REAL COMMUTATIVE ALGEBRA I. PLACES (*)

by

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The principal theorem is the extension theorem (theorem 3). The specialization at an inner point on a real algebraic variety $V \mid k$ always extends to a real place of the function field $F \mid k$ of V. [Inner points are the members of the strong closure of the set of all simple points.]

Gleyzal rank, of an order and of an ordered field, is defined.

Theorem 1 asserts that for a real place h on a function field $F \mid k$ with residual field F_h,

$$\text{Gl. rank}\,(F_h \mid k) + \text{rank}\, h = \text{Gl rank}\,(F \mid k)$$

Theorem 2 is an existence theorem for places and orders with prescribed ranks.

The subject of real places was introduced in 1932 by Krull [3] based on the work of E. Artin [1] (and to a lesser extent, R. Baer [2]). Our «Gleyzal rank» honors the incredible ingenuity of A. Gleyzal who created a concept of Archimedian completions with almost no algebraic tools. Lang, in 1953, proved an extension theorem [4] quite different from ours, namely, given a real place $K \longrightarrow \Delta$, where K is a function field, there exists, for some real closed field containing K, an extension (real, of course) of the place. Lang also gives examples to show the difficulties in the way of extending a specialization to a real place.

The set of all $\mathbb{R}$-places is the object of study in [7] (as well as in [5], but language of [5] is different) with a topology which is induced by Harrinson's topology on the set of all orders. There and also in [10] (there is an oversight in the latter and statements about sums of squares being positive are valid only for inner points) analogues of Hilbert's 17^{th} problem are proved. See also [6].

The geometric extension theorem, from specialization to real place, is based on our Reelnullstellensatz [8] (published in December 1969); the first proof appears in the unpublished Technical Report of 1971 [10], along with the preliminary material on Gleyzal rank. The remainder of the results were obtained in December, 1979.

1. Gleyzal Rank.—For non-negative infinitesimals x and y in the ordered field $K \mid k$, the relations «$x \lhd y$» and «$y \rhd x$» signify that for every positive integer n, x is less than y^n. If neither $x \lhd y$ nor $y \lhd x$ holds then we write «$x \sim y$», read «x is equivalent to y». The following lemma shows that this is truly an equivalence relation. Let $L(y)$, for any infinitesimal y, denote the set $\{x : \mid x^m \mid \leqslant y$, for some $m\}$.

Lemma 1.—Assume $K \mid k$ is an ordered field, with B and J denoting the ring of all finite elements and ideal of all infinitesimal elements, respectively. Then:

(a) Every ideal of B is real and saturated ($0 < \mid x \mid < y \in B$ implies $x \in B$).

(b) An ideal in B is prime if and only if it is a union of sets $L(y)$; in particular each $L(y)$ is a real prime ideal, and a prime ideal P is equal to $\cup \{L(y) : y \in P\}$.

(c) $L(x) \subset L(y)$ if and only if $x \in L(y)$.

(d) $L(x) < L(y)$ if and only if $x \lhd y$.

(e) (Trichotomy) For all x and y in J, exactly one of the following is true:

$$L(x) = L(y), \quad L(x) < L(y), \quad L(x) > L(y).$$

(f) A chain $0 \lhd x_s \lhd \ldots \lhd x_j \in J$ is *satured* in the sense that no more terms may be inserted, even at the top, if and only if the corresponding chain of prime ideals, $(0) < L(x_s) < \ldots < L(x_j) = J$, is satured.

(g) There exists one and only one satured chain of $L(x_i)$.

The proof, whose details appear in [10], is omitted; it is straightforward.

Definitions.—The *length of the chain* $0 < x_s \lhd \ldots \lhd x_s \in J$ is s. The *Gleyzal rank* of the ordered field $K \mid k$ is the supremum of the lengths of such chains. In case K | k is Archimedean its Gleyzal rank is zero. Abbreviation: Gl. rank K | k.

LEMMA 2.—Assume K | k is an ordered field with

$$0 < x_t \lhd \dots \lhd x_1 \in J.$$

Then:

(a) Each x_m is infinitesimal over $k(x_1, \dots, x_{m-1})$.

(b) The system $\langle x_1, \dots, x_t \rangle$ is algebraically independent over k.

