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Polynomial Invariants of 2-component Links

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1. Introduction

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this paper, we will define two different types of polynomials which are ambient isotopic invariants of L. One is associated with a cyclic cover branched along one of their components, and the other is associated with a metabelian cover of L. These invariants are defined for any link unless the linking number, $lk(X, Y)$, is ± 1.

The invariants a_n^*, h_n^* defined in [5] can be considered one of the special cases of our polynomial invariants. In fact, we can prove that a_n^* depends only on $lk(X, Y)$; therefore, for all n, a_n^* coincides for two links with the same linking number. (See Theorem 5.7.)

It should be noted that our metabelian representation of the link group differs completely from those studied in [2], [3], [9] or [10], where in most of the cases there exist only finitely many metabelian representations. We will prove in this paper that every 2-component link L with $lk(X, Y) \neq \pm 1$ has infinitely many metabelian coverings. In particular, if $lk(X, Y)$ is even, then the link group $G(L)$ has a representation on the dihedral group of order 2^{k+1} for each $k \geqslant 1$. (See Proposition 3.3 and Theorem 4.1.) Our polynomials are, in fact, the covering linkage invariants associated with these (infinite) sequences of coverings.

In this paper, some of the basic formulas involving Fox free differential calculus [1] will be used without proofs, since they have already been proved in [9] or are easy consequences of the results in [9].

2. Group Actions

Let F be a free group (of rank 2) freely generated by x and y.

Let $P = Z[[t]]$ be the ring of formal power series over the ring of integers Z. Denote by $Sym(S)$ the group of permutations on a set S.

Associated with an ordered sequence $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ of 1 or 2 is an action ϕ of F on P; that is, a homomorphism $\phi\colon F \to Sym(P)$ defined as follows:

$$\phi(x)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + \delta(i)a_{i-1})t^i$$
$$\phi(y)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + (1 - \delta(i))a_{i-1})t^i \tag{2.1}$$

where $\delta(i) = 1$ or 0 according to $j_i = 1$ or 2.

Throughout this paper, we do not distinguish between an action $\phi\colon F \times P \to P$ and the homomorphism $\phi\colon F \to Sym(P)$, associated with ϕ, and therefore the same symbol will be used.

EXAMPLE 2.1. Let $s = \{1, 2, 1, 2, \ldots\}$, where $j_k = 1$ if and only if k is odd. Then for $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P$,

$$\phi(x)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i} t^{2i+1}$$
$$\phi(y)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i+1} t^{2i+2}.$$

Using Fox free derivative [1], we can now express the action ϕ more precisely. The following proposition is essentially Proposition 3.1 in [9].

Proposition 2.1. *Let $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ be an ordered sequence of* 1 *or* 2. *For $u \in F$, write $\phi(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i$. Then*

(1) $b_0 = a_0$,

(2) For $q \geqslant 1$,

$$b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial z_q}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial z_{q-1}\partial z_q}\right)^o + \ldots + a_0\left(\frac{\partial^q u}{\partial z_1 \ldots \partial z_q}\right)^o, \quad (2.2)$$

where z_i is x or y according to $j_i = 1$ or 2, and o denotes the trivializer.

For the proof, see Proposition 3.1 in [9].

Let $\hat{F}$ be the free group freely generated by $\{x_{f(t)}, y_{f(t)} \mid f(t) \in P\}$. Let $\{\mathfrak{D}_f(t)\}$ be the Reidemeister-Schreier rewriting function of $\hat{F}$ associated with the action ϕ ([4] or [9]). $\mathfrak{D}_{f(t)}: F \to \hat{F}$ is characterized by the following two properties: For any $f(t) \in P$ and $u, v \in F$,

(1) $\mathfrak{D}_{f(t)}(x) = x_{f(t)}$ and $\mathfrak{D}_{f(t)}(y) = y_{f(t)}$,
(2) $\mathfrak{D}_{f(t)}(uv) = \mathfrak{D}_{f(t)}u \cdot \mathfrak{D}_{\phi(u)f(t)}v$. (2.3)

The following properties will easily be proved from (2.3).

(1) $\mathfrak{D}_{f(t)}(u^{-1}) = (\mathfrak{D}_{\phi(u^{-1})f(t)}(u))^{-1}$
(2) If $\phi(uv^{-1}) = 1$, then $\mathfrak{D}_{f(t)}(uv^{-1}) = (\mathfrak{D}_{f(t)}u) \cdot (\mathfrak{D}_{f(t)}v)^{-1}$. (2.4)

Now let $\psi: \hat{F} \to P$ be a homomorphism defined by $\psi(x_{f(t)}) = 0$ and $\psi(y_{f(t)}) = f(t)$.

Proposition 2.2. *Let $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ be an ordered sequence of* 1 *or* 2. *For $u \in F$ and $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P$, write $\psi\mathfrak{D}_{f(t)}u = \sum_{i=0}^{\infty} b_i t^i$. Then*

(1) $b_0 = a_0$,
(2) For $q \geqslant 1$,

$$b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial z_q \partial y}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial z_2 \ldots \partial z_q \partial y}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial z_1 \ldots \partial z_q \partial y}\right)^o, \quad (2.5)$$

where z_i is x or y according to $j_i = 1$ or 2.

For a proof, see Proposition 6.1 in [9].

In this paper we are particularly interested in the group action associated with a sequence $\{1, 1, \ldots, 1, \ldots\}$ or $\{1, 2, 2, \ldots, 2, \ldots\}$. Our approach, however, is different from what we did in [9] and hence, we obtain different representations of the link groups. To be more precise, we define two actions σ and τ of F on P.

Definition 2.1. *For $f(t) \in P$, define*

$$\begin{aligned} \sigma(x)f(t) &= (1+t)f(t) \\ \sigma(y)f(t) &= f(t) \end{aligned} \tag{2.6}$$

$$\begin{aligned} \tau(x)f(t) &= f(t) + f(0)t \\ \tau(y)f(t) &= (1+t)f(t) - f(0)t. \end{aligned} \tag{2.7}$$

Let P^* be the set of power series $f(t)$ for which $f(0) = 1$. Then σ and τ induce actions of F on P^*, since for any $u \in F$, $[\sigma(u)f(t)]_{t=0} = [\tau(u)f(t)]_{t=0} = 1$. Furthermore, let $q(t)$ be an element of P and $\langle q(t) \rangle$ the ideal of P generated by $q(t)$. Denote by R the quotient ring $P/\langle q(t) \rangle$.

Proposition 2.3.
(1) *σ induces an action $\bar{\sigma}$ of F on R.*
(2) *If $q(0) = 0$, then τ induces an action $\bar{\tau}$ of F on R.*

PROOF. It suffices (and is easy) to show that $\langle q(t) \rangle$ is closed under the actions σ and τ.

Remark 2.1. Let $q(t) = \sum_{i=0}^{\infty} s_i t^i$ and let s_m be the first non zero coefficient of $q(t)$. Then every element $\bar{f}(t)$ in R has a unique representative $\bar{f}(t)$ of the form: $\bar{f}(t) = \bar{a}_0 + \bar{a}_1 t + \ldots + \bar{a}_k t^k + \ldots$, where $\bar{a}_0, \ldots, \bar{a}_{m-1}$ are integers, and if s_m is positive, then $\bar{a}_k (k \geqslant m)$ is a non-negative integer less than s_m, but if s_m is negative, $\bar{a}_k (k \geqslant m)$ is a non-positive integer greater than s_m. We call this unique representative $\bar{f}(t)$ the *normal* form of $\bar{f}(t)$.

