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A Littlewood-Paley Inequality for Arbitrary Intervals

José L. Rubio de Francia

1. Introduction

For every interval $I \subset \mathbb{R}$ we denote by S_I the partial sum operator: $(S_I f)\hat{} = \hat{f}\chi_I$. Given a sequence $\{I_k\}$ of disjoint intervals, we form the quadratic expression

$$\Delta f(x) = \Big(\sum_k |S_{I_k} f(x)|^2\Big)^{1/2} \tag{1.1}$$

We aim to prove here the following

Theorem 1.2. *For every p with* $2 \leqslant p < \infty$, *there exists* $C_p > 0$ *such that, for every sequence* $\{I_k\}$ *of disjoint intervals, the operator* Δ *defined by* (1.1) *satisfies*

$$\|\Delta f\|_p \leqslant C_p \|f\|_p \qquad (f \in L^p(\mathbb{R})). \tag{1.3}$$

Two particular cases of this result were previously known:

1.4. When $\{I_k\}$ is a lacunary sequence: $I_k = [a_{k-1}, a_k]$ with (say) $(a_{k+1} - a_k) \geqslant 2(a_k - a_{k-1})$, then (1.3) holds for all $1 < p < \infty$, and a converse inequality: $C_p\|\Delta f\|_p \leqslant \|\Delta f\|_p$ is also verified by every f such that supp

$(\hat{f}) \subset U_k I_k$. This is a classical theorem due to Littlewood and Paley [11], which is sometimes a good substitute for Plancherel's theorem in L^p, $p \neq 2$ (see [16], [5]).

1.5. When all the intervals I_k have the same length, then inequality (1.3) holds for $2 \leqslant p < \infty$, and this is best possible as it is shown by the example: $I_k = [k-1, k]$, $k = 1, 2, \ldots, N$ and $\hat{f} = \chi_{[0,N]}$ (with N large enough). This result was first proved by L. Carleson [1], and a different proof was given by A. Córdoba [3], who used it in order to obtain L^p estimates for Bochner-Riesz multipliers, [4].

In the proof presented below, we first reduce the problem to the case where the intervals $\{I_k\}$, after suitably dilated do not overlap too much. Once we are in this situation, it is possible to regularize the partial sum operators, obtaining, instead of Δf, its smooth version Gf, which is easier to handle as a vector valued singular integral. The estimates required for the kernel of G are a combination of classical Littlewood-Paley theory and the ones used in a simplified proof of the case (1.5), given in [14]. In the last three sections, we discuss some variants of the main result: weighted estimates, results in L^p with $p < 2$, and n-dimensional analogues.

This problem came to my knowledge through A. Córdoba, who was always firmly convinced of the truth of such a general statement. My finding the proof was greatly stimulated by conversations with L. Carleson, P. W. Jones, J. P. Kahane, M. Reimann, P. Sjögren and P. Sjölin, during a delightful stay in Sweden.

2. Reduction to the well-distributed case

All the intervals considered will be of finite length. For every interval I and $c > 0$, we denote by cI the interval with the same center as I and length: $|cI| = c|I|$.

Definition 2.1. *A sequence of intervals $\{I_k\}$ is well distributed if the doubles of the intervals have bounded overlapping, i.e.*

$$\sum_k \chi_{2I_k}(x) \leqslant C \qquad (x \in \mathbb{R})$$

Now, we define the Whitney decomposition $W(I)$ of an interval I as follows: First of all, the definition is invariant under translations and dilations, and if $I = [0, 1]$, then $W(I)$ consists of the intervals:

$$\left\{[a_{k+1}, a_k]\right\}_{k=0}^{\infty}; \quad \left[\frac{1}{3}, \frac{2}{3}\right]; \quad \left\{[1 - a_k, 1 - a_{k+1}]\right\}_{k=0}^{\infty}$$

where $a_k = 2^{-k}/3$. Observe that the intervals of $W(I)$ form a disjoint covering of I, and:

$$\begin{cases} 2H \subset I \quad \text{for every} \quad H \in W(I) \\ \sum_{H \in W(I)} \chi_{2H}(x) \leqslant 5 \qquad \text{for all } x \end{cases} \tag{2.2}$$

