If X has a periodic orbit at infinity, i.e. if S^1 is a periodic orbit of $\mathcal{P}(X)$, then it is hyperbolic if and only if

$$\mu = \int_0^{2\pi} R_n(X, \theta) A_n^{-1}(X, \theta)\, d\theta \neq 0.$$

Actually, the derivative $\Pi'(0)$ of the Poincaré return map Π associated to a transversal segment is given by

$$\log \Pi'(0) = (-1)\sigma\mu, \tag{3.3}$$

where σ denotes the sign of the orientation of the orbit relative to the canonical orientation of S^1.

In fact, from (3.1) the trajectories of $\mathcal{P}(X)$ near S^1 satisfy the following differential equation

$$\frac{d\rho}{d\theta} = \frac{-\rho(\Sigma\rho^i R_{n-i}(\theta))}{\Sigma\rho^i A_{n-i}(\theta)}, \qquad i = 0, 1, \ldots, n$$

Denote by $\rho = \rho(\rho_0, \theta)$ the solution of this equation, with initial condition $\rho(\rho_0, 0) = \rho_0$. The Poincaré return map is therefore given by $\Pi(\rho_0) = \rho(\rho_0, 2\pi)$. Therefore,

$$\Pi'(0) = \frac{\partial\rho}{\partial\rho_0}(0, 2\pi) = \exp\left[-\int_0^{\sigma 2\pi} R_n A_n^{-1}\, d\theta\right],$$

as follows from (3.3) and a well known formula for the derivative of solutions with respect to initial conditions. Since, in this case,

$$\int_0^{\sigma 2\pi} = \sigma \int_0^{2\pi},$$

this proves (3.3).

Remark 3.2. The derivative of μ in the direction of

$$V = T\frac{\partial}{\partial x} + U\frac{\partial}{\partial y} \in \chi_n$$

is given by

$$D\mu_X(V) = \int_0^{2\pi} (T_n Q_n - U_n P_n) A_n^{-2}(X, \theta)\, d\theta,$$

which is not null. In particular, if

$$V = (x^2 + y^2)^k \left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \qquad n = 2k + 1, \qquad D\mu(V) = \int_0^{2\pi} A_n^{-1}(\theta)\, d\theta \neq 0.$$

This shows that the space of vector fields in χ_n, $n = 2k + 1$, with a non hyperbolic orbit at infinity, is an analytic hypersurface.

The proof of *c*) can be finished as follows.

For $X \in A$ take a neighborhood $V \subset A$, such that the Poincaré return map Π_i: $V \times L_i \to L_i$ of $\mathcal{P}(Y)$, $Y \in V$, is defined on a segment L_i-transversal to each periodic orbit γ_i of X. Write $0 = L_i \cap \gamma_i$.

Let n_i be the multiplicity of γ_i as a periodic orbit of X; that is, n_i is the order of the zero of $\Pi_i(X, x) - x$, at $0 \in L_i$. Using the Weierstrass Preparation Theorem [29], write $\Pi_i(Y, x) - x = U_i(Y, x) P_i(Y, x)$, where

$$P_i = x^{n_i} + a_{n_i - 1}^{(i)}(Y) x^{n_i - 1} + \ldots + a_0^{(i)}(Y),$$

with $a_j^{(i)}$ and U_i analytic functions, $U_i \neq 0$ in $V \times L_i$ and $a_j^{(i)}(X) = 0$.