EXAMPLE 2.2. Let $q(t) = 2 + 3t$. Then the normal form of $\bar{f}(t) = 3 + 6t - t^2 - 3t^3 + 3t^4$ is $1 + t + t^3$.

Since Propositions 2.1 and 2.2 for $\phi = \sigma$ or τ will be used quite extensively in this paper, it will be convenient to state them as separate propositions.

Proposition 2.4. *For $u \in F$, write*

$$\sigma(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for any $q \geqslant 0$,

$$(1)\ b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial x}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial x^2}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial x^{q-1}}\right)^o + a_0\left(\frac{\partial^q u}{\partial x^q}\right)^o$$

$$(2)\ c_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o + a_0\left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o. \tag{2.8}$$

In particular, $a_0 = b_0 = c_0$.

Proposition 2.5. *Let* $\mathfrak{D}^{\sigma}_{f(t)}$ *and* $\mathfrak{D}^{\tau}_{f(t)}$ *be the Reidemeister-Schreier rewriting functions associated with the actions* σ *and* τ, *respectively. For*

$$f(t) = \sum_{i=0}^{\infty} a_i t^i \in P,$$

write

$$\psi \mathfrak{D}^{\sigma}_{f(t)}(u) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \psi \mathfrak{D}^{\tau}_{f(t)}(u) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for $q \geqslant 0$,

$$(1)\ b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial x \partial y}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial x^{q-1}\partial y}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o,$$

$$(2)\ c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right) + \ldots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \quad (2.9)$$

3. Representations of a Free Group

For an integer n (positive, negative or 0), let

$$q_n(t) = \sum_{i=1}^{\infty} \binom{n}{i} t^i = (1+t)^n - 1.$$

As usual,

$$\binom{n}{i} \quad \text{denotes} \quad \frac{n(n-1)\ldots(n-i+1)}{i!}.$$

Lemma 3.1. *If* $m \equiv 0 \pmod{n}$, *then* $q_m(t) \equiv 0 \pmod{q_n(t)}$.

A proof is easy.

Now let $R(n)$ be the quotient ring $P/\langle q_n(t)\rangle$, and let $R^*(n)$ be the set of elements $f(t)$ in $R(n)$ such that $f(0) = 1$. Since $q_n(0) = 0$, it follows from Propositions 2.3 and 2.4 that σ and τ, respectively, induce actions σ_n and τ_n of F on $R^*(n)$. Let $\Omega_\sigma(n)$ and $\Omega_\tau(n)$ denote the orbits of 1 in $R^*(n)$ under σ_n and τ_n respectively. Namely,

$$\Omega_\sigma(n) = \{\sigma_n(u)(1) \mid u \in F\} \quad \text{and} \quad \Omega_\tau(n) = \{\tau_n(u)(1) \mid u \in F\}.$$

σ_n and τ_n define homomorphisms

$$\sigma_n: F \to Sym(\Omega_\sigma(n)) \quad \text{and} \quad \tau_n: F \to Sym(\Omega_\tau(n)).$$

Proposition 3.2.

(1) *$\sigma_n(F)$ is a cyclic group of order $|n|$.*

(2) *$\tau_n(F)$ is a metabelian group.*

PROOF. (1) Since $(1+t)^n \equiv 1 \pmod{q_n(t)}$, $\Omega_\sigma(n)$ consists of exactly $|n|$ elements $\{1, 1+t, (1+t)^2, \ldots, (1+t)^{|n|-1}\}$. Since $\sigma_n(1+t)^k = (1+t)^{k+1}$, it follows that $\sigma(x^n) = 1$, but $\sigma(x^k) \neq 1$ for $1 \leqslant k < |n|$, and hence $\sigma_n(F)$ is a cyclic group of order $|n|$.

(2) Let $G = \tau_n(F)$. As a special case of Proposition 9.1 in [9], we see that $G'' = [G', G'] = 1$. Therefore, G is metabelian.

Generally, $\Omega_\tau(n)$ is not a finite set. Therefore, to obtain a finite representation of F, we need to "truncate" higher terms of $f(t)$. Let I_{k+1} be the ideal of P generated by t^{k+1} and $q_n(t)$. Let $R_k(n) = P/I_{k+1}$ and let $R_k^*(n)$ be the set of elements $f(t)$ in $R_k(n)$ such that $f(0) = 1$. An element of $R_k^*(n)$ is a polynomial of degree at most k, and it has the (unique) normal form of degree $\leqslant k$. (See Remark 2.1.) Obviouly, τ_n induces an action $\tau_{k,n}$ of F on $R_k^*(n)$.

Let $\Omega_k(n)$ be the orbit of 1 under $\tau_{k,n}$; i.e., $\Omega_k(n) = \{\tau_{k,n}(u)(1) \mid u \in F\}$. $\tau_{k,n}$ defines a (transitive) homomorphism $\tau_{k,n}: F \to Sym(\Omega_k(n))$.

Proposition 3.3.

(1) *$\tau_{k,n}(F)$ is nilpotent of class at mos k.*

(2) *If n is a prime p, then $\tau_{k,n}(F)$ is a finite p-group.*

(3) *In particular, if $n = 2$, then $\tau_{k,n}(F)$ is the dihedral group of order 2^{k+1}.*

PROOF.

(1) follows from Proposition 3.2 in [9];

(2) since a proof will be done by an easy induction on k, the details will be omitted. Note that $(\tau_{k,p}(x))^{p^k} = 1$ and $(\tau_{k,p}(y))^p = 1$;

(3) denote $X = \tau_{k,n}(x)$ and $Y = \tau_{k,n}(y)$. Then a straight-forward calculation shows that for any $f(t) \in P^*$

$$\begin{aligned} &(1)\ X^{2^k}(f(t)) \equiv f(t) \bmod I_{k+1} \\ &(2)\ Y^2(f(t)) \equiv f(t) \bmod I_{k+1} \\ &(3)\ (XY)^2(f(t)) \equiv f(t) \bmod I_{k+1}. \end{aligned} \tag{3.1}$$

Therefore, $\tau_{k,2}(F)$ is a quotient group of the dihedral group

$$D_{2^k} = \langle A, B \mid A^{2^k} = B^2 = (AB)^2 = 1 \rangle.$$

But it is easy to see that they are, in fact, isomorphic. The details will be omitted.

Remark 3.1. We can prove, further, that for a prime p,

$$\begin{cases} \tau_{k,p}[x, y, x] = 1 \\ \tau_{k,p}[x, \underbrace{y, y, \ldots, y}_{k \text{ times}}] = 1, \end{cases} \tag{3.2}$$

where $[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1}$ and $[u_1, u_2, \ldots, u_m] = [[u_1, u_2, \ldots, u_{m-1}], u_m]$.
In particular, $\tau_{2,p}$ is isomorphic to the group

$$M(p) = \langle x, y \mid x^p = y^p = [x, y]^p = 1,\ [x, y, x] = [x, y, y] = 1 \rangle.$$

Remark 3.2. p-group representations of F obtained in Proposition 3.3 (2) and (3) are quite different from those discussed in [9, §10] or [10, §§2-3].