Lemma 2.3. *Given disjoint intervals* $\{I_k\}$, *let* $\Delta f(x)$ *be defined as in* (1.1), *and let*

$$\Delta_k f(x) = \Big(\sum_{H \in W(I_k)} |S_H f(x)|^2 \Big)^{1/2}$$

Then for all $1 < p < \infty$, *we have the equivalence*

$$\|\Delta f\|_p \sim \Big\| \Big(\sum_k (\Delta_k f)^2 \Big)^{1/2} \Big\|_p \qquad (f \in L^p)$$

PROOF: This is essentially known, and a more general (weighted) version of it will be given in 6.3 below. Here is however a short sketch of proof: The operators Δ_k are uniformly bounded in $L^2(w)$ if $w \in A_2$ (see [10]), from which it follows that

$$\Big\| \Big(\sum_k (\Delta_k f_k)^2 \Big)^{1/2} \Big\|_p \leqslant C_p \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p \tag{2.4}$$

for all $1 < p < \infty$. When we choose $f_k = S_{I_k} f$ in (2.4), we obtain the inequality $\geqslant$ in the Lemma. Since there is equality of norms when $p = 2$, the usual duality argument proves the converse inequality $\leqslant$.

It follows that Theorem 1.2 holds for the sequence $\{I_k\}$ if and only if it holds for the sequence

$$\bigcup_k W(I_k) = \{H / H \in W(I_k) \quad \text{for some} \quad k\}$$

But this last sequence is well distributed according to (2.2), and we arrive at

Lemma 2.5. *In proving Theorem* 1.2, *it is no restriction to assume that the given sequence of intervals* $\{I_k\}$ *is well distributed.*

3. The smooth operator and the basic estimate

We start with a well distributed sequence, and we divide each interval I into seven consecutive intervals of equal length

$$I = I^{(1)} \cup I^{(2)} \cup \ldots \cup I^{(7)}, \qquad |I^{(i)}| = |I|/7$$

so that $8I^{(i)} \subset 2I$. It suffices to prove the theorem for each one of the families $\{I^{(i)} \mid I \in \text{initial sequence}\}$. Therefore, we can assume from the beginning that we are given a sequence $\mathbf{I}$ of disjoint intervals such that

$$\sum_{I \in \mathbf{I}} \chi_{8I}(x) \leqslant C \qquad (x \in \mathbb{R}) \tag{3.1}$$

It will be convenient to label the intervals of the sequence according to their length. Thus, for each integer k, let

$$\{I_k^j\}_j = \{I \in \mathbf{I} \mid 2^k \leqslant |I| < 2^{k+1}\}$$

For every k, j, let n_k^j be the first integer such that $n_k^j 2^k \in I_k^j$, and fix a Schwartz function $\psi(x)$ whose Fourier transform satisfies

$$\chi_{[-2,2]} \leqslant \hat{\psi} \leqslant \chi_{[-3,3]}$$

Then we define

$$\psi_k^j(x) = 2^k \psi(2^k x) \exp(2\pi i n_k^j 2^k x)$$

so that the Fourier transform of ψ_k^j is adapted to I_k^j, i.e.

$$(\psi_k^j)\hat{}\,(\xi) = \hat{\psi}(2^{-k}\xi - n_k^j) = \begin{cases} 1 & \text{if} \quad \xi \in I_k^j \\ 0 & \text{if} \quad \xi \notin 8I_k^j \end{cases} \tag{3.2}$$

Definition 3.3. *The smooth operator G associated to a sequence of intervals satisfying* (3.1) *is*

$$Gf(x) = \Big(\sum_{k \in \mathbb{Z}} \sum_j |\psi_k^j * f(x)|^2\Big)^{1/2} =$$

$$= \Big\{\sum_{k,j} \Big| \int 2^k \psi(2^k(x-y)) \exp(-2\pi i n_k^j 2^k y) f(y)\, dy \Big|^2\Big\}^{1/2}$$