There are two cases:

a) If γ_i is a periodic orbit on $S^2 - S^1$, $D_X a_0 \neq 0$. In fact, $a_0 = \Pi_i(\cdot, 0) U^{-1}(\cdot)$, and by [23; 1p. 383], if $V = T(\partial/\partial x) + U(\partial/\partial y)$,

$$D_X a_0(V) = U^{-1}(X, 0) \int_0^{\tau} \exp\left[-\int_0^t \operatorname{div} X \right] (PU - QT)\, dt,$$

where τ is the period of γ_i.

b) If γ_i is the periodic orbit at infinity, $a_0 \equiv 0$, and $P_i = x Q_i$, where

$$Q_i = x^{n_i - 1} + a_{n_i - 1}(Y) x^{n_i - 2} + \ldots + a_1^i(Y),$$

where n_i is odd and $a_1^{(i)}(Y) = U_i^{-1}(Y, 0)\{\exp[(-1)\sigma\mu(Y)] - 1\}$, according to Remark 3.1.

The non hyperbolic periodic orbits of $\mathcal{P}(Y)$ are near some γ_i. In case *a*) they intersect L_i at points where the quasi-polynomial $P_i(\cdot, x)$ has a multiple root. In case *b*), this happens when $a_1^{(i)}(Y) = 0$, for non hyperbolicity at infinity, and at multiple roots of $Q_i(\cdot, x)$, for non hyperbolicity on $S^2 - S^1$.

These sets, defined by the condition of having multiple roots are semianalytic. In fact, they are inverse inmages by the analytic maps $Y \to (a_{n_i-1}^{(i)}(Y), \ldots, a_0^{(i)}(Y))$, for case *a*), and $Y \to (a_{n_i-2}^{(i)}(Y), \ldots, a_1^{(i)}(Y))$, for case *b*), of the discriminant locus of the generic polynomials of correspondent degree, which is a semi-algebraic set [27].

d) The semi-algebraic set $\chi_n - B(a)$ has finitely many connected components $C_1, C_2, \ldots, C_l$ [28]. On each such component C, the saddle singular points $p_j(X)$ of $\mathcal{P}(X)$ as well as its four separatrices $S_i^j(X, s_i)$, $i = 1, 2, 3, 4$, parametrized by arc length s_i with origin in $p_j(X)$, are well defined analytic functions of the two variables. Take $S_1^j(Y, s_1)$ and $S_2^k(Y, s_2)$ two such separatrices, the first unstable and the second stable, which correspond to saddle points $p_j(Y)$ and $p_k(Y)$, which may be equal.

The set B_{jk} of $(Y, l) \in C \times R_+$ for which $S_1^j(Y)$ and $S_2^k(Y)$ form a saddle connection of lenght l is an analytic submanifold of dimension $N - 1$ in $C \times R_+$ whose projection into C is a one-to-one immersion.

In fact, for $(X, l_0) \in B_{jk}$, take a small segment L transversal to X through a point $p_0 = S_1^j(X, s_1(0)) = S_2^k(X, s_2(0))$. There are analytic functions $s_i(Y)$, $i = 1, 2$, implicitly defined by $S_1^j(Y, s_1(Y)) \in L$, $S_2^k(Y, s_2(Y)) \in L$ and such that $s_i(X) = s_i(0)$.

It was shown in [23], see also [2,18], that the derivative of the function $S = S_1^j(Y, s_1(Y)) - S_2^k(Y, s_2(Y))$ is given by

$$Ds_X(Z) = \int_{-\infty}^{\infty} \exp\left[-\int_0^t \operatorname{div} X\right](RT - QU)\,dt,$$

where $Z = T(\partial/\partial x) + U(\partial/\partial y)$, and the integral is computed on the saddle connection. Without loss of generality assume that the saddle connection does not contain $(0, 0, 1)$, and the coordinates (ρ, θ) of (3.1) can be used.

Writing $X = P(\partial/\partial x) + Q(\partial/\partial y)$ and $X^{\perp} = -Q(\partial/\partial x) + P(\partial/\partial y)$ in these coordinates and applying the above integral formula, one gets an expression of the form

$$DS_X(X^{\perp}) = \int_{-\infty}^{\infty} g(\rho, \theta)\rho\,dt,$$

where $g(\rho, \theta)$ is strictly positive. This shows that $DS_X \neq 0$.

