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The Ternary Goldbach Problem

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1. Introduction

The object of this paper is to present new proofs of the classical "ternary" theorems of additive prime number theory. Of these the best known is Vinogradov's result on the representation of odd numbers as the sums of three primes; other results will be discussed later. Earlier treatments of these problems used the Hardy-Littlewood circle method, and are highly "analytical". In contrast, the method we use here is a (technically) elementary deduction from the Siegel-Walfisz Prime Number Theorem. It uses ideas from Linnik's dispersion method, together with Vaughan's identity.

It is convenient to quote the Siegel-Walfisz Theorem here. (See Walfisz [17; Hilfssatz 3] or Davenport [6; Chapter 22] for example.)

For any constant $A > 0$ there exists $C(A) > 0$ such that

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) = \frac{x}{\phi(k)} + O(x \exp(-C(A)(\log x)^{1/2})), \tag{1.1}$$

uniformly for $(l, k) = 1$ and $k \leqslant (\log x)^A$.

We now state our results.

Theorem 1. *For $x \geqslant 2$ define*

$$N_2(m) = \sum_{\substack{p \leq x \\ p + p' = m}} (\log p)(\log p'),$$

where p, p' run over primes. Set

$$\mathfrak{S}(m) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid m \\ p \geq 3}} \left(\frac{p-1}{p-2}\right)$$

for even m, and $\mathfrak{S}(m) = 0$ for odd m. Then for any $C > 0$ we have

$$\sum_{2x < m \leq 3x} |N_2(m) - x\mathfrak{S}(m)| \ll x^2(\log x)^{-C}.$$

Corollary 1. *For any $C > 0$ there are at most $O(x(\log x)^{-C})$ even integers $m \leqslant x$ which are not the sum of two primes.*

Corollary 2. *Every sufficiently large odd number is the sum of three primes.*

Corollary 3. *There are infinitely many sets of three distinct primes in arithmetic progression.*

Corollary 2 is the famous result of Vinogradov [15] and [16]. Proofs of Corollary 1 (via forms of Theorem 1) were given independently by van der Corput [3], Čudakov [4], [5], and Estermann [8], all using Vinogradov's method. Heilbronn [9] also discovered the result independently. It is not clear who was the first to state Corollary 3 explicitly.

Sharper versions of Corollary 1 have been obtained more recently by Vaughan [13], and by Montgomery and Vaughan [12]. In particular, the latter work proves that the exceptional set in Corollary 1 has cardinality $O(x^{1-\delta})$ for some fixed positive δ. Our results are all ineffective, since the Siegel-Walfisz Theorem (1.1) is itself ineffective. However, the estimate of Montgomery and Vaughan [12] gives an effective version of Corollary 1, and hence also of Corollaries 2 and 3.

As a by-product of our argument we shall obtain the following version of the "Barban-Davenport-Halberstam" Theorem.

Theorem 2. *For any $C > 0$ we have*

$$\sum_{k \leq x(\log x)^{-C}} \sum_{\substack{l=1 \\ (l, k)=1}}^{k} \left| \sum_{\substack{n \leq x \\ n \equiv l (\bmod k)}} \Lambda(n) - \frac{x}{\phi(k)} \right|^2 \ll x^2(\log x)^{8-C/3}.$$

Results of this type were first obtained by Barban [1], [2], and rediscovered by Davenport and Halberstam [7]. In [2] Barban obtained the asymptotic formula

$$\sum_{k \le Q} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \pi(x; k, l) - \frac{\mathrm{Li}(x)}{\phi(k)} \right|^2 =$$
$$= Q\mathrm{Li}(x) + O(x^2(\log x)^{-A}) + O(Qx(\log x)^{-2}\log(x/Q))$$

for $\exp(c(\log x)^{1/2}) \leqslant Q \leqslant x$ (where A may be any positive constant, and c is an absolute constant). Moreover, when $Q = x$, he showed that the right-hand side may be replaced by

$$x\mathrm{Li}(x) + E(\mathrm{Li}(x))^2 + O(x^2(\log x)^{-A})$$

for a suitable constant E. This work anticipated some of the results of Montgomery [11; Chapter 17]. It should be noted that our proof of Theorem 2 does not use the large sieve.