(c) Gl. rank K | k is at most equal to the transcendence degree of K | k:

$$\text{Gl. rank } K \mid k \leq \text{tr. deg. } K \mid k$$

PROOF.—By induction on t. For $t = 1$, assertion (a) is given and (b) asserts merely that algebraic extensions are relatively Archimedean. Assume validity for t. It will be shown that if $0 < y \lhd x_t \lhd \dots \lhd x_1$ then y is infinitesimal over $k(x_1, \dots, x_t)$. The induction will then be complete for both (a) and (b), from which (c) is an immediate consequence. So, let $a(x)$ and $b(x)$ be polynomials in $k[x_1, \dots, x_t]$, both of them positive. It must be shown merely that

$$y < \frac{a(x)}{b(x)} \quad \text{i. e. that} \quad y\, b(x) < a(x).$$

The induction hypotheses shows that x_t is a minimal element of $k(x_1, \dots, x_t)$, i. e., the chain $0 < x_t$ is saturated. It may therefore be assumed that $a(x) = x^n_t$ for some positive integer n. Let $b_m(x)$ be the homogeneous component of $b(x)$ of degree m. Let (v) be a t-tuple of nonnegative integer components $(v) = (v_1, \dots, v_t)$.

Then $b_m(x)$ can be written as

$$b_m = \Sigma_v\, b_m^{(v)} \cdot x_1^{v_1} \dots x_t^{v_t} \quad \text{where} \quad \Sigma\, v_i = m.$$

Then:

$$0 < y\, b(x) \leq y \sum_m |\, b_m(x) \,| \leq y \cdot \sum_{m,\, v} |\, b_m^{(v)} x_1^{v_1} \dots x_t^{v_t} \,| \leq$$

$$\leq y \sum_m \sum_{(v)} b_m^{(v)} x_1^m = y \sum_m \lambda_m x_1^m < \rho\, y$$

for some positive ρ in k. But since $y \lhd x_t$ is assumed, the last term is less than $x^n{}_t = a(x)$, whence $y\, b(x) < a(x)$ as was required to complete the induction.

NOTE.—Let $k \subset F \subset F'$. Then:

$$\text{Gl. rank } F \mid k + \text{Gl. rank } F' \mid F \leq \text{Gl. rank } F' \mid k$$

Equality may fail. Let x and y be independent variables. Order $\mathbb{R}(x, y)$ so $0 < x \lhd y \in J$; let $k = \mathbb{R}$, $F = \mathbb{R}(x)$, $F' = \mathbb{R}(x, y)$.

2. REAL PLACES.—Let $K \mid k$ and $\Delta \mid k$ be real fields over the ordererfield k. Let P be an order of $K \mid k$. The *canonical place*, or *Krull place*, asociated with P is denoted h_P; its valuation ring is B_P, its maximal ideal is J_P, where, as usual, B_P and J_P are the sets (respectively) of all finite (infinitesimal) elements of $K \mid k$ according to order P. The residual $K_p \mid k \equiv B_P/J_P$ has a unique order *compatible* with h_P (i. e. an order by which h_P is order-preserving). Suppose now that some place h on $K \mid k$ onto $\Delta \mid k \cup \{\infty\}$ is given, and that P is an order of $K \mid k$ such that $B_P \subset R_h \equiv$ valutation ring of h. Then again $\Delta \mid k$ has a unique order compatible with h. If $\Delta \mid k$ is already ordered and if some place h of $K \mid k$ onto $\Delta \mid k \cup \{\infty\}$ is given then there exist *associated orders* P of $K \mid k$ which are compatible (i. e. h is order-preserving) and for any such P, we have $M_h \subset J_P \subset B_P \subset R_h$, where M_h is the maximal ideal of h; $B_P = R_h$ if and only if $\Delta \mid k$ is Archimedean. Two orders P, Q which are both associated with h, need not be equal, but they agree as to finiteness and infinitesimality, $B_Q = B_P$, $J_Q = J_P$. For a chain of prime ideals $\{L_i\}$ in $B \equiv B_P$, let $A_i = B_{L_i} \equiv$ local ring of B_P at L_i. Then A_i is the valuation ring of a place whose maximal ideal is L_i

$$0 < L_g < \ldots < L_j = J \subset B = A_1 < \ldots < A_g$$

These ideas go back to Krull [3]. Cf. [10].

THEOREM 1.—Let h be an order-preserving place of the ordered field $K \mid k$ onto $\Delta \mid k \cup \{\infty\}$. If Gl. rank $K \mid k$ is finite, then:

$$\text{rank } h + \text{Gl. rank } \Delta \mid k = \text{Gl. rank. } K \mid k$$

and if Gl. rank $K \mid k$ is infinite then also rank h is infinite.

Proof.—Assume Gl. rank K | k is finite, say g is the Gleyzal rank. Let $0 < x_1 \lhd \ldots \lhd x_g \in J$ be a saturated chain. Let $L_i = L(x_i)$. Then $0 < L_1 < \ldots < L_g = J$ is saturated. By Lemma 1 the maximal ideal M_h of h fits into the chain of L_i somewhere, so if rank $h = r$ (clearly h has finite rank) then $M_h = L_r$. From Lemma 1a we see that M_h contains $x_1, \ldots, x_r$ and no other x_i. Straightforward computations show that in any case *a real place* h *preserves the relation* $\lhd$ *in the sense that for* u *and* v *in J but not in* M_h, u $\lhd$ v *is valid if and only if* h u $\lhd$ h v *holds in* Δ | k. *Moreover* h *preserves infinitesimals.* From this it follows that the chain $0 < h\, x_{r+1} \lhd \ldots \lhd h\, x_g$ is saturated in Δ | k. This shows that Gleyzal rank of Δ | k is $g - r$, as was to be proved.

