ON τ_δ-COMPLETENESS OF $\mathcal{H}(U)$ (*)

by

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Introduction

Let E be a complex locally convex vector space, U a non void open subset of E, and $\mathcal{H}(U)$ the vector space of all holomorphic functions $f: U \longrightarrow \mathbb{C}$ endowed with some of its natural topologies τ_ω or τ_δ. The question of the τ_ω-completeness of $\mathcal{H}(U)$, although not yet completely solved, has been treated by several authors (see [1], [2], [5]), and at the present moment almost nothing is known concerning its τ_δ-completeness.

Assuming that U is balanced, and using Schauder decomposition tecniques, we prove that whenever $\mathcal{H}(U)$ is τ_ω-complete it is τ_δ-complete.

We shall systematically use standard symbols in Theory of Holomorphy [7].

Now, let X be a Hausdorff locally convex vector space and $(X_n)_{n \in N}$ a sequence of non trivial subspaces of X; then

Definition 1.—$(X_n)_{n \in N}$ is an equischauder decomposition of X if each $x \in X$ has an unique expressión

$$x = \sum_{n=0}^{\infty} x_n$$

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where $x_n \in X_n$ for $n \in N$, the series being convergent in X, and the sequence of the canonical projectors

$$\pi_n : x \to \sum_{k=0}^{n} x_k$$

is equicontinuous.

We denote by X′ and X'_n the corresponding topological duals endowed with their respective strong topologies; then

DEFINITION 2.—$(X_n)_{n \in N}$ is a shrinking decomposition of X if $(X'_n)_{n \in N}$ is an equischauder decomposition of X′.

Let

$$(x_n)_{n \in N} \in \prod_{n \in N} X_n$$

and denote by

$$s_n = \sum_{k=0}^{n} x_n$$

the sequence of the corresponding partial sums; then

DEFINITION 3.—A space X with an equischauder decomposition $(X_n)_{n \in N}$ is γ-complete if whenever

$$(x_n)_{n \in N} \in \prod_{n \in N} X_n$$

and $(s_n)_{n \in N}$ is bounded in X, the series $\sum_{n=0}^{\infty} x_n$ is convergent is X.

Now, the following result is known [4].

PROPOSITION 1.—A space X with a shrinking decomposition $(X_n)_{n \in N}$ is complete if and only if the following conditions are satisfied:

a) Each one of the X_n, $n \in N$, is complete.

b) X is γ-complete.

We shall now apply that for studying ($\mathcal{H}$ (U), τ) where E is locally convex, $U \subset E$ is balanced and $\tau = \tau_\omega$ or τ_δ. It is known [3] that.

PROPOSITION 2.—$\mathcal{P}\,({}^nE)_{n \in N}$ is a shrinking decomposition of $(\mathcal{H}\,(U), \tau)$.

DEFINITION 4.—U is Taylor series τ-complete if for every sequence of polynomials $P_n \in \mathcal{P}\,({}^nE)$, $n \in N$, such that

$$\sum_{n=0}^{\infty} q\,(P_n) < \infty$$

whenever q is a τ-continuous seminorm on $\mathcal{H}\,(U)$, we have

$$\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U).$$

Clearly, Taylor series τ-completeness of U is equivalent to γ-completeness of $(\mathcal{H}\,(U), \tau)$.

PROPOSITION 3.—If U is balanced and $(\mathcal{H}\,(U), \tau_\omega)$ is complete, then

a) U is τ_ω and τ_δ-Taylor series complete.

b) $\mathcal{H}\,(U)$ is τ_δ-complete.

Indeed, by proposition 1 the assumption of τ_ω-completeness of $\mathcal{H}\,(U)$ implies that:

1. All the space $\mathcal{P}\,({}^nE)$ are $\tau_{\omega \mid \mathcal{P}({}^nE)}$-complete.

2. U is τ_ω-Taylor series complete.

Since it is known [3] that $\tau_{\omega \mid \mathcal{P}({}^nE)} = \tau_{\delta \mid \mathcal{P}({}^nE)}$ for all $n \in N$, and $\tau_\delta \geqslant \tau_\omega$, we get:

1'. All the spaces $\mathcal{P}\,({}^nE)$ are $\tau_{\delta \mid \mathcal{P}({}^nE)}$-complete.

2'. U is τ_δ-Taylor series complete.

REMARK.—Our situation is not vacuous since it is known [5] that for all normed spaces, and more generaly, for all metrizable spaces satisfying the so named Barroso's condition B, $(\mathcal{H}\,(U), \tau_\omega)$ is complete even though U were not balanced.

In [6] Mujica and Avilés have proved τ_ω-completeness of $\mathcal{H}\,(U)$ when U is an open set of a quasi-normable locally convex vector space E (in particular, a Schwartz space). Also in [3] Dineen has proved τ_ω-completeness of $\mathcal{H}\,(U)$ for some spaces E not included above.

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