4. Representations of Link Groups

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this section, we will define a homomorphism from the link group $G(L)$ onto the group $\sigma_n(F)$ or $\tau_{k,n}(F)$ for various n and k.

For the first group $\sigma_n(F)$, such a homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$ always exists, since $\sigma_n(F)$ is cyclic of order $|n|$. In fact, let m_x and m_y be meridian elements of X and Y, respectively. Then for any integer n, a mapping $\Sigma_n: G(L) \to \sigma_n(F)$ defined by

$$\begin{cases} \Sigma_n(m_x) = \sigma_n(x) \\ \Sigma_n(m_y) = \mathrm{id} \end{cases} \tag{4.1}$$

gives a required homomorphism. However, it will be seen later that Σ_n is only interesting in our purpose when n divides $lk(X, Y)$, the linking number between X and Y.

On the other hand, the second group $\tau_{k,n}(F)$ is not an obvious group. In fact, $\tau_{k,n}(F)$ is metabelian, but not abelian. Nevertheless, for any k, we can find a homomorphism from $G(L)$ onto $\tau_{k,n}(F)$ when $lk(X, Y)$ is divisible by n.

In this section, we will prove the following theorem.

Theorem 4.1. *Let n be an integer. Suppose $lk(X, Y) \equiv 0 \pmod n$. Then for each $k \geqslant 1$, there is a homomorphism*

$$T_{k,n}: G(L) \to \tau_{k,n}(F) \subset Sym(\Omega_k(n))$$

such that $T_{k,n}(m_x) = \tau_{k,n}(x)$ *and* $T_{k,n}(m_y) = \tau_{k,n}(y)$. n *can be* 0 *only when* $lk(X, Y) = 0$.

PROOF. Since there is no essential difference in proving the theorem we may assume that n is a positive integer. Also we may assume *w.l.o.g.* that $lk(X, Y) \geqslant 0$.

Now, if $k = 1$, then $\tau_{k,n}(F)$ is a cyclic group of order n, and the theorem is trivially true. Therefore, we assume that $k \geqslant 2$.

Let $G(L) = \langle x_i, y_j \mid r_i, s_j \rangle$, $1 \leqslant i \leqslant \lambda$, $1 \leqslant j \leqslant \mu$, be a "modified" Wirtinger presentation of $G(L)$ in the following sense.

x_1 and y_1 correspond to prescribed meridian elements m_x and m_y, respectively, and relators are of the form

$$\begin{cases} r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ r_\lambda = \eta x_1 \eta^{-1} x_1^{-1}, & \end{cases}$$

$$\begin{cases} s_j = v_j y_1 v_j^{-1} y_{j+1}^{-1}, & 1 \leqslant j \leqslant \mu - 1 \\ s_\mu = \xi y_1 \xi^{-1} y_1^{-1}, & \end{cases}$$

where u_i, v_j are words in $\{x_i, y_j\}$, and η and ξ represent longitudes of X and Y, respectively, so that $\{x_1, \eta\}$ and $\{y_1, \xi\}$ form peripheral subgroups of $G(L)$.

Let F^* be the free group freely generated by $\{x_i, y_j, 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu$. As before, F denotes the free group $\langle x, y \mid \rangle$. Let $\rho: F^* \to F^*$ and $\nu: F^* \to F$ be homomorphisms defined by

$$\begin{cases} \rho(x_1) = x \quad \text{and} \quad \rho(y_1) = y \\ \rho(x_{i+1}) = u_i x_1 u_i^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ \rho(y_{j+1}) = v_j v_1 v_j^{-1}, & 1 \leqslant j \leqslant \mu - 1. \end{cases}$$

$$\begin{cases} \nu(x_i) = x, & 1 \leqslant i \leqslant \lambda \\ \nu(y_j) = y, & 1 \leqslant j \leqslant \mu. \end{cases}$$

Using ρ and ν, we define the third homomorphism

$$\theta_{k+1} = \nu\rho^k: F^* \to F \quad \text{for} \quad k \geqslant 0.$$

(θ_{k+1} will be called the Chen-Milnor homomorphism.)

Let $T = \tau_{k,n}\theta_{k+1}$ be a homomorphism from F^* to $\tau_{k,n}(F)$. Then T will induce the homomorphism $T_{k,n}: G(L) \to \tau_{k,n}(F)$ if

$$T(r_i) = 1, \qquad 1 \leqslant i \leqslant \lambda, \quad \text{and} \quad T(s_j) = 1, \qquad 1 \leqslant j \leqslant \mu. \tag{4.2}$$

Now, Proposition 5.1 in [9] proves (4.2) except the last two relations $T(r_\lambda) = 1$ and $T(s_\mu) = 1$. Therefore, it only remains to show that

$$T[\eta, x_1] = 1 \quad \text{and} \quad T[\xi, y_1] = 1. \tag{4.3}$$

Since one of the relations in (4.2) is redundant, it is enough to show that $T[\eta, x_1] = 1$.

For simplicity, write $\theta_{k+1}(\eta) = h$. Since $\theta_{k+1}(x_1) = x$, it suffices to prove that

$$\tau_{k,n}(hx) = \tau_{k,n}(xh). \tag{4.4}$$

Denote $u = hx$ and $w = xh$, and write

$$\tau_{k,n}(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau_{k,n}(w)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then, since $a_0 = 1$, it follows from (2.8) (2) that for $q \geqslant 1$,

$$\text{(1)}\ b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y}\right)^o + \dots + a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o \tag{4.5}$$

and

$$\text{(2)}\ c_q = a_q + a_{q-1}\left(\frac{\partial w}{\partial y}\right)^o + \dots + a_1\left(\frac{\partial^{q-1} w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

Now

$$\left(\frac{\partial^r u}{\partial y^r}\right)^o = \left(\frac{\partial^r w}{\partial y^r}\right)^o = \binom{m}{r}, \quad \text{where} \quad m = \left(\frac{\partial h}{\partial y}\right)^o = lk(X, Y),$$

[1], and $(\partial h/\partial x)^o = 0$. Further,

$$\left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o$$

and

$$\left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(x\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, it follows from (4.5) that

$$b_0 = c_0 = 1, \qquad b_1 = \left(\frac{\partial u}{\partial x}\right)^o = 1 \quad \text{and} \quad c_1 = \left(\frac{\partial w}{\partial x}\right)^o = 1,$$

and hence,

$$\sum_{q=0}^{\infty} c_q t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \left(\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)^o t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \binom{m}{q-1} t^q.$$

Since n devides m by the assumption, it follows from Lemma 3.1 that

$$\sum_{q=2}^{\infty} \binom{m}{q-1} t^q \equiv 0 \pmod{q_n(t)},$$

and hence

$$\sum_{q=0}^{\infty} c_q t^q \equiv \sum_{q=0}^{\infty} b_q t^q \pmod{I_{k+1}}.$$

This proves Theorem 4.1.