Clearly this ends the proof of *d*). In fact, when $S(Y) = 0$, the lenght of the saddle connection is given by $l(Y) = s_1(Y) + s_2(Y)$ which is also analytic. Therefore, B_{jk} is an analytic manifold of dimension $N - 1$, which projects regularly into C. The set $B(c)$ in C is the union of finitely many images of such projections.

The proof of (A, b) follows from Proposition 3.1, by the stratification of semi-algebraic and semi-analytic sets into analytic manifolds. See Lojasiewicz [11] and Whitney [28].

The proof of Theorem A, *a*) is straight:

1) If $X \in \mathcal{S}_n$ the constructions of topological equivalences in [1, 7] all produce r-stability. Therefore, $X \in \Sigma_n^r$, $r = 1, \ldots, \omega$.

2) If $X \in \Sigma_n^r$, $r = 1, \ldots, \omega$, from A, *b*) X must be topologically equivalent to an element of $\mathcal{S}_n$ and therefore the singularities and periodic orbits of $\mathcal{P}(X)$ must be finite and there must not be saddle connections contained in $S^2 - S^1$. The r-stability condition forces the singularities and, particularly, the periodic orbits to be hyperbolic. Actually, for the hyperbolicity of singular points and infinite periodic orbits it is sufficient to impose topological stability.

4. Final Remarks

1) For the study of stable smooth vector fields on non compact domains, the reader is refered to Nitecky et al [8] and the references quoted in this work. Here, perturbations with compact support are allowed and stability is not a generic property.

2) The set $A = (\chi_n - B(a)) - C$ in Proposition 3.1 is related to the class of polynomial vector fields studied by Poincaré [15, Theorem 17], for which the finiteness of limit cycles was first proved.

For extensions and further developments of this finiteness Theorem, the reader is refered to Chicone-Shafer [3], Paterlini-Sotomayor [12], Iliashenko [9], Ye Yanquian [30], Pugh-Françoise [5] and references quoted in these works.

3) Using Thom's Transversality Theorem [26], it can be asserted that the generic one parameter family of elements in χ_n has at most countably many bifurcations. The idea of Corollary 1.2 was suggested to the author by his previous work [24] and by the reading of Pontrjagin [16].

4) Although Theorem A expresses the meagerness of the bifurcation set in analytical terms and implies, through Corollary 1.1, that a vector field in χ_n is probabilistically almost surely stable, i.e. on $\mathcal{S}_n$, it cannot be regarded as the ultimate result on this line of ideas. In fact, it does not give any estimate on the cost involved in deciding whether or not a given vector field in χ_n is stable, i.e. on $\mathcal{S}_n$, in the sense of complexity theory, a la Smale [22].

A key step for such estimate amounts to the study, in terms of n and R, of the volume of a tube of radius R of the set $\chi_n^{r,1} \cap S^{N-1}$ relative to the volume of the unitary sphere S^{N-1} of χ_n. The study can be done for the part of the tube around $B(a)$ in viev of the algebraic nature of this set, using ideas of integral geometry, as suggested by Smale [22], and results of Demmel [4]. The analysis for the part of the tube on $B(b)$ and $B(c)$ does not seem to be straight. The set $B(b)$ is not semi-algebraic, as follows from results of Illiashenko [10]. Also the set $B(c)$ is not semi-algebraic, as is easy to verify at least for n big. For $n = 2$, this is not known [30].

These remarks indicate that new different techniques and expectations should be devised in connection with the possibility of developing a complexity theory for the stability and bifurcations in χ_n.

5. Aknowledgement

The author aknowledges the hospitality of the University of California, Berkeley, where this paper was written and where he had stimulating conversations with S. Smale and C. Pugh.

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J. Sotomayor
University of Toronto
Toronto, Canada, M551A1

Partially supported by CNPq, Brazil, and the J. S. Guggenheim Memorial Foundation.