The techniques used in this paper draw on ideas from Linnik's dispersion method, and from Barban [2] and Hooley [10]. Vaughan's identity [14] also plays a crucial part. In addition we shall use the function

$$\Lambda_Q(n) = \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d \mid (n,q)} d\mu(d) = \sum_{q \le Q} \frac{\mu(q)}{\phi(q)} c_q(n),$$

where $c_q(n)$ is the Ramanujan sum. The function $\Lambda_Q(n)$ is so constructed as to copy $\Lambda(n)$ in its distribution over arithmetic progressions.

We shall use the notation $L = \log x$ throughout the proof. The implied constants in the $O(.)$ and $\ll$ notations may depend on A, B and C. In general they are ineffective.

2. The distribution of $\Lambda_Q(n)$ in arithmetic progressions

In this section we investigate the properties of the function $\Lambda_Q(n)$, and show that it mimics $\Lambda(n)$. As a by-product we will establish Theorem 2.

We first note some well-known bounds that will be required from time to time. We have

$$\phi(q) \gg q(\log q)^{-1}, \qquad \sigma(q) \ll q(\log q) \tag{2.1}$$

and

$$\sum_{k \le K} (d(k))^t \ll K(\log K)^{2^t - 1}, \qquad (t = 1, 2, 3). \tag{2.2}$$

Since $d(ab) \leqslant d(a)d(b)$, we also have

$$\sum_{n \le N} (n,r)d(n) = \sum_{a|r} a \sum_{\substack{n \le N \\ (n,r)=a}} d(n) \leqslant \sum_{a|r} a \sum_{ab \le N} d(ab) \tag{2.3}$$
$$\leqslant \sum_{a|r} ad(a) \sum_{b \le N/a} d(b)$$
$$\ll \sum_{a|r} ad(a)(Na^{-1}(\log N))$$
$$\ll N(\log N) \sum_{a|r} d(a)$$
$$\ll d(r)^2 N(\log N),$$

by (2.2) with $t = 1$.

Before starting the main part of the argument we shall put (1.1) into a more convenient form, by weakening the error term to $O(xL^{1-A})$. The condition $k \leqslant L^A$ may then be dropped, since the sum on the left of (1.1) is automatically $O((1 + xk^{-1})L)$. Moreover if $(l, k) > 1$ then $p^e \equiv l (\bmod k)$ requires $p|k$. There are then $O(\log k)$ available primes p and $O(L)$ possible exponents e. Hence

$$\sum_{\substack{n \equiv l (\bmod k) \\ n \le x}} \Lambda(n) \ll L^3, \qquad ((l,k) > 1, k \leqslant x),$$

and clearly this is true also when $k > x$. After replacing A by $A + 1$ we can now put (1.1) into the more useful form

$$\sum_{\substack{n \equiv l (\bmod k) \\ n \le x}} \Lambda(n) = E_{k,l} \frac{x}{\phi(k)} + O(xL^{-A}) \tag{2.4}$$

uniformly for all k, l; here we have defined

$$E_{k,l} = \begin{cases} 1, & (k,l) = 1, \\ 0, & (k,l) > 1. \end{cases}$$

We now turn to $\Lambda_Q(n)$, and start by looking at its size. Using (2.1) we have

$$|\Lambda_Q(n)| = \left| \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{\substack{d|q \\ d|n}} d\mu(d) \right|$$
$$\leqslant \sum_{d|n} d \sum_{\substack{q \le Q \\ d|q}} \frac{1}{\phi(q)}$$
$$\ll (\log Q) \sum_{d|n} d \left(\sum_{\substack{d \le Q \\ d|q}} q^{-1} \right)$$
$$\ll (\log Q) \sum_{d|n} d(d^{-1}(\log Q)).$$

Thus

$$\Lambda_Q(n) \ll d(n)(\log Q)^2. \tag{2.5}$$

Next we show that in any given arithmetic progression the functions $\Lambda_Q(n)$ and $\Lambda(n)$ behave very similarly. It is convenient to write $\Delta_Q(n) = \Lambda_Q(n) - \Lambda(n)$.