It follows from (3.1) and (3.2) that $\sum_{k,j} |(\psi_k^j)\hat{}\,(\xi)|^2 \leqslant C$, which, by Plancherel's theorem, implies that Gf is well defined in $L^2(\mathbb{R})$ and satisfies

$$\|Gf\|_2 \leqslant C\|f\|_2 \tag{3.4}$$

Our objective is the corresponding L^p inequality, $2 < p < \infty$. This will be a consequence of the main estimate for Gf stated below. We denote by $(\cdot)^{\#}$ the sharp maximal operator of Fefferman and Stein [6], and also,

$$M_q f(x) = \{M(|f|^q)(x)\}^{1/q} \qquad (1 \leqslant q < \infty)$$

where $M = M_1$ stands for the Hardy-Litewood maximal function. Then, we have for every $f \in L_C^\infty(\mathbb{R}) = \{\text{bounded functions with compact support}\}$

$$(Gf)^{\#}(x) \leqslant CM_2 f(x) \qquad (x \in \mathbb{R}) \tag{3.5}$$

The next two sections will be devoted to the proof of (3.5). We wish to observe here that this will complete the proof of Theorem 1.2, since for all $f \in L_C^\infty$ and $2 < p < \infty$

$$\Big\| \Big(\sum_{k,j} |S_{I_k^j} f|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \| Gf \|_p \leqslant C_p' \| (Gf)^{\#} \|_p \leqslant CC_p' \| M_2 f \|_p \leqslant C_p'' \| f \|_p$$

(the first inequality follows by the usual truncation argument which can be seen in [5], [16], [17], because $S_{I_k^j} f = S_{I_k^j}(\psi_k^j * f)$).

4. A lemma for vector-valued singular integrals

Let H be a separable Hilbert space, and let $K(x, y)$ be an H-valued function defined in $\mathbb{R}^2$ such that $\| K(x, \cdot) \|_H$ is locally integrable for each fixed $x \in \mathbb{R}$. Then

$$Tf(x) = \int f(y) K(x, y)\, dy$$

is well defined for every $f \in L_C^\infty(\mathbb{R})$. Given $x, z \in \mathbb{R}$, we denote

$$I_m(x, z) = \{ y \in \mathbb{R} : 2^m |x - z| < |y - z| \leqslant 2^{m+1} |x - z| \}$$

where m is an integer.

Lemma 4.1. *Suppose that T, defined as above, is a bounded operator from $L^2(\mathbb{R})$ to $L_H^2(\mathbb{R})$, and that the kernel $K(x, y)$ satisfies, for some $A > 0$, $\alpha > 1$, the condition*

$$\int_{I_m(x, z)} |\langle K(x, y) - K(z, y), \lambda \rangle|^2\, dy \leqslant A^2 \frac{2^{-\alpha m} \| \lambda \|_H^2}{|x - z|} \tag{4.2}$$

for every $x, z \in \mathbb{R}$, $\lambda \in H$, and $m \geqslant 1$. Then, for the operator $Gf(x) = \| Tf(x) \|_H$ we have the estimate

$$(Gf)^{\#}(x) \leqslant C(A, \alpha) M_2 f(x) \qquad (f \in L_C^\infty)$$

Proof. It is essentially a repetition of the argument in [6]. Given $x \in \mathbb{R}$ and an interval I centered at x, we define the vector

$$h_I = \int_{y \notin 2I} f(y) K(x, y)\, dy \in H$$

so that, if $\bar{f} = f \chi_{2I}$

$$Tf(z) - h_I = T\bar{f}(z) + \int_{y \notin 2I} f(y) [K(z, y) - K(x, y)]\, dy$$

Denoting by $g(z)$ an arbitrary H-valued function with $\|g(z)\|_H \leqslant 1$ for all $z \in I$, we can write

$$\frac{1}{|I|}\int_I \|Tf(z) - h_I\|_H\,dz \leqslant \frac{1}{|I|}\int_I \|T\bar{f}(z)\|_H\,dz +$$

$$+ \sup_g \frac{1}{|I|}\left|\int_I \langle g(z), \int_{y \notin 2I} f(y)[K(z,y) - K(x,y)]\,dy\rangle\,dz\right| = (1) + (2)$$