Now, let $g = \theta_{k+1}(\xi)$ and $v = gy$ and $z = yg$. Note that $(\partial g/\partial y)^o = 0$. Since $T_{k,n}[\xi, y_1] = 1$, we have $\tau_{k,n}(v) = \tau_{k,n}(z)$. Write

$$\tau_{k,n}(v)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i, \qquad a_0 = 1$$

and

$$\tau_{k,n}(z)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i, \qquad a_0 = 1.$$

Then by (2.8) (2) we obtain

$$(1)\ a_0 = b_0 = c_0 = 1 \tag{4.6}$$

For $q \geqslant 1$,

$$(2)\ b_q = a_q + a_{q-1}\left(\frac{\partial v}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} v}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o$$

and

$$(3)\ c_q = a_q + a_{q-1}\left(\frac{\partial z}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} z}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o.$$

Since $(\partial v/\partial y)^o = (\partial z/\partial y)^o = 1$, it follows that for $q \geqslant 1$,

$$\left(\frac{\partial^q v}{\partial y^q}\right)^o = \left(\frac{\partial^q z}{\partial y^q}\right)^o = \binom{1}{q}$$

and hence

$$\left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o + \left(\frac{\partial^{q-1} g}{\partial x \partial y^{q-2}}\right)^o$$

and

$$\left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, $\tau_{k,n}(v) = \tau_{k,n}(z)$ yields the following

Proposition 4.2. *Let* $g = \theta_{k+1}(\xi)$. *Then for* $k \geqslant 1$,

$$\left(\frac{\partial g}{\partial x}\right)^o t^2 + \left(\frac{\partial^2 g}{\partial x \partial y}\right)^o t^3 + \dots + \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o t^{q+1} + \dots \equiv 0 \qquad (\text{mod } I_{k+1}). \tag{4.7}$$

Remark 4.1. A homomorphism $\Sigma_n \colon G(L) \to \sigma_n(F)$ is formally given as follows. First, define a homomorphism $\Sigma^* \colon F^* \to \sigma_n(F)$ by

$$\begin{cases} \Sigma^*(x_i) = \sigma_n(x) & \text{for} \quad i = 1, 2, \dots, \lambda, \\ \Sigma^*(y_j) = id & \text{for} \quad j = 1, 2, \dots, \mu. \end{cases} \tag{4.8}$$

Then $\Sigma^*(r_i) = \Sigma^*(s_j) = 1$ for any i, j. Therefore, Σ^* induces the homomorphism $\Sigma_n \colon G(L) \to \sigma_n(F)$. This rather obvious observation will be used in the next section.

5. Covering Space (I) Cyclic Covering

In the previous section we found representations Σ_n and $T_{k,n}$ of $G(L)$ on $\sigma_n(F)$ and $\tau_{k,n}(F)$.

To each finite representation ϕ we can associate a (unbranched) covering space M. Let $U(X)$ and $U(Y)$ denote tubular neighborhoods of X and Y in S^3, respectively. Then the covering space M associated with ϕ is a compact 3-manifold with boundary consisting of tori.

Suppose we have a homomorphism

$$\phi \colon \pi_1(M) \to A$$

from $\pi_1(M)$ to an abelian group A. Then ϕ induces the homomorphism $\hat{\phi}$ from $H_1(M)$ to A. The most characteristic element of $H_1(M)$ is a "longitude" $\tilde{\xi}$ of each boundary torus of M. In many cases, such a "longitude" can be realized as a "lift" of a longitude ζ of $\partial U(X)$ or $\partial U(Y)$, and then $\phi(\hat{\zeta})$ will be an invariant of the original link type L. By taking A as the polynomial ring $R_k(n)$, we will obtain our polynomial invariants.

In this section, we define such polynomial invariants for a finite representation $\Sigma_n: G(L) \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$.

Let M_σ be the (unbranched) covering space of $S^3 - L$ associated with Σ_n. M_σ is in fact the n-fold cyclic covering space of $S^3 - X$.

Let $\mathfrak{D}^{*\sigma}_{f(t)}$ be the Reidemeister-Schreier rewriting function associated with the action $\Sigma^*: F^* \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$, where Σ^* is defined in Remark 4.1. (4.8).

Now the set $S_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_i), \mathfrak{D}^{*\sigma}_{f(t)}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_\sigma(n)\}$ generates a free group F^*_σ and $\pi_1(M_\sigma)$ has a presentation $\langle S_\sigma: R_\sigma, U_\sigma\rangle$ where $R_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(r_i), \mathfrak{D}^{*\sigma}_{f(t)}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega(n)\}$ and $U_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_1), f(t) \in \Omega_\sigma(n)\}$.

Theorem 5.1. *Let n be an integer. Suppose $lk(X, Y) \equiv 0 \pmod n$. Then for $k \geqslant 1$, there exists a homomorphism $\Phi_\sigma: \pi_1(M_\sigma) \to R_k(n)$ such that for any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = 0 \quad \textit{and} \quad \Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = f(t).$$

Remark 5.1. n can be 0 only if $lk(X, Y) = 0$, and then M_σ is an infinite cyclic covering space of $S^3 - X$.

PROOF OF THEOREM 5.1. To prove the theorem it suffices to define a homomorphism $\Phi^*_\sigma: F^*_\sigma \to R_k(n)$ such that $\Phi^*_\sigma(w) = 0$ for $w \in R_\sigma$ or U_σ.

Now let F_σ be the free group freely generated by a set $\{x_{f(t)}, y_{f(t)} \mid f(t) \in \Omega_\sigma(n)\}$, and let ψ_σ be a homomorphism from F_σ to $R_k(n)$ given by

$$\psi_\sigma(x_{f(t)}) = 0 \quad \text{and} \quad \psi_\sigma(y_{f(t)}) = f(t). \tag{5.2}$$

Using ψ_σ, we define, for $f(t) \in \Omega_\sigma(n)$ and for any i, j,

$$\begin{cases} \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_i)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(x_i) \\ \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_j)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(y_j). \end{cases} \tag{5.3}$$

Note that

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(x_1) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)}(x) = \psi_\sigma(x_{f(t)}) = 0,$$

and

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)} \theta_{k+1}(y_1) = \psi_\sigma \mathfrak{D}^\sigma_{f(t)}(y) = \psi_\sigma(y_{f(t)}) = f(t),$$

and therefore, Φ_σ^* satisfies (5.1). We will prove further, for any $u \in F^*$,

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) = \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma} \theta_{k+1}(u). \tag{5.4}$$

A proof will be done by induction on the length $l(u)$ of u.

If $l(u) = 1$, then $u = x_i^{\pm 1}$ of $y_j^{\pm 1}$. If $u = x_i$ or y_j, (5.4) is trivially true. Suppose $u = x_i^{-1}$. Since $\sigma_n(x_i) = \sigma_n(\theta_{k+1}(x_i))$, it follows from (2.4) (1) that

$$\begin{aligned}\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_i^{-1}) &= \Phi_\sigma^*[\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^{*\sigma}(x_i)]^{-1} \\ &= \psi_\sigma[\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^{\sigma}\theta_{k+1}(x_i)]^{-1} \\ &= \psi_\sigma[\mathfrak{D}_{\sigma_n(\theta_{k+1}(x_i)^{-1})f(t)}^{\sigma}\theta_{k+1}(x_i)]^{-1} \\ &= \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}(\theta_{k+1}(x_i))^{-1} \\ &= \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(x_i^{-1}).\end{aligned}$$

Similarly, (5.4) holds for $u = y_j^{-1}$.