Lemma 1. *We have*

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Lambda_Q(n) = \frac{x}{\phi(k)} + O(QL^2) + O(xL(kQ)^{-1}d(k)), \tag{2.6}$$

for $(k, l) = 1$, *and*

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Delta_Q(n) \ll_A QL^2 + xL(kQ)^{-1}(k, l)d(k) + xL^{-A}, \tag{2.7}$$

for any l, *uniformly for* $1 \leqslant Q, k \leqslant x$.

By definition we have

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Lambda_Q(n) = \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\mu(d) \# \{n \leqslant x; d|n, n \equiv l (\mathrm{mod}\, k)\}. \tag{2.8}$$

The conditions $d|n$ and $n \equiv l(\mathrm{mod}\, k)$ are compatible only when $(d, k)|l$, in which case they define a unique residue class to modulus $kd/(d, k)$. Hence (2.8) is

$$\sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|r} d\mu(d)\{(kd)^{-1}(d, k)x + O(1)\}$$
$$= k^{-1}x \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|r} \mu(d)(d, k) + O\left(\sum_{q \le Q} \frac{\sigma(q)}{\phi(q)}\right),$$

where r is the product of those primes $p|q$ for which $(p, k)|l$. The error term of (2.9) is $O(QL^2)$ by (2.1).

Since $\mu(d)(d, k)$ is a multiplicative function of d we have, for $\mu(q) \neq 0$,

$$\sum_{d|r} \mu(d)(d, k) = \prod_{p|r} (1 - (p, k)) = \begin{cases} \mu((q, l))\phi((q, l)), & q|k, \\ 0, & q\nmid k. \end{cases}$$

We write $f(q) = \mu(q)^2\mu((q, l))\phi((q, l))/\phi(q)$, so that $f(q)$ is multiplicative.

Then

$$\sum_{\substack{q|k\\ q\le Q}} f(q) = \sum_{q|k} f(q) + O\Bigg(\sum_{\substack{q|k\\ q>Q}} |f(q)|\Bigg) = \tag{2.10}$$

$$= \sum_{q|k} f(q) + O(Q^{-1}\sum_{q|k} q|f(q)|)$$

$$= \prod_{p|k}(1+f(p)) + O(Q^{-1}\prod_{p|k}(1+p|f(p)|)).$$

However

$$f(p) = \begin{cases} (p-1)^{-1}, & p|k, p\nmid l,\\ -1, & p|k, p|l. \end{cases}$$

Hence (2.10) is

$$E_{k,l}\frac{k}{\phi(k)} + O\left(Q^{-1}(k,l)d(k)\frac{\sigma(k)}{k}\right),$$

and (2.8) becomes

$$\sum_{\substack{n\equiv l(\mathrm{mod}\,k)\\ n\le x}} \Lambda_Q(n) = E_{k,l}\frac{x}{\phi(k)} + O(QL^2) + O\left(x(kQ)^{-1}(k,l)d(k)\frac{\sigma(k)}{k}\right).$$

The estimates (2.6) and (2.7) now follow, using (2.1) and (2.4).

Our next lemma is an analogue of Theorem 2 for $\Delta_Q(n)$. For convenience we define

$$\delta_t(x,k,Q) = \sum_{l=1}^{k}\Bigg|\sum_{\substack{n\equiv l(\mathrm{mod}\,k)\\ n\le x}} \Delta_Q(n)\Bigg|^t, \qquad (t=1,2).$$

We then have:

Lemma 2. *Let $Q = L^B$ and $K \leqslant xQ^{-1}$. Then*

$$\sum_{k\le K} k^{-1}\delta_1(x,k,Q) \ll_B xQ^{-1/2}L^7 \tag{2.11}$$

for any fixed $B > 0$.