Now, the first term is easy to estimate

$$(1) \leqslant C\left(\frac{1}{|I|}\int_{2I} |f|^2\right)^{1/2} \leqslant C\sqrt{2}\,M_2 f(x)$$

and in the second term, the value corresponding to each fixed g is majorized by

$$\frac{1}{|I|}\int_I \sum_{m=1}^{\infty} \int_{I_m(z,x)} |f(y)|\,|\langle g(z), K(z,y) - K(x,y)\rangle|\,dy\,dz \leqslant$$

$$\leqslant \sup_{z \in I} \sum_{m=1}^{\infty} \left(\int_{I_m(z,x)} |f(y)|^2\,dy\right)^{1/2} A2^{-\alpha m/2}|x - z|^{-1/2}$$

where we have used (4.2) and the fact that $\|g(z)\|_H \leqslant 1$. Thus,

$$(2) \leqslant 2A \sum_{m=1}^{\infty} 2^{(1-\alpha)m/2} M_2 f(x)$$

and the series converges because $\alpha > 1$. Since

$$(Gf)^{\#}(x) \leqslant C \sup_I \frac{1}{|I|}\int_I \|Tf(z) - h_I\|_H\,dz$$

the proof is ended.

It is easy to formulate generalizations of this lemma: One can consider kernels defined in $\mathbb{R}^n \times \mathbb{R}^n$ with values $K(x,y) \in L(A,B)$, for some Banach spaces A, B, and replace the exponent 2 in our initial assumptions: $\|Tf\|_2 \leqslant C\|f\|_2$ and (4.2), by different exponents p, q. Some of these variants are considered in [15]. The simple case stated here is precisely what we need for our present problem.

5. Proof of the basic estimate

Here we shall use the preceding lemma in order to prove the pointwise estimate (3.5), thus finishing the proof of Theorem 1.2. We must therefore consider the l^2-valued kernel

$$K(x,y) = \{2^k \psi(2^k x - 2^k y) \exp(-2\pi i n_k^j 2^k y)\}_{k,j}$$

where ψ and n_k^j are defined in §3, and we must prove that $K(x,y)$ satisfies (4.2). It suffices to do so when $\lambda = \{\lambda_k^j\}_{k,j} \in l^2$ has unit norm, and for every such λ, we let

$$K_\lambda(x,y) = \langle K(x,y), \lambda \rangle = \sum_{k,j} \bar{\lambda}_k^j 2^k \psi(2^k x - 2^k y) \exp(-2\pi i n_k^j 2^k y) =$$

$$= \sum_k 2^k \psi(2^k x - 2^k y) q_k(2^k y)$$

where, for each $k \in \mathbb{Z}$, q_k is a 1-periodic function defined by its Fourier series

$$q_k(t) = \sum_j \bar{\lambda}_k^j \exp(-2\pi i n_k^j t)$$

Observe that $n_k^j \neq n_k^{j'}$ if $j \neq j'$, so that each q_k satisfies

$$\int_a^{a+1} |q_k(t)|^2 \, dt \leqslant 1 \qquad (a \in \mathbb{R}; k \in \mathbb{Z}) \tag{5.1}$$

and this is the only property of the functions q_k that we shall use, so that we disregard the fact that they also depend on λ. Our problem is then reduced to establishing the inequality

$$\int_{I_m(x,z)} |K_\lambda(x,y) - K_\lambda(z,y)|^2 \, dy \leqslant A 2^{-\alpha m} |x - z|^{-1} \tag{5.2}$$

with $\alpha > 1$. We can assume that $z = 0$, since this amounts to translating q_k by $2^k z$, so that (5.1) is preserved. On the other hand, replacing x by $2x$ does not change the inequality (5.2) at all, and thus, we can also assume that $1 \leqslant |x| < 2$. Writing $I_m(x,0) = I_m(x)$ we have by changing variables