Now suppose (5.4) holds for any element u with $l(u) < d$. Let w be an element of F^* with $l(w) = d$. Then $w = ux_i^{\pm 1}$ or $uy_j^{\pm 1}$ for some u with $l(u) = d - 1$.

Consider the case $w = ux_i$. Then

$$\begin{aligned}\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(w) &= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(ux_i) \\ &= \Phi_\sigma^*[(\mathfrak{D}_{f(t)}^{*\sigma}u) \cdot (\mathfrak{D}_{\sigma_n(u)f(t)}^{*}(x_i))] \\ &= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) + \Phi_\sigma^* \mathfrak{D}_{\sigma_n(u)f(t)}^{*\sigma}(x_i) \\ &= \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(u) + \psi_\sigma \mathfrak{D}_{\sigma_n(u)f(t)}^{\sigma}\theta_{k+1}(x_i) \\ &= \psi_\sigma[\mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(u) \cdot \mathfrak{D}_{\sigma_n(u)f(t)}^{\sigma}\theta_{k+1}(x_i)].\end{aligned}$$

Since $\sigma_n(u) = \sigma_n(\theta_{k+1}(u))$, the last expression becomes

$$\psi_\sigma[\mathfrak{D}_{f(t)}^{\sigma}(\theta_{k+1}(u) \cdot \theta_{k+1}(x_i))] = \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(ux_i). \tag{5.5}$$

Since similar computations provide the proofs for other cases, the details will be omitted.

Now we must show

$$\begin{aligned}\Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma} r_i) = 0, \qquad & i = 1, 2, \ldots, \lambda \\ \Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma} s_j) = 0, \qquad & j = 1, 2, \ldots, \mu.\end{aligned} \tag{5.6}$$

First consider $r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}$, $1 \leqslant i < \lambda$. Since $\theta_{k+1}(x_{i+1}) \equiv \theta_{k+2}(x_{i+1})$ $(\mathrm{mod}\, F_{k+2})$ by Proposition 5.1 in [9], Propositions 2.5 and (5.1) in [1] imply that

$$\psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(x_{i+1}) = \psi_\sigma \mathfrak{D}_{f(t)}^{\sigma}\theta_{k+2}(x_{i+1}).$$

Since $\theta_{k+2}(x_{i+1}) = \theta_{k+1}(u_i x_1 u_i^{-1})$ by the definition, we obtain

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u_i x_1 u_i^{-1}),$$

and hence

$$\Phi_\sigma^*[(\mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}))(\mathfrak{D}_{f(t)}^{*\sigma}(u_i x_1 u_i^{-1}))^{-1}] = 0,$$

which is equal to

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}(u_i x_1 u_i^{-1})^{-1}) = 0,$$

by (2.4) (2).

Similarly, we can prove $\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(s_j) = 0$ for $j \neq \mu$.

Now it remains to show that

$$\begin{aligned}&(1)\ \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}[\eta, x_1] = 0, \quad \text{or} \\ &(2)\ \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}[\xi, y_1] = 0.\end{aligned} \tag{5.7}$$

Since $\sigma_n[\xi, y_1] = id$, (5.7) (2) is equivalent to

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi y_1) = \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(y_1 \xi). \tag{5.8}$$

To prove (5.8), we compute both sides separately. Note that $\theta_{k+1}(y_1) = y$ and $\sigma_n(y_1) = id$. Then

$$\begin{aligned}\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi y_1) &= \Phi_\sigma^*[\mathfrak{D}_{f(t)}^{*\sigma}(\xi) \cdot \mathfrak{D}_{\sigma_n(\xi)f(t)}^{*\sigma}(y_1)] \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \Phi_\sigma^*\mathfrak{D}_{\sigma_n(\xi)f(t)}^{*\sigma}(y_1) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \psi_\sigma\mathfrak{D}_{\sigma_n(\xi)f(t)}^{\sigma}\theta_{k+1}(y_1) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \sigma_n(\xi)f(t).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(y_1\xi) &= \Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma}y_1 \cdot \mathfrak{D}_{\sigma_n(y_1)f(t)}^{*\sigma}\xi) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}y_1 + \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) \\ &= f(t) + \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi).\end{aligned}$$

Therefore, it suffices to prove

$$f(t) \equiv \sigma_n(\xi)f(t) \qquad (\operatorname{mod} I_{k+1}). \tag{5.9}$$

Since $m = lk(X, Y) \equiv 0 \pmod n$, it follows from Lemma 4.1 that $q_m(t) \equiv 0$ $(\operatorname{mod} q_n(t))$. Also, since $\sigma_n(\xi) = \sigma_n(x^m)$, $\sigma_n(x^m)f(t) = (1+t)^m f(t) \equiv f(t)$ $(\operatorname{mod} q_n(t))$.

This now completes the proof of Theorem 5.1.

A straightforward computation of the relation $\Phi_\sigma^*\mathfrak{D}_{f(t)}^*[\eta, x_1] = 0$ yields the following

Corollary 5.2. *Let $h = \theta_{k+1}(\eta)$. Then*

$$\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} h}{\partial x^q \partial y}\right)^o t^{q+1} \equiv 0 \qquad (\operatorname{mod} I_{k+1}). \tag{5.10}$$

Remark 5.2. (5.10) can be considered as the "dual" form to (4.7).

Now before our polynomial invariants are introduced, we need a few propositions.

Proposition 5.3. *For any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\eta) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta).$$

PROOF. Let $u = \theta_{k+1}(\eta)$. Then $\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma(u)$. For $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P^*$, write $\psi_\sigma \mathfrak{D}_1^\sigma(u) = \sum_{i=0}^{\infty} b_i t^i$ and $\psi_\sigma \mathfrak{D}_{f(t)}^\sigma(u) = \sum_{i=0}^{\infty} c_i t^i$.

Then Proposition 2.5 yields, since $a_0 = 1$,

$$(1) \quad \sum_{i=0}^{\infty} b_i t^i = \sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o t^q$$

$$(2) \quad \sum_{i=0}^{\infty} c_i t^i = \sum_{j=1}^{\infty} a_j \left[\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o t^{q+j}\right] + \sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o t^q. \tag{5.11}$$

By (5.10), for $j > 0$,

$$\sum_{q=0}^{\infty} \left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o t^{q+j} \equiv 0 \qquad (\operatorname{mod} q_n(t))$$

and hence,

$$\sum_{i=0}^{\infty} b_i t^i \equiv \sum_{i=0}^{\infty} c_i t^i \qquad (\operatorname{mod} q_n(t)).$$

This proves Proposition 5.3.

Proposition 5.4. *For any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\xi) = f(t)[\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi)].$$

PROOF. Since $f(t) \in \Omega_\sigma(n)$, $f(t)$ is of the form $(1+t)^r$ for some $0 \leqslant r < n$. (We assume that n is non-negative.)