The proof falls into two parts. First we bound the sum on the left of (2.11) in terms of

$$S = \sum_{K<k\le 2K} \delta_2(x,k,Q),$$

and then we use Lemma 1 to estimate S. The second stage follows the idea used by Barban [2]. Naturally it suffices to consider the case $K = xQ^{-1}$.

For any $j \geqslant 1$ we have

$$\left|\sum_{n \equiv l (\mathrm{mod}\, k)}\right| = \left|\sum_{m=1}^{j} \sum_{n \equiv l + mk (\mathrm{mod}\, jk)}\right| \leqslant \sum_{m=1}^{j}\left|\sum_{n \equiv l + mk (\mathrm{mod}\, jk)}\right|.$$

By summing over $l(\mathrm{mod}\, k)$ we deduce that

$$\delta_1(x, k, Q) \leqslant \delta_1(x, jk, Q).$$

We proceed to average this over those j for which $jk \in (K, 2K]$. Since the number of such j is of exact order Kk^{-1} we obtain

$$Kk^{-1}\delta_1(x, k, Q) \ll \sum_{\substack{K < h \leq 2K \\ k \mid h}} \delta_1(x, h, Q).$$

On summing for $k \leqslant K$ this yields

$$K \sum_{k \leq K} k^{-1}\delta_1(x, k, Q) \ll \sum_{K < h \leq 2K} d(h)\delta_1(x, h, Q).$$

To obtain an estimate in terms of S we apply Cauchy's inequality, in conjunction with the case $t = 2$ of (2.2). This leads to

$$K \sum_{k \leq K} k^{-1}\delta_1(x, k, Q) \ll (K(\log K)^3)^{1/2}\Big(\sum_{K < h \leq 2K} \delta_1(x, h, Q)^2\Big)^{1/2}. \tag{2.12}$$

However, by Cauchy's inequality again, we have

$$\delta_1(x, h, Q)^2 \leqslant h\delta_2(x, h, Q) \ll K\delta_2(x, h, Q),$$

and so (2.12) yields

$$\sum_{k \leq K} k^{-1}\delta_1(x, k, Q) \ll L^{3/2}S^{1/2}. \tag{2.13}$$

We proceed to bound S. We have

$$\begin{aligned}\delta_2(x, k, Q) &= \sum_{\substack{m, n \leq x \\ k \mid m - n}} \Delta_Q(m)\Delta_Q(n) \\ &= \sum_{n \leq x} \Delta_Q(n)^2 + 2 \sum_{\substack{m < n \leq x \\ k \mid m - n}} \Delta_Q(m)\Delta_Q(n).\end{aligned} \tag{2.14}$$

From (2.5) we have $\Lambda_Q(n) \ll Ld(n)$, whence $\Delta_Q(n) \ll Ld(n)$. The diagonal terms in (2.14) therefore total $O(xL^5)$, by the case $t = 2$ of (2.2). It follows that

$$S = 2 \sum_{m < n \leq x} \Delta_Q(m)\Delta_Q(n) \,\#\, \{k, t; n - m = kt, K < k \leqslant 2K\} + O(xKL^5)$$

$$S = 2 \sum_{1 \leq t \leq xK^{-1}} \sum_{m \leq x} \Delta_Q(m) \sum_{n \equiv m (\mathrm{mod}\, t)} \Delta_Q(n) + O(xKL^5).$$

In the innermost sum n runs over a subinterval of $(0, x]$, so that (2.7) of Lemma 1 can be applied. This yields

$$S \ll xKL^5 + \sum_t \sum_m |\Delta_Q(m)| \{xL(tQ)^{-1}(t, m)d(t)\}. \tag{2.15}$$

Note here that

$$xL(tQ)^{-1}(t, m)d(t) \gg x(tQ)^{-1} \gg x(xK^{-1}Q)^{-1} \gg xL^{-2B} \gg QL^2 + xL^{-A},$$

on taking $A = 2B$, as indeed we may. Thus the second term on the right of (2.7) is the dominant one.