$$\|K_\lambda(x,\cdot) - K_\lambda(0,\cdot)\|_{L^2(I_m(x))} \leqslant$$

$$\leqslant \sum_{k \in \mathbb{Z}} 2^{k/2} \left(\int_{I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)|^2 |q_k(y)|^2 \, dy \right)^{1/2}$$

$$\leqslant \sum_k 2^{k/2} \left\{ \sup_{y \in I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)| \right\} (2^{k+m+2} + 1)^{1/2}$$

$$= \sum_{k=-h}^{\infty} + \sum_{k=-\infty}^{-h-1}$$

where we choose $h = [2m/3]$. For the terms in the first sum we use the fact that $|\psi(y)| \leqslant C|y|^{-2}$, so that

$$\sup_{y \in I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)| \leqslant C 2^{-2k-2m}$$

and then,

$$\sum_{k=-h}^{\infty} (\ldots) \leqslant C \sum_{k=-h}^{\infty} 2^{-k-3m/2} \leqslant C 2^{-5m/6}$$

For the second sum we use the majorization

$$\sup_{y \in I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)| \leqslant C 2^k x \leqslant C 2^{k+1}$$

and we obtain (since $k + m < m - h \leqslant m/3$)

$$\sum_{k=-\infty}^{-h-1} (\ldots) \leqslant C \sum_{k=-\infty}^{-h-1} 2^{3k/2} 2^{m/6} \leqslant C 2^{-5m/6}$$

Combining everything, we have proved the desired inequality (5.2) with $\alpha = \frac{5}{3} > 1$.

Remarks. The initial computations involving λ_k^j's are rather formal, and serious convergence problems may arise. However, everything becomes correct if we define a truncated smooth operator G_F by allowing only a finite set F of k's and j's in the definition. The final estimates are independent of the set F and so, a limiting argument proves the same result for the whole operator G.

A somewhat shorter computation is needed to show that

$$\int_{|y-z| \geq 2|x-z|} |K_\lambda(x, y) - K_\lambda(z, y)| \, dy \leqslant C \tag{5.3}$$

(instead of (5.2)). The analogue of Lemma 4.1 under this weaker assumption shows that $\| Gf \|_{BMO} \leqslant C \| f \|_\infty$, which is certainly weaker than (3.5) but still enough to prove our theorem, since interpolation with (3.4) gives $\| Gf \|_p \leqslant C_p \| f \|_p$, $2 < p < \infty$.

However, for the weighted analogues of Theorem 1.2 which we shall obtain in the next section, the full force of the basic estimate (3.5) is required.

6. Weighted inequalities

The following extension of the theorem just proved holds.

Theorem 6.1. *If* $2 < p < \infty$, *and if the weight* $w(x)$ *(in* $\mathbb{R}$*) belongs to the class* $A_{p/2}$, *then, the operator* Δ *defined by* (1.1) *for an arbitrary sequence of dis-*

joint intervals satisfies

$$\int [\Delta f(x)]^p w(x)\, dx \leqslant C_p(w) \int |f(x)|^p w(x)\, dx$$

PROOF. Let us consider first the smooth operator G associated to a sequence of intervals satisfying (3.1). Then, for all $w \in A_{p/2}$ $(2 < p < \infty)$ and f good enough

$$\int [Gf(x)]^p w(x)\, dx \leqslant C_{p,w} \int [(Gf)^{\#}(x)]^p w(x)\, dx \leqslant$$
$$\leqslant CC_{p,w} \int [M_2 f(x)]^p w(x)\, dx \leqslant C'_{p,w} \int |f(x)|^p w(x)\, dx$$

On the other hand, for arbitrary intervals $\{I_k\}$, the inequality

$$\Big\|\Big(\sum_k |S_{I_k} f_k|^2\Big)^{1/2}\Big\|_{L^p(w)} \leqslant C_{p,w} \Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{L^p(w)} \tag{6.2}$$

($w \in A_p$, $1 < p < \infty$) holds, because it holds for the Hilbert transform (see [10] for details). Thus, the usual truncation argument can be applied, i.e.: If $\{I_k\}$ is the given sequence of intervals, and the associated somooth operator is $Gf = \left(\sum_k |\psi_k * f|^2\right)^{1/2}$, with $\hat{\psi}_k = 1$ on I_k, then we define $f_k = \psi_k * f$ and use (6.2) to obtain

$$\int [\Delta f(x)]^p w(x)\, dx \leqslant C_{p,w} \int [Gf(x)]^p w(x)\, dx$$

($w \in A_p$; $1 < p < \infty$). Putting everything together, the theorem is proved for well distributed sequences of intervals.