Write

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) = \psi_\sigma\mathfrak{D}_{f(t)}^{\sigma}\theta_{k+1}(\xi) = \sum_{i=0}^{\infty} c_i t^i \text{ and } \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma\mathfrak{D}_1^{\sigma}\theta_{k+1}(\xi) = \sum_{i=0}^{\infty} b_i t^i.$$

Denote $w = \theta_{k+1}(\xi)$. Then Proposition 2.5 yields again

$$(1)\ \sum_{i=0}^{\infty} b_i t^i = \sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^q, \quad \text{and}$$

$$(2)\ \sum_{i=0}^{\infty} c_i t^i = \sum_{j=0}^{\infty} a_j\left[\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^{q+j}\right]. \tag{5.12}$$

Since

$$f(t) = (1+t)^r = \sum_{i=0}^{\infty} a_i t^i \quad \text{and} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0,$$

it follows from (5.12) (1) (2) that

$$\begin{aligned} f(t)\sum_{i=0}^{\infty} b_i t^i &= \sum_{j=0}^{\infty} a_j t^j\left(\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^q\right) \\ &= \sum_{j=0}^{\infty} a_j\left[\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}w}{\partial x^q \partial y}\right)^o t^{q+j}\right] \\ &= \sum_{i=0}^{\infty} c_i t^i. \end{aligned}$$

Definition 5.1. *Let n be an integer that divides $lk(X, Y)$. Then for any integer $k \geqslant 1$, define the polynomials $\eta_k^{(n)}(t)$ and $\xi_k^{(n)}(t)$ in $R_k(n)$ by*

$$\begin{aligned} \eta_k^{(n)}(t) &= \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\eta), \quad \textit{and} \\ \xi_k^{(n)}(t) &= \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\xi). \end{aligned} \tag{5.13}$$

Theorem 5.5. *For any $k \geqslant 1$, $\eta_k^{(n)}(t)$ and $\{f(t)\xi_k^{(n)}(t), f(t) \in \Omega_\sigma(n)\}$ are invariants of an oriented link type L.*

Remark 5.3. For any $u \in F$, $\theta_{k+1}(u) \equiv \theta_{k+2}(u) \bmod F_{k+2}$, and hence, for any $l \geqslant k$,

$$\begin{aligned} \eta_l^{(n)}(t) &\equiv \eta_k^{(n)}(t) \pmod{I_{k+1}} \\ \xi_l^{(n)}(t) &\equiv \xi_k^{(n)}(t) \pmod{I_{k+1}}. \end{aligned} \tag{5.14}$$

Since k can be taken arbitrarily large, these invariants are formal power series.

Corollary 5.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\eta_k^{(n)}$. Then for any $i \geqslant 0$, $\bar{a}_i = a_i^*$, where a_i^* is the invariant defined in* [5].

Now, as we did in [4] or [9], these invariants can be interpreted as the "linking number" between one component of the lifts of X or Y and the characteristic link defined in [9]. Using this geometric interpretation, we can obtain more information on $\eta_k^{(n)}(t)$.

Let $\tilde{X}$ and $\tilde{Y} = \tilde{Y}_0 \cup \ldots \cup \tilde{Y}_{n-1}$ be the lifts of X and Y, respectively, in the covering space M_σ associated with the homomorphism $\Sigma_n: G(L) \to \to \mathrm{Sym}(\Omega_\sigma(n))$.

Theorem 5.7. *Suppose $lk(X, Y) = rn$, $n > 0$. Then, for any $k \geqslant 1$,*

$$\eta_k^{(n)}(t) = r\left\{\binom{n}{1} + \binom{n}{2}t + \ldots + \binom{n}{n}t^{n-1}\right\}. \tag{5.15}$$

In particular, the invariant a_i^* defined in [5] *is completely determined by the linking number $lk(X, Y)$.*

PROOF. By [4] or [9], the characteristic link associated with the linking function or the homomorphism $\Phi_\sigma: \pi_1(M_\sigma) \to R_k(n)$ is a 1-cycle

$$\bar{Y} = \sum_{i=0}^{n-1} (1 + t)^i \tilde{Y}_i \quad \text{in} \quad H_1(\tilde{Y}; R_k(n)),$$

and $\bar{Y}$ bounds a 2-chain $\tilde{D}$ in $C_2(M_\sigma; R_k(n))$. Then by [4], $\eta_k^{(n)}(t) = \mathrm{Int}(\tilde{X}, \tilde{D})$, where Int denotes the intersection number. Since $\tilde{X}$ bounds a 2-chain $\tilde{C}$ in M_σ, $\mathrm{Int}(\tilde{X}, \tilde{D}) = lk(\tilde{X}, \bar{Y}) = \mathrm{Int}(\tilde{C}, \bar{Y})$. Obviously, $\mathrm{Int}(\tilde{C}, \tilde{Y}_i) = r$ and hence

$$\eta_k^{(n)}(t) = r\sum_{i=0}^{n-1} (1 + t)^i = r\sum_{j=1}^{n} \binom{n}{j} t^{j-1}.$$

This proves (5.15).

Corollary 5.8. *If $lk(X, Y) = 0$, then for any n and k, $\eta_k^{(n)}(t) = 0$.*

Corollary 5.9. $\xi_k^{(n)}(0) = 0$ *and* $\eta_k^{(n)}(0) = lk(X, Y)$.

PROOF. By Remark 5.3, $\xi_k^{(n)}(0) = \xi_0^{(n)}(0)$. Since $\theta_1(\xi) = x^m$, $m = lk(X, Y)$, we have

$$\xi_0^{(n)}(t) = \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma \mathfrak{D}_1^\sigma \theta_1(\xi) = \psi_\sigma \mathfrak{D}_1^\sigma(x^m) = 0.$$

Therefore, $\xi_k^{(n)}(0) = 0$. The second part follows from (5.15).

Suppose $lk(X, Y) = 0$ and take $n = 0$. Then, since $q_n(t) = 0$, $\xi_k^{(0)}(t)$ is a rational function on t. Furthermore, if we let $s = 1 + t$ and express $\xi_k^{(0)}(t)$ as a Laurent polynomial on s, then it is essentially the η-function defined in [6]. In fact, we can prove the following theorem

Theorem 5.10. *Suppose that Y is contractible $S^3 - X$. Let $\eta(L, X, Y; s)$ be the polynomial defined in* [6]. *Write $\xi_k^{(0)}(t)$ as a Laurent polynomial $\tilde{\xi}_k(s)$ on $s = 1 + t$. Then for a sufficiently large k,*

$$\tilde{\xi}_k(s) \doteq \eta(L, X, Y; s),$$

where $A \doteq B$ means that A and B are equal up to a unit in $Z[s, s^{-1}]$.

A proof follows from the definition of $\xi_k^{(0)}(t)$ and Theorem 2 in [7].

6. Covering Space (II) Metabelian Covering

In this section, we consider the other representation $T_{k,n}: G(L) \to Sym(\Omega_k(n))$ and the covering space M_T of $S^3 - L$ associated with $T_{k,n}$.