We rearrange the double sum in (2.15) as

$$xLQ^{-1} \sum_t \sum_m |\Delta_Q(m)| t^{-1}(t, m)d(t) \leqslant xLQ^{-1} \sum_{m \leq x} |\Delta_Q(m)| \sum_{t \leq x} t^{-1}(t, m)d(t).$$

The inner sum is $O(d(m)^2 L^2)$, by (2.3). Thus, since $\Delta_Q(m) \ll Ld(m)$ as before, (2.15) becomes

$$\begin{aligned} S &\ll xKL^5 + xL^3Q^{-1} \sum_{m \leq x} |\Delta_Q(m)| d(m)^2 \\ &\ll xKL^5 + xL^4Q^{-1} \sum_{m \leq x} d(m)^3 \\ &\ll xKL^5 + x^2L^{11}Q^{-1}, \end{aligned}$$

by (2.2) with $t = 3$. Lemma 2 now follows from (2.13), given our condition on K.

We can now derive Theorem 2. It follows from Lemma 2 that

$$\begin{aligned} &\sum_{k \leq K} k^{-1} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) - \frac{x}{\phi(k)} \right| \ll \\ &\ll \sum_{k \leq K} k^{-1} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda_Q(n) - \frac{x}{\phi(k)} \right| + xQ^{-1/2}L^7. \end{aligned}$$

By (2.6) of Lemma 1 the right hand side is

$$\begin{aligned} &\ll xQ^{-1/2}L^7 + \sum_{k \leq K} \{QL^2 + xL(kQ)^{-1}d(k)\} \\ &\ll xQ^{-1/2}L^7 + KQL^2 + xQ^{-1}L^3 \\ &\ll xQ^{-1/2}L^7 + KQL^2, \end{aligned}$$

on using (2.2) with $t = 1$. However

$$\left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) - \frac{x}{\phi(k)} \right| \ll xk^{-1}L,$$

so that

$$\sum_{k\le K}\sum_{\substack{l=1\\(l,k)=1}}^{k}\left|\sum_{\substack{n\equiv l(\mathrm{mod}\,k)\\n\le x}}\Lambda(n)-\frac{x}{\phi(k)}\right|^2 \ll xL\sum_{k\le K}k^{-1}\sum_{\substack{l=1\\(l,k)=1}}^{k}\left|\sum_{\substack{n\equiv l(\mathrm{mod}\,k)\\n\le x}}\Lambda(n)-\frac{x}{\phi(k)}\right|$$
$$\ll xL(xQ^{-1/2}L^7+KQL^2)$$
$$\ll x^2L^{8-C/3},$$

on choosing $K=xL^{-C}$, $Q=L^B$, $B=2C/3$. This proves Theorem 2.

3. Application of Vaughan's identity

In this section we use Vaughan's identity to estimate the sum

$$\Sigma=\sum_{2x<m\le 3x}\Big(\sum_{n\le x}\Delta_Q(n)\Lambda(m-n)\Big).$$

Here we shall take $Q=L^B$ with a large constant value for B. The identity states that for any $u,v\geqslant 1$ we have

$$\sum_{v<n\le N}f(n)\Lambda(n)=S_1-S_2-S_3,$$

with

$$S_1=\sum_{c\le u}\mu(c)\sum_{r\le N/c}(\log r)f(cr),$$
$$S_2=\sum_{k\le uv}c_k\sum_{r\le N/k}f(kr),\qquad c_k=\sum_{\substack{c\le u\\cn=k}}\sum_{n\le v}\mu(c)\Lambda(n),$$
$$S_3=\sum_{\substack{r>u\\rn\le N}}\sum_{n>v}d_r\Lambda(n)f(rn),\qquad d_r=\sum_{\substack{c|r\\c\le u}}\mu(c). \tag{3.1}$$

We shall take $N=3x$, $u=Q$, $v=xQ^{-2}$ and

$$f(n)=\begin{cases}\Delta_Q(m-n), & m-x\leqslant n<m,\\ 0, & \text{otherwise.}\end{cases}$$

We proceed to estimate

$$\Sigma_i=\sum_m|S_i|,$$

for $i=1,2,3$.