Now, for the reduction to the well-distributed case, we argue as in §2, and we only need to prove the weighted analogue of 2.3, namely

Lemma 6.3. *Given a sequence of disjoint intervals $\{I_k\}$, let $W(I_k)$ be the Whitney decomposition of each I_k. Then, for all $w \in A_p$, $1 < p < \infty$, we have the equivalence*

$$\Big\|\Big(\sum_k |S_{I_k} f|^2\Big)^{1/2}\Big\|_{L^p(w)} \sim \Big\|\Big(\sum_k \sum_{H \in W(I_k)} |S_H f|^2\Big)^{1/2}\Big\|_{L^p(w)}$$

for every $f \in L^p(w)$.

PROOF. Let Δf be defined as in (1.1), and let $\Delta_k f$ be the corresponding operator for the sequence $W(I_k)$. As we mentioned in Lemma 2.3, the operators Δ_k are uniformly bounded in $L^2(w)$ if $w \in A_2$, and more precisely (see [10]) if $\mathrm{supp}(\hat{f}) \subset I_k$

$$C_w^{-1} \int |f|^2 w \leqslant \int (\Delta_k f)^2 w \leqslant C_w \int |f|^2 w \qquad (w \in A_2)$$

with C_w independent of k. By the extrapolation theorem for A_p-weights (see [8], [13]) this implies

$$\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{L^p(w)} \sim \Big\|\Big(\sum_k [\Delta_k f_k]^2\Big)^{1/2}\Big\|_{L^p(w)}$$

for all $w \in A_p$ and $f_k \in L^p(w)$, $1 < p < \infty$, and taking $f_k = S_{I_k} f$ we get the desired equivalence

$$\|\Delta f\|_{L^p(w)} \sim \Big\|\Big(\sum_k [\Delta_k f]^2\Big)^{1/2}\Big\|_{L^p(w)}$$

which completes the proof of the lemma and the theorem.

The theorem is best possible for $p > 2$ in the sense that Δ cannot be bounded in $L^p_{(w)}$ for all $w \in A_q$ if $q > p/2$ (since this would imply that Δ is bounded in $L^{2-\epsilon}(\mathbb{R})$, which is false for some sequences $\{I_k\}$ of intervals). It is natural to expect, however, that

$$\int \sum_k |S_{I_k} f|^2 w \leqslant C_w \int |f|^2 w \qquad (w \in A_1) \tag{6.4}$$

for every sequence $\{I_k\}$ of disjoint intervals, since this is the limiting case of 6.1, and it is known to be true in the extremal cases considered in (1.4) and (1.5). It would suffice to obtain the same inequality for the smooth operator G, but the basic estimate: $(Gf)^{\#} \leqslant CM_2 f$ is not enough to prove it.

7. Some results in L^p, $p < 2$

Given a sequence $\{I_k\}$ of disjoint intervals, one may ask more generally for which values of p and q does the inequality

$$\Big\|\Big(\sum_k |S_{I_k} f|^q\Big)^{1/q}\Big\|_p \leqslant C\|f\|_p \tag{7.1}$$

hold. The example in (1.5) shows that a necessary condition (not only for arbitrary $\{I_k\}$, but even for equal length intervals) is: $q \geqslant \max(2, p')$. Thus, we have proved in Theorem 1.2 the best possible result for $2 \leqslant p < \infty$, and it is natural to expect that, for $1 < p < 2$, the best possible inequality is also true, namely.