Theorem 6.1. *Let n be an integer and suppose $lk(X, Y) \equiv 0 \pmod{n}$. Then for each $k \geqslant 1$, there exists a homomorphism*

$$\Phi_\tau: \langle S_\tau: R_\tau \rangle \to R_k(n)$$

such that, for any $f(t) \in \Omega_k(n)$,

$$\Phi_\tau(\mathfrak{D}^{*\tau}_{f(t)} x_1) = 0 \quad and \quad \Phi_\tau(\mathfrak{D}^{*\tau}_{f(t)} y_1) = f(t), \tag{6.1}$$

where $\mathfrak{D}^{*\tau}_{f(t)}$ denotes the Reidemeister-Schreier rewriting function associated with *$T_{k,n}$ and $S_\tau = \{\mathfrak{D}^{*\tau}_{f(t)}(x_i), \mathfrak{D}^{*\tau}_{f(t)}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$ and $R_\tau = \{\mathfrak{D}^{*\tau}_{f(t)}(r_i), \mathfrak{D}^{*\tau}_{f(t)}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$.*

PROOF. We can use the same argument employed in the proof of Theorem 5.1 using the Reidemeister-Schreier rewriting functions $\mathfrak{D}^{*\tau}_{f(t)}$, $\mathfrak{D}^{\tau}_{f(t)}$ and homomorphisms Φ^*_τ, ψ_τ instead of $\mathfrak{D}^{*\sigma}_{f(t)}$, $\mathfrak{D}^{\sigma}_{f(t)}$, Φ^*_σ, ψ_σ. What we need to prove here is the formula (6.2) below corresponding to (5.7) (1).

$$\Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta x_1) = \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(x_1 \eta). \tag{6.2}$$

First we compute both sides separately. The left hand side is

$$\begin{aligned} \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta x_1) &= \Phi^*_\tau(\mathfrak{D}^{*\tau}_{f(t)}\eta \cdot \mathfrak{D}^{*\tau}_{T_{k,n}(\eta)f(t)} x_1) \\ &= \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta) + \Phi^*_\tau \mathfrak{D}^{*\tau}_{T_{k,n}(\eta)f(t)}(x_1) \\ &= \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta) \\ &= \psi_\tau \mathfrak{D}^{\tau}_{f(t)} \theta_{k+1}(\eta). \end{aligned}$$

On the other hand, the right hand side is

$$\begin{aligned}\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(x_1\eta) &= \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(x_1) + \Phi_\tau^* \mathfrak{D}_{T_{k,n}(x_1)f(t)}^{*\tau}(\eta)\\ &= \Phi_\tau^* \mathfrak{D}_{T_{k,n}(x_1)f(t)}^{*\tau}(\eta)\\ &= \psi_\tau \mathfrak{D}_{\tau_{k,n}(x)f(t)}^{\tau}\theta_{k+1}(\eta).\end{aligned}$$

To compare these terms, let $u = \theta_{k+1}(\eta)$ and $f(t) = \sum_{i=0}^{\infty} a_i t^i$.

Write

$$\psi_\tau \mathfrak{D}_{f(t)}^{\tau}\theta_{k+1}(\eta) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \psi_\tau \mathfrak{D}_{\tau_{k,n}(x)f(t)}^{\tau}\theta_{k+1}(\eta) = \sum_{i=0}^{\infty} c_i t^i.$$

Then by (2.9) (2), we have, since $a_0 = 1$,

(1) $b_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m = lk(X, Y)$

(2) For $q \geqslant 1$,

$$b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \tag{6.3}$$

Now $\tau_{k,n}(x) = f(t) + t = 1 + (1 + a_1)t + \sum_{i=2}^{\infty} a_i t^i$ and hence, (2.9) yields, again,

(1) $c_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m$

(2) $c_1 = (1 + a_1)\left(\dfrac{\partial u}{\partial y}\right)^o + \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^o$

(3) For $q \geqslant 2$,

$$\begin{aligned}c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_2\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o +\\ + (1 + a_1)\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o.\end{aligned} \tag{6.4}$$

Therefore, for $q \geqslant 1$, $c_q - b_q = (\partial^q u/\partial y^q)^o$ and hence

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = \sum_{i=1}^{\infty} \left(\frac{\partial^i u}{\partial y^i}\right)^o t^i.$$

Since

$$\left(\frac{\partial^i u}{\partial y^i}\right)^o = \binom{(\partial u/\partial y)^o}{i} = \binom{m}{i}$$

for $i \geqslant 1$, it follows from Lemma 3.1 that

$$\sum_{i=1}^{\infty}\left(\frac{\partial^i u}{\partial y^i}\right)^o t^i = \sum_{i=1}^{\infty}\binom{m}{i}t^i = q_m(t) \equiv 0 \qquad (\operatorname{mod} q_n(t)),$$

and therefore,

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = 0.$$

This proves (6.2).

Proposition 6.2. *For any $f(t) \in \Omega_k(n)$,*

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta). \tag{6.5}$$

PROOF. Let $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and write $\Phi_\tau^*\mathfrak{D}_1^{*\tau}(\eta) = \sum_{i=0}^{\infty} b_i t^i$ and $\Phi_\tau^*\mathfrak{D}_{f(t)}^{*\tau}(\eta) = \sum_{i=0}^{\infty} c_i t^i$. Then by (2.9) (2) we have, for $q \geqslant 0$ and $u = \theta_{k+1}(\eta)$,

(1) $b_q = \left(\dfrac{\partial^{q+1}u}{\partial x \partial y^q}\right)^o$ and

$$\text{(2)} \quad c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1}u}{\partial x \partial y^q}\right)^o. \tag{6.6}$$

Therefore, $b_0 = c_0 = m = lk(X, Y)$ and

$$\begin{aligned}\sum_{q=0}^{\infty} c_q t^q - \sum_{q=0}^{\infty} b_q t^q &= \sum_{q=1}^{\infty}\left\{a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o\right\}t^q \\ &= \sum_{j=1}^{\infty} a_j\left[\sum_{i=1}^{\infty}\binom{m}{i}t^{i+j-1}\right] \\ &\equiv 0 \quad (\operatorname{mod} q_n(t)),\end{aligned}$$

since for $j \geqslant 1$,

$$\sum_{i=1}^{\infty}\binom{m}{i}t^{i+j-1} \equiv 0 \quad (\operatorname{mod} q_n(t)).$$

Proposition 6.3. *$T_{k,n}(\xi) = id$.*

PROOF. Let $w = \theta_{k+1}(\xi)$ and write $T_{k,n}(\xi)(\sum_{i=0}^{\infty} a_i t^i) = \sum_{i=0}^{\infty} b_i t^i$, where $a_0 = 1$. Then by (2.8) (2),

$$b_q = a_q + a_{q-1}\left(\frac{\partial w}{\partial y}\right)^o + \dots + a_1\left(\frac{\partial^{q-1}w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

However,

$$\left(\frac{\partial^i w}{\partial y^i}\right)^o = \binom{(\partial w/\partial y)^o}{i} = 0, \quad \text{since} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0.$$

Therefore,

$$b_q = a_q + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

By Proposition 4.2 (4.7),

$$\sum_{q \geq 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \pmod{q_n(t)}$$

and hence

$$\sum_{q \geq 0} b_q t^q - \sum_{q \geq 0} a_q t^q = \sum_{q \geq 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \pmod{q_n(t)}.$$

Proposition 6.4. *For any $f(t) \in \Omega_k(n)$,*

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\xi).$$

PROOF. The details will be omitted, since a proof can be obtained, using similar computations shown in the proofs of Propositions 6.2 and 6.3.