In case Gl. rank K | k is infinite the argument above shows that h has infinite rank.

Theorem 2.—For a real function field k (V) = F of dimension d, let $0 < r + m = g < d$, r, m, and g being otherwise arbitrary non-negative integers. There exists an order of F of Gleyzal rank g and an order-preserving place h of rank r, whose residual field has Gleyzal rank m.

Proof.—Let $T = k(x_1, \ldots, x_d)$ be a pure transcendental extension of degree d. Assume, as will be proved shortly, that T admits an order whose Gleyzal rank is g. Let $\bar{T}$ be a real-closure of T relative to such an order. It is possible to embed k (V) in $\bar{T}$; there results an induced order of k (V). Now $\bar{T}$ is algebraic over T and over k (V). Let h be the Krull place on $\bar{T}$ for the given order, let h' be the induced place on k (V) and h'' the place on T. Each of these places is the Krull place for the induced order (the key is that the infinitesimal ideal for the larger field lies over the infinitesimal ideal in the lower). The rank of each place is equal to the Gleyzal rank of the corresponding orders. But since the extensions are algebraic, the ranks are all equal to each other. Hence the Gleyzal ranks are the same, namely g. In k (V), then, we have:

$$0 < P_1 < \ldots < P_g = J \subset B = A_g < \ldots < A_1$$

The place $h_r : A_r \longrightarrow A_r/P_r$ has rank r, and, by Theorem 1, the residual field has rank $g - r = m$.

Thus the problem is reduced to the problem of existence of an order of rank g on a pure transcendental extension of degree $d > g$. Successively adjoin $X_1, \ldots, X_{g-1}$, making each variable infinitesimal over the preceding field; then there is the saturated chain:

$$X_{g-1} \lhd \ldots \lhd X_1 \in J.$$

Let $K = k(X_1, \ldots, X_{g-1})$. Now choose (cf. § 15 [11]) algebraically independent over K formal power series of the form: $z(t) = \Sigma a_i t^i$, with $a_i \in K$, say $z_g, \ldots, z_d$. For any member of $K(X_g, \ldots, X_d)$ say $f(X_g, \ldots, X_d)$, substitute $z_i(t)$ for X_i and factor out of the resulting Laurent series the minimum power of t:

$$f \longrightarrow f(z_g(t), \ldots, z_d(t)) = t^N \bar{f}(t)$$

where $\bar{f}(t)$ is a power series $f_0 + f_1 t + \ldots$, with $f_0 \neq 0$. Tus $\bar{f}(0)$ is a non-zero member of K (except when $f = 0$) because of the algebraic independence of the $z_i(t)$.

Define P as the set of all f such that $f = 0$ or $\bar{f}(0) > 0$. This defines an order of $K(X_g, \ldots, X_d) \mid K$ whose Gleyzal rank is one. In fact, the chain

$$0 < X_g \lhd \ldots \lhd X_1 \in J$$

is saturated in $k(V)$. The theorem is proved.

NOTE 1.—If *F were merely of finite transcendence degree the conclusion follows*; same proof, regardless of the algebraic degree.

NOTE 2.—The case $g = d$ corresponds to a trivial place.

PROPOSITION.—(Extract from the proof of theorem 2).

Gleyzal rank is not altered in algebraic extensions

3. PLACES ON FUNCTION FIELDS.—An *inner point* of a real algebraic variety is any member of the strong closure of the set of all simple points. Let k be an ordered field. $V \mid k$ a real variety in $K^{(n)}$ where $K \mid k$ is real-closed; let $k[x_1, \ldots, x_n]$ be the coordinate ring.

THEOREM 3.—For any inner point Q on V there is a real place on

k (V) centered at Q, V being a real variety over the ordered field k.

PROOF.—The standard reduction to the case where dim Q = 0 works here. It must merely be observed that after the reduction by change of ground field Q is still an inner point. See, e. g. Theorem 16 [2].

Now assume that Q is an inner point of dimension zero. In the argument following it is assumed merely that $k(Q) \mid k$ is Archimedean. The condition that Q be an inner point of V, i. e. a member of the strong closure of the set of all simple points of V, is equivalent to the condition *that every strong neighborhood of Q is Zariski-dense in V*. For if Q is not in the strong closure then there is an oriented box U in $K^{(n)}$ which contains Q but no simple points, whence $U \cap V$ is contained in the proper subvariety S of all singular points. There exist a polynomial which vanishes over S and hence over $U \cap V$ but which is not in the ideal of V. This shows that $U \cap V$ is not Zariski dense in V. For the converse the case $K = \mathbb{R}$ was proved in [3], Theorem 4.9. Tarski's principle thus gives the result for arbitrary real closed K.