To bound S_1 we use partial summation in conjunction with (2.7) of Lemma 1. This yields

$$\sum_{r \le N/c} (\log r) f(rc) = \sum_{(n-x)/c \le r < m/c} (\log r) \Delta_Q(m - rc)$$
$$\ll L \operatorname{Max}_{y \le x} \left| \sum_{\substack{s \equiv m (\bmod c) \\ s \le y}} \Delta_Q(s) \right|$$
$$\ll L(xL(cQ)^{-1}(c, m)d(c)).$$

Note that, as before, the second term on the right of (2.7) dominates the other two, since A can be taken arbitrarily large. It now follows that

$$\Sigma_1 \ll \sum_{2x < m \le 3x} \sum_{c \le u} xL^2(cQ)^{-1}(c, m)d(c).$$

However (2.3) yields

$$\sum_{c \le u} c^{-1}(c, m)d(c) \ll d(m)^2(\log u)^2 \ll d(m)^2 L.$$

Moreover

$$\sum_{2x < m \le 3x} d(m)^2 \ll xL^3$$

by (2.2) with $t = 2$. Combining these estimates yields

$$\Sigma_1 \ll x^2 L^6 Q^{-1}. \tag{3.2}$$

We turn next to Σ_2. Since

$$|c_k| \leqslant \sum_{n | k} \Lambda(n) = \log k \ll L,$$

we have

$$S_2 \ll L \sum_{k \le uv} \left| \sum_{(m-x)/k \le r < m/k} \Delta_Q(m - kr) \right|$$
$$= L \sum_{k \le uv} \left| \sum_{\substack{n \equiv m (\bmod k) \\ n \le x}} \Delta_Q(n) \right|.$$

As m runs over the interval $(2x, 3x]$, each congruence class (mod k) is covered $O(xk^{-1})$ times. It follows that

$$\Sigma_2 \ll Lx \sum_{k \le uv} k^{-1} \delta_1(x, k, Q).$$

We may now apply Lemma 2 to obtain

$$\Sigma_2 \ll x^2 Q^{-1/2} L^8.$$

Lastly we examine Σ_3. We split the ranges for r and n into intervals $r \in (U, 2U]$, $n \in (V, 2V]$, where $U = u2^i$, $V = v2^j$. Since the corresponding subsum is empty unless

$$x \leqslant 4UV, \qquad UV \leqslant 3x, \tag{3.4}$$

there can be only $O(L)$ pairs of values U, V to be considered. It follows that

$$\Sigma_3 \ll L \sum_{2x < m \leq 3x} \sum_{V < n \leq 2V} \Lambda(n) \Big| \sum_{U < r \leq 2U} d_r f(rn) \Big|$$

for some U, V. Since $\Lambda(n) \ll L$ we can use Cauchy's inequality to obtain

$$\begin{aligned} \Sigma_3{}^2 &\ll L^2 x V L^2 \sum_{m,n} \Big| \sum_{U < r \leq 2U} d_r f(rn) \Big|^2 \\ &= xVL^4 \sum_{U < r_i \leq 2U} d_{r_1} d_{r_2} \sum_{2x < m \leq 3x} \sum_{V < n \leq 2V} f(r_1 n) f(r_2 n). \end{aligned} \tag{3.5}$$

The innermost sum here is

$$S(r_1, r_2, m) = \sum_{n \in I} \Delta_Q(m - r_1 n) \Delta_Q(m - r_2 n),$$

where I is the interval

$$I = (V, 2V] \cap \left[\frac{m - x}{r_1}, \frac{m}{r_1}\right) \cap \left[\frac{m - x}{r_2}, \frac{m}{r_2}\right).$$

Let us first suppose that $r_1 = r_2$. Then, by (2.5), we have

$$S(r_1, r_1, m) \ll L \sum_{\substack{s \equiv m (\operatorname{mod} r_1) \\ s \leq x}} d(s)^2.$$

As before, if we sum over m, the residue classes $(\operatorname{mod} r_1)$ are each covered $O(xr_1^{-1}) = O(xU^{-1})$ times. Thus (2.2) with $t = 2$ yields

$$\sum_m S(r_1, r_1, m) \ll xU^{-1} L \sum_{s \leq x} d(s)^2 \ll x^2 U^{-1} L^4. \tag{3.6}$$

We now examine $S(r_1, r_2, m)$ when $r_1 < r_2$, the case $r_1 > r_2$ being essentially identical. We write $r = r_2 - r_1$ and $j = m - r_2 n$. Then

$$\begin{aligned} \sum_m S(r_1, r_2, m) &= \sum_{m,n} \Delta_Q(m - r_1 n) \Delta_Q(m - r_2 n) \\ &= \sum_j \Delta_Q(j) \sum_n \Delta_Q(j + rn), \end{aligned} \tag{3.7}$$

where the conditions $2x < m \leqslant 3x$, $n \in I$ translate as $0 < j \leqslant x$ and

$$n \in (V, 2V] \cap (-j/r, (x - j)/r] \cap ((2x - j)/r_2, (3x - j)/r_2].$$

By (2.7) of Lemma 1 we have

$$\sum_n \Delta_Q(j + rn) \ll xL(rQ)^{-1}(r, j)d(r),$$

since, as before, the middle term on the right of (2.7) dominates. Now, by (2.5) and (2.3), equation (3.7) yields

$$\begin{aligned} \sum_m S(r_1, r_2, m) &\ll xL(rQ)^{-1}d(r) \sum_{j \le x} |\Delta_Q(j)|(r, j) \qquad (3.8) \\ &\ll xL^2(rQ)^{-1}d(r) \sum_{j \le x} d(j)(r, j) \\ &\ll x^2L^3(rQ)^{-1}d(r)^3 \\ &= x^2L^3|r_1 - r_2|^{-1}Q^{-1}d(|r_1 - r_2|)^3, \qquad (r_1 \ne r_2). \end{aligned}$$

It is clear from the definition (3.1) that $|d_r| \leqslant d(r)$. Moreover, since (3.4) requires that

$$U \ll xv^{-1} = Q^2 = L^{2B},$$

we have

$$d(r) \ll r^{2/(2B)} \ll L, \qquad (r \ll U).$$

Hence, using (3.5), (3.6) and (3.8) we find

$$\begin{aligned} \Sigma_3{}^2 &\ll xVL^4\Bigg(\sum_{U < r \le 2U} |d_r|^2 x^2 U^{-1} L^4 + \\ &\qquad + \sum_{\substack{U < r_i \le 2U \\ r_1 \ne r_2}} |d_{r_1} d_{r_2}| x^2 L^3 |r_1 - r_2|^{-1} Q^{-1} d(|r_1 - r_2|)^3 \Bigg) \\ &\ll xVL^4(x^2L^6 + x^2L^8Q^{-1}\Sigma|r_1 - r_2|^{-1}) \\ &\ll xVL^4(x^2L^6 + x^2L^9UQ^{-1}) \\ &\ll x^4L^{10}U^{-1} + x^4L^{13}Q^{-1} \\ &\ll x^4L^{13}Q^{-1}. \end{aligned}$$

This last estimate may be combined with (3.2) and (3.3) to give

$$\Sigma \ll \Sigma_1 + \Sigma_2 + \Sigma_3 \ll x^2Q^{-1/2}L^8. \qquad (3.9)$$

4. Completion of the proof of Theorem 1 and its Corollaries

To complete the proof of Theorem 1 we need to know about

$$\sum_{n \le x} \Lambda_Q(n)\Lambda(m - n) \qquad (4.1)$$

for $2x < m \leqslant 3x$. By the definition of $\Lambda_Q(n)$ in conjunction with (2.4) this is

$$\sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\mu(d) \sum_{\substack{d|n \\ n \le x}} \Lambda(m-n) =$$

$$= x \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} \frac{d\mu(d)}{\phi(d)} E_{d,m} + O\left(xL^{-A} \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\right).$$

Since $d\mu(d)\phi(d)^{-1}E_{d,m}$ is a multiplicative function of d, the innermost sum in the main term is

$$\prod_{\substack{p|q \\ p\nmid m}} \left(1 - \frac{p}{p-1}\right) = \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)},$$

if q is square-free. Moreover the error term is

$$\ll xL^{-A} \sum_{q \le Q} \frac{\sigma(q)}{\phi(q)} \ll xL^{2-A}Q \ll xQ^{-1},$$

by (2.1), since we may take $A = 2B + 2$. It follows that (4.1) is

$$x \sum_{q \le Q} \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)^2} + O(xQ^{-1}) =$$

$$= x \sum_{1}^{\infty} \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)^2} + O\left(x \sum_{q > Q} \frac{(q,m)}{\phi(q)^2}\right) + O(xQ^{-1}).$$

The main term here is

$$x \prod_{p\nmid m} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|m} \left(1 + \frac{1}{(p-1)}\right) = x\mathfrak{S}(m)$$

and first error term is $O(xQ^{-1}Ld(m)^2)$ by (2.3), since

$$\frac{(q,m)}{\phi(q)^2} \ll \frac{(q,m)d(q)}{q^2}.$$

Now, using (3.9) together with the case $t = 2$ of (2.2), we see that

$$\sum_{2x < m \le 3x} \left| \sum_{n \le x} \Lambda(n)\Lambda(m-n) - x\mathfrak{S}(m) \right| \ll$$

$$\ll x^2Q^{-1/2}L^8 + x^2Q^{-1}L^4 \ll x^2Q^{-1/2}L^8.$$

Since the number of prime powers $p^e \leqslant 3x$ with $e \geqslant 2$ is $O(x^{1/2})$ we have

$$\sum_{n \le x} \Lambda(n)\Lambda(m-n) = \sum_{\substack{p' \le x \\ p'+p''=m}} (\log p')(\log p'') + O(x^{1/2}L^2).$$

Thus

$$\sum_{2x < m \le 3x} |N_2(m) - x\mathfrak{S}(m)| \ll x^2 Q^{-1/2} L^8,$$

and, as $Q = L^B$ with B arbitrary, Theorem 1 follows.

The corollaries require little comment. Since

$$\mathfrak{S}(m) \geqslant 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \gg 1$$

whenever m is even, there can be only $O(xL^{-C})$ even numbers m counted in Theorem 1 for which $N_2(m) = 0$. This gives Corollary 1. Next let n be odd, and take $x = n/3$. Then the numbers $n - p$, for odd primes $p < x$, are all even, and there are asymptotically xL^{-1} of them. Since only $O(xL^{-C})$ such numbers can have $N_2(n - p) = 0$ there must be at least one solution of $n - p = p' + p''$, if n is large enough. This proves Corollary 2. Similarly, since the number of integers $m = 2p$ in the range $2x < m \leqslant 3x$ is asymptotically $\frac{1}{2}xL^{-1}$, and only $O(xL^{-C})$ such integers can have $N_2(m) = 0$, there must be solutions of $2p = p' + p''$ with $p' \leqslant x$. Since this entails $p' \neq p(\neq p'')$, Corollary 3 is proved.

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