Conjecture 7.2. *For arbitrary disjoint intervals* $\{I_k\}$ *and for* $1 < p < 2$, *the inequality*

$$\Big\|\Big(\sum_k |S_{I_k} f|^{p'}\Big)^{1/p'}\Big\|_p \leqslant C_p\|f\|_p$$

holds for *every* $f \in L^p(\mathbb{R})$.

As supporting evidence for this conjecture, apart from Theorem 1.2, we mention two partial results:

a) If $\{I_k\}$ is well distributed, $f \longrightarrow \left(\sum_k |S_{I_k}f|^{p'}\right)^{1/p'}$ is an operator of weak type (p,p), $1 < p < 2$.

b) If $1 < p < 2$ and $q > p'$ then (7.1) holds for arbitrary disjoint intervals $\{I_k\}$.

PROOF OF (*a*). The Hilbert transform H admits a vector valued extension: $\tilde{H}((f_k)) = (Hf_k)$ which is bounded in $L^p(l^q)$ for all $1 < p, q < \infty$, and expressing every partial sum operator in terms of H (as in [16], for instance) we obtain

$$\left\|\left(\sum_k |S_{I_k}f_k|^q\right)^{1/q}\right\|_p \leqslant C_{p,q}\left\|\left(\sum_k |f_k|^q\right)^{1/q}\right\|_p \tag{7.3}$$

Now, we define ψ_k so that $\hat{\psi}_k$ is adapted to I_k, i.e.

$$\chi_{I_k} \leqslant \hat{\psi}_k \leqslant \chi_{2I_k}$$

Moreover, all ψ_k can be defined in terms of a fixed Schwartz function ψ, so that $|\psi_k(x)| = l_k|\psi(l_k x)|$ with $l_k = |I_k|$. Then, the operator

$$f \rightarrow (\psi_k * f)_{k\in\mathbb{N}}$$

is bounded from L^1 to weak-$L^1(l^\infty)$, because $\sup_k |\psi_k * f| \leqslant CMf$, and it is also bounded from L^2 to $L^2(l^2)$ due to the fact that the intervals $\{I_k\}$ are well distributed. By interpolation

$$\left\|\left(\sum_k |\psi_k * f|^{p'}\right)^{1/p'}\right\|_{p,\infty} \leqslant C_p \|f\|_p \qquad (1 < p < 2) \tag{7.4}$$

and we only have to apply (7.3) with $f_k = \psi_k * f$ and $q = p'$.

PROOF OF (*b*). We interpolate between the obvious inequality

$$\left\|\left(\sum_k |S_{I_k}f|^2\right)^{1/2}\right\|_2 \leqslant \|f\|_2 \qquad (f \in L^2)$$

and the following consequence of the Carleson-Hunt theorem ([2], [9])

$$\left\|\sup_k |S_{I_k}f|\right\|_{1+\varepsilon} \leqslant C_\varepsilon \|f\|_{1+\varepsilon} \qquad (f \in L^{1+\varepsilon}; \varepsilon > 0).$$

8. n-dimensional results

By an interval in $\mathbb{R}^n$, we shall mean the product of n one-dimensional intervals: $I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$. We would like to state the analogue of Theorem 1.2 for an arbitrary sequence of disjoint intervals in $\mathbb{R}^n$, but in order to adapt the argument developed in sections §2 – §5, we should need a lemma similar to 4.1 for product-type vector-valued kernels. No such result seems to be known so far, though one may hope that the methods of [7] could be suitably modified to this end.

What one can prove by standard reiteration techniques is a theorem for «cross-partitions»: *A cross-partition of* $\mathbb{R}^n$ *is a family* $\{I_k\}_{k\in\mathbb{N}^n}$ *of disjoint n-dimensional intervals such that*

$$I_k = I^{(1)}_{k_1} \times I^{(2)}_{k_2} \times \ldots \times I^{(n)}_{k_n} \qquad (k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n)$$

where, for each $i = 1, 2, \ldots, n$, *the sequence of intervals* $\{I^{(i)}_j\}_{j\in\mathbb{N}}$ *form a partition of* $\mathbb{R}$.