Definition 6.1. *Let n be an integer that divides $lk(X, Y)$. Then for any integer $k \geqslant 1$, define the polynomials $\bar{\eta}_k^{(n)}(t)$ and $\bar{\xi}_k^{(n)}(t)$ in $R_k(n)$ as follows:*

$$\begin{aligned} \bar{\eta}_k^{(n)}(t) &= \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta), \\ \bar{\xi}_k^{(n)}(t) &= \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi). \end{aligned} \tag{6.7}$$

Theorem 6.5. *For any $k \geqslant 1$, $\bar{\eta}_k^{(n)}(t)$ and $\bar{\xi}_k^{(n)}(t)$ are invariants for an oriented link type L.*

Remark 6.1. As is stated in Remark 5.3, for $l \geqslant k$, $\bar{\eta}_l^{(n)}(t) \equiv \bar{\eta}_k^{(n)}(t) \pmod{I_{k+1}}$ and $\bar{\xi}_l^{(n)}(t) \equiv \bar{\xi}_k^{(n)}(t) \pmod{I_{k+1}}$, and therefore, these invariants are formal power series.

Now, it follows from Proposition 7.1 in [4] that these invariants also can be interpreted as linking numbers between two cycles in the covering space M_τ associated with $T_{k,n}$. Since $T_{k,n}(\xi) = \text{id}$, every lift $\tilde{\xi}$ of a longitude ξ of Y in M_τ is a simple closed curve in M_τ and hence, $\bar{\xi}_k^{(n)}(t)$ is interpreted as the intersection number between $\tilde{\xi}$ and a 2-chain in $C_2(M_\tau; R_k(n))$ which bounds the characteristic link. On the other hand, $T_{k,n}(\eta)$ may not be an identity, and therefore, a lift $\tilde{\eta}$ of a longitude η of X in M_τ may not be a closed curve. Let r be the smallest positive integer such that $T_{k,n}(\eta^r) = \text{id}$. Then Proposition 6.2 shows that for any $f(t) \in \Omega_k(n)$, $\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta^r) = r\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta)$ and therefore, $\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \bar{\eta}_t^{(n)}(t)$ can be considered the "linking number" between a "longitude" of a covering torus and the characteristic link in M_τ.

Corollary 6.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\bar{\eta}_k^{(n)}(t)$. Then $\bar{a}_0 = n$ and $\bar{a}_i = h_i^*$, where h_i^* is the invariant defined in* [5].

Corollary 6.7. *If the Alexander polynomial of L is* 0, *then all invariants $\eta(t)$, $\xi(t)$, $\bar{\eta}(t)$, $\bar{\xi}(t)$ vanish.*

Corollary 6.8. $\bar{\xi}_k^{(n)}(0) = 0$ *and* $\bar{\eta}_k^{(n)}(0) = lk(X, Y)$.

PROOF. The proof of the first part is similar to that of Corollary 5.9. On the other hand, $\bar{\eta}_k^{(n)}(0) = \bar{\eta}_0^{(n)}(0)$ and $\bar{\eta}_0^{(n)}(t) = \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \psi_\tau \mathfrak{D}_1^\tau \theta_1(\eta) = \psi_\tau \mathfrak{D}_1^\tau(y^m) = = \psi_\tau(y_1^m) = m$, since $\tau_{0,n}(y) = \text{id}$ and $\theta_1(\eta) = y^m$, where $m = lk(X, Y)$.

Finally, we study the behavior of these invariants under simple transformations of the link. The following two propositions are easy to prove, and therefore, the details will be omitted.

Proposition 6.9. *Let L' be the mirror image of an oriented link L. Then for any n and k,*

$$\begin{aligned} \eta_k^{(n)}(t)_{L'} &= -\eta_k^{(n)}(t)_L \\ \xi_k^{(n)}(t)_{L'} &= -\xi_k^{(n)}(t)_L \\ \bar{\eta}_k^{(n)}(t)_{L'} &= -\bar{\eta}_k^{(n)}(t)_L \\ \bar{\xi}_k^{(n)}(t)_{L'} &= -\bar{\xi}_k^{(n)}(t)_L. \end{aligned}$$

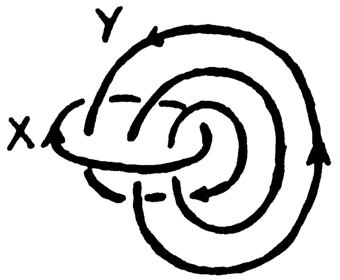
Proposition 6.10. *Let L^* be the link obtained from L by reversing the orientation of one component, X say. The n for any n and k,*

$$\begin{cases} \eta_k^{(n)}(t)_{L^*} = -\eta_k^{(n)}(t)_L \\ \xi_k^{(n)}(t)_{L^*} = \xi_k^{(n)}(t)_L. \end{cases}$$

7. Examples

In this section, we compute our invariants for two simple 2-component links.

Example 1. Torus link of type (6.2).



$G(L) = \langle x, y \colon [\eta, x] = 1,\ [\xi, y] = 1 \rangle$
where $\eta = x^{-1}yxyxyx^{-1}$ and
$\xi = xyxyxy^{-2}$.

$lk(X, Y) = 3$ and $\Delta(x, y) = 1 + xy + x^2y^2$.
Let $n = 3$ and $q_n(t) = 3t + 3t^2 + t^3$. Then for $k \geqslant 4$.

$$\begin{cases} \eta_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = t^2 + 2t^3 + t^4. \end{cases}$$

Example 2. Whitehead link.

$G(L) = \langle x, y \colon [\eta, x] = 1,\ [\xi, y] = 1 \rangle$
where $\eta = y^{-1}xyx^{-1}yxy^{-1}x^{-1}$ and
$\xi = x^{-1}yxy^{-1}xyx^{-1}y^{-1}$.

$lk(X, Y) = 0$ and $\Delta(x, y) = (1 - x)(1 - y)$.
Let $n = 3$ and $q_3(t) = 3t + 3t^3 + t^3$. Then for $k \geqslant 4$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = 0. \end{cases}$$

Let $n = 0$ and $q_0(t) = 0$. Then for any $k \geqslant 0$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = (1 + t)^{-1} - 2 + (1 + t) \\ \qquad\ \ = t^2 - t^3 + t^4 - \dots . \end{cases}$$

References

[1] R. H. Fox. Free Differential Calculus, I, *Ann. of Math.* **57** (1953), 547-560.
[2] —. Metacyclic Invariants of Knots and Links, *Can. J. Math.* **22** (1970), 193-201.
[3] R. Hartley. Metabelian Representations of Knot Groups, *Pacific J. Math.* **82** (1979), 93-104.
[4] R. Hartley and K. Murasugi. Covering Linkage Invariants, *Can. J. Math.* **29** (1977) 1312-1339.
[5] R. Holmes and N. Smythe. Algebraic Invariants of Isotopy of Links, *Amer. J. Math.* **88** (1966) 646-654.
[6] S. Kojima and M. Yamasaki. Some New Invariants of Links, *Invent. Math.* **54** (1979), 213-228.
[7] T. Maeda and K. Murasugi. Covering Linkage invariants and Fox's Problem 13, *Contemporary Math.* **20** (1983) 271-283.
[8] J. Milnor. Isotopy of Links, *Lefschetz Symposium, Princeton Mathematical Series, 12* (1957) 280-306.
[9] K. Murasugi, Nilpotent coverings of Links and Milnor's Invariant, (to appear).
[10] —. On the Height of 2-Component Links, (to appear).

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