The first step in the construction is to construct an order P of k (V) which is centered at Q: this means that P contains every $f(x)$ which is defined and positive at Q. Let Δ be the set of all $f(x)$ in k [V] which are positive (> 0) at Q, let Σ be the set of all sums of the form $\Sigma\, d_i u_i^2$ with d_i in Δ, u_i arbitrary in k (V). To prove that there exists an order of k (V) which is centered at Q it suffices to find an order which contains Σ. If there were no such order then there would exist d_i and u_i with $u_i \neq 0$, and $\Sigma\, d_i u_i^2 = 0$ ([2], [3]). Write f_i/g_i for u_i, with f_i and g_i in k [V], both nonzero. Set

$$G = \Pi\, g_i, \qquad G_i = \Pi_{j \neq i}\, g_j.$$

The latter is G/g_i which is not zero, i. e. G_i is not zero. Then:

$$0 = \Sigma\, d_i u_i^2 = G^2\, \Sigma\, d_i (f_i^2/g_i^2) = \Sigma\, d_i^2\, G_i^2 f_i^2$$

Since d_i belongs to Δ, $d_i(Q)$ is strictly positive. By continuity there exists a strong neighborhood U of Q in V such that $d_i(Q') > 0$ is valid for all Q′ in U. Then for all Q′ in U,

$$0 = \Sigma\, d_i(x)\, G_i(x)^2 \cdot f_i(x)^2 = \Sigma\, d_i(Q') \cdot G_i(Q')^2 f_i(Q')^2$$

Since $d_i(Q')$ is positive, $G_i(Q') \cdot f_i(Q')$ is zero for all i and all Q' in U. By the previons paragraph, the assumption that Q is an inner point implies that the neighborhood U is Zariski-dense in V, which with the just deduced condition that $G_i(x) \cdot f_i(x)$ vanishes all over U, implies $G_i(x) \cdot f_i(x) = 0$. By hypotheses $f_i(x) \neq 0$ whence $G_i(x) = 0$, contrary to the condition deduced above.

Thus we have an order P which is centered at Q, with finite ring B, infinitesimal ideal J. Let J_Q be the set of all infinitesimals in $k[Q]$, let the letter σ represent the specialization at Q and let I_Q be the Kernel of σ. We now show that

$$I_Q \subset J_Q \cap k\,|\,V\,| = \sigma^{-1} J_Q$$

Let $f(x)$ be any member of I_Q. Then $f(Q) = 0$ and for all positive m in k $m \pm f(Q) > 0$. Hence $m \pm f(x) > 0$, which shows that $f(x)$ belongs to J. As to the equality it is enough to observe that the condition for membership of an $f(x)$ (in $k[V]$) in J is that $m \pm f(x) > 0$ hold for all positive m in k, which is the same as the condition for membership of $f(x)$ in $\sigma^{-1} J_Q$.

By assumption, $k[Q]$ contains no infinitely large elements. Let h be the canonical map of B onto B/J, with the latter field ordered compatibly. Let σ be the specialization (as before) and define μ so as to make the diagram commutative:

$$\begin{array}{ccccc} J & \subset & B & \xrightarrow{\;h\;} & B/J \\ \Big| & & \Big| & & \Big\uparrow \mu \\ I_Q & \subset & k[V] & \xrightarrow{\;\sigma\;} & k[Q] \end{array}$$

Now we show that both σ and μ are order-preserving maps.

For the first assertion, let $f(x)$ be any member of $k[V]$. By assumption $f(Q)$ belongs to B. Choose a positive m in k with $m \pm f(Q) > 0$. Since our order of $k(V)$ is centered at Q, $m \pm f(x) > 0$, so $f(x)$ itself belongs to B. Thus $k[V] \subset B$. Now the centering of our order at Q implies that σ is order-preserving. For $f(x)$ in $k[V]$, if $f(Q) \geqslant 0$ then $f(x)$ is either positive or a member of J, and in either case $\mu f(Q) = h f(x) \geqslant 0$, since h preserves order. This shows that μ preserves order.

Next, we note that $k[Q]$ contains no infinitesimals. Hence J_Q lies over I_Q and, therefore, μ is injective for from the condition $J_Q = (0)$ it follows that $I_Q = J \cap k[V]$, since $I_Q = \sigma^{-1} J_Q = \sigma^{-1}(0)$, J lies over I_Q.

We now see that h is isomorphic with a real place centered at Q, wich completes the proof of the reduced case, and with that, the theorem.

NOTE.—Extract from the proof. Q is an inner point of V if and only if every strong neighborhood of Q is Zariski-dense.

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