Theorem 8.1. *If* $\{I_k\}_{k\in\mathbb{N}^n}$ *is a cross-partition of* $\mathbb{R}^n$ *and* $2 \leqslant p < \infty$, *then for all* $f \in L^p(\mathbb{R}^n)$

$$\left\| \left(\sum_k |S_{I_k} f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p$$

PROOF. For notational simplicity, we shall assume $n = 2$. Let $I_{j,k} = I'_j \times I''_k (j, k \in \mathbb{N})$ be the given family of intervals, and let $S_{j,k}$, S'_j and S''_k denote, respectively, the partial sum operators in $\mathbb{R}^2$ corresponding to the intervals $I_{j,k}$, $I'_j \times \mathbb{R}$ and $\mathbb{R} \times I''_k$. By the one-dimensional result and Fubini's theorem, we have

$$\left\| \left(\sum_j |S'_j f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p \qquad (f \in L^p; 2 \leqslant p < \infty) \tag{8.2}$$

and similarly for S''_k, $k \in \mathbb{N}$. Thus, the operator

$$S'' : f \to S'' f = (S''_k f)_{k\in\mathbb{N}}$$

is bounded from $L^p(\mathbb{R}^2)$ to $L^p_H(\mathbb{R}^2)$, where $H = l^2$, and the theorem of Marcinkiewicz and Zygmund [12] (which is also valid for Hilbert space-valued functions) gives

$$\int \left(\sum_j \|S'' f_j(x)\|^2_H \right)^{p/2} dx \leqslant C^p_p \int \left(\sum_j |f_j(x)|^2 \right)^{p/2} dx \tag{8.3}$$

Now, given $f \in L^p(\mathbb{R}^2)$, $2 \leqslant p < \infty$, we apply (8.3) with $f_j = S'_j f$ taking into account (8.2) and the fact that $S''_k S'_j f = S_{j,k} f$.

The same inequality holds in $L^p(w)$ if $w \in A^*_{p/2}$ = [$A_{p/2}$ – weights with respect to all n-dimensional intervals], $2 < p < \infty$. Another partial result is the following.

Theorem 8.4. *Let* $\{Q_j\}$ *be a sequence of well distributed cubes (in the sense of* 2.1) *in* $\mathbb{R}^n$. *Then*

$$\left\| \left(\sum_j |S_{Q_j} f|^2 \right)^{1/2} \right\|_p \leqslant C_p \| f \|_p \qquad (2 \leqslant p < \infty)$$

The proof is a repetition of the arguments in §3, §4 and §5. More generally, if l_i, q_i are fixed positive numbers, one can prove the same result for a family of intervals $\{I_j\}$ such that I_j has dimensiones $l_1 \delta_j^{q_1} \times l_2 \delta_j^{q_n}$ for some $\delta_j > 0$. In this case, the definition of well distributed sequence is made in terms of the non-isotropic dilations: $\delta \cdot x = (\delta^{q_1} x_1, \delta^{q_2} x_2, \ldots, \delta^{q_n} x_n)$.

By putting both theorems together and using the general arguments of §2 (see also [14]), one can find a huge variety of configurations of intervals in $\mathbb{R}^n$ for which the inequality stated in 8.1 turns out to be true, but the general n-dimensional analogue of Theorem 1.2 seems to be still out of reach.

Added in proof. Since the result proved in this paper was known, several authors became interested in it making some contributions. Thus, another proof of the basic estimte (3.5) was given by P. Sjölin, and a different approach to the problem was found by J. Bourgain yielding, for a sequence of disjoint intervals covering $\mathbb{R}$, the inequality

$$\| f \|_p \leqslant \left\| \left(\sum_k |S_{I_k} f|^2 \right)^{1/2} \right\|_p \qquad (1 \leqslant p \leqslant 2)$$

(which, for $1 < p \leqslant 2$, is equivalent to theorem 1.2). Finally, J. L. Journé has been able to prove recently the general n-dimensional version of our theorem, namely, the analogue of theorem 8.1 for an arbitrary partition of $\mathbb{R}^n$ into n-dimensional intervals.

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José L. Rubio de Francia
División de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid