ON A GENERATING FUNCTION OF ULTRASPHERICAL POLYNOMIALS

by

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SUMMARY

S. K. Chatterjea has recently proved a class of generating relations involving ultraspherical polynomials from the view point of continuous transformation-groups. The object of the present paper is to point out that this class of generating relations implies the explicit representation, the addition and the multiplication formulas, in addition to the usual generating relation for the ultraspherical polynomials.

1. In a recent paper [1] Chatterjea has proved the following generating function for the Ultraspherical polynomials from the view point of continuous Transformation - groups.

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} [t(\mu - \sqrt{\mu^2-1})]^n P_n^\lambda (\mu x - (x-t)\sqrt{\mu^2-1}), \tag{1.1}$$

where $P^\lambda_n(x)$ is defined by the Rodrigues' formula

$$P_n^\lambda(x) = \frac{(-1)^n (2\lambda)_n (1-x^2)^{\frac{1}{2}-\lambda}}{2^n n! \left(\lambda+\frac{1}{2}\right)_n} D^n (1-x^2)^{n+\lambda-\frac{1}{2}},$$

where

$$D = \frac{d}{dx}$$

and μ is quite arbitrary.

In particular, when $\mu = 1$, one obtains the usual generating function

$$(1 - 2\,x\,t + t^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{\lambda}(x)\, t^n \tag{1.2}$$

The object of this note is to point out that one can make elementary verification of Chatterjea's result (1.1) by means of any one of the following formulas of Ultraspherical polynomials:

(A) Explicit representation [2]

$$P_n^{\lambda}(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m} (2\,x)^{n-2m}}{m\,!\,(n-2\,m)\,!} \tag{1.3}$$

(B) Addition Theorem [3]

$$(1 + 2\,x y + y^2)^{n/2} \cdot P_n^{\lambda}\left(\frac{x+y}{\sqrt{1+2\,x\,y+y^2}}\right)\frac{n\,!}{(2\,\lambda)_n} =$$

$$= \sum_{m=0}^{n} \binom{n}{m} \frac{m\,!}{(2\,\lambda)_m} P_m^{\lambda}(x)\,(y)^{n-m} \tag{1.4}$$

For our purpose, we proceed to show that (1.4) is equivalent to (1.5).

Putting

$$\frac{x}{\sqrt{1+2\,x\,y+y^2}} = u,$$

and

$$\frac{y}{\sqrt{1+2\,x\,y+y^2}} = v,$$

so trat $\dfrac{u}{v} = \dfrac{x}{y}$ and therefore

$$\lambda = \frac{u}{\sqrt{1-v^2-2\,u\,v}}, \quad y = \frac{v}{\sqrt{1-v^2-2\,u\,v}}.$$

Thus the above addition theorem can be stated as follows ;

$$P_n^\lambda(u+v) = \sum_{m=0}^{n} \binom{n}{m} p_n \frac{(2\lambda)_n\, m!}{n!\,(2\lambda)_m} \cdot P_m^\lambda\left(\frac{u}{p}\right)\left(\frac{v}{p}\right)^{n-m} \qquad (1.5)$$

where

$$p = \sqrt{1 - v^2 - 2uv}$$

(C) Multiplication theorem [4]

$$P_n^\lambda(\mu x) = \sum_{m=0}^{[n/2]} \frac{(\lambda)_m}{m!} \mu^n \left(1 - \frac{1}{\mu^2}\right)^m \cdot P_{n-2m}^{(\lambda+m)}(x) \cdot \qquad (1.6)$$

In other words, we wish to show that Chatterjea's generating function (1.1) contains many properties of Ultraspherical polynomials, viz, the explicit representation, the addition formula, the multiplication formula, in addition to the usual generating function (1.2).

2. For our purpose, we may write (1.1) in the form.

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} \left(\frac{t}{\xi}\right)^n P_n^\lambda\left(\xi x + \frac{1-\xi^2}{2\xi} t\right) \qquad (2.1)$$

First we notice that

$$\begin{aligned}
(1 - 2xt + t^2)^{-\lambda} &= \left(1 + \frac{t^2}{\xi_2}\right)^{-\lambda} \cdot \left[1 - \frac{2xt\xi^2 + t^2 - \xi^2 t^2}{\xi^2 + t^2}\right]^{-\lambda} = \\
&= \sum_{m=0}^{\infty} \frac{(\lambda+n)_m (-t^2/\xi^2)^m}{m!} \cdot \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(2xt + \frac{1-\xi^2}{\xi^2} t^2\right)^n = \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda+n)_m (-t^2/\xi^2)^m}{m!} \cdot \frac{(\lambda)_n}{n!} \left(2xt + \frac{1-\xi^2}{\xi^2} t^2\right)^n = \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{t}{\xi}\right)^{n+2m} \frac{(-1)^m (\lambda)_{n+m} \left(2\xi x + \frac{1-\xi^2}{\xi} t\right)^n}{m!\, n!} = \\
&= \sum_{n=0}^{\infty} \left(\frac{t}{\xi}\right)^n \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m} \left(2\xi x + \frac{1-\xi^2}{\xi} t\right)^{n-2m}}{m!\,(n-2m)!}
\end{aligned} \qquad (2.2)$$

It follows therefore from (2.1) and (2.2) that

$$P_n^\lambda\left(\xi x + \frac{1-\xi^2}{2\xi}t\right) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m} \left(2\xi x + \frac{1-\xi^2}{\xi}\right)}{m!\,(n-2m)!},$$

which is the explicit representation (1.3) for the Ultraspherical polynomials.

Next we have:

$$(1-2xt+t^2)^{-\lambda} =$$

$$= \left[\frac{\{2\xi^2-(1-\xi^2)t^2\}^2}{4\xi^4}\right]^{-\lambda} \left[1 - \frac{4\xi^2 x t}{2\xi^2-(1-\xi^2)t^2} + \frac{\sigma^2 t^2}{\{2\xi^2-(1-\xi^2)t^2\}^2}\right]^{-\lambda}$$

where

$$\sigma = [4\xi^2 - 4\xi^2 x t(1-\xi^2) - (1-\xi^2)^2 t^2]^{\frac{1}{2}} =$$

$$= \left[\frac{2\xi^2-(1-\xi^2)t^2}{2\xi^2}\right] \cdot \sum_{m=0}^{\infty} \left(\frac{\sigma t}{2\xi^2-(1-\xi^2)t^2}\right)^m P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right) =$$

$$= \sum_{m=0}^{\infty} \left(\frac{\sigma t}{2\xi^2}\right)^m P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right)\left[1 - \frac{(1-\xi^2)t^2}{2\xi^2}\right] =$$

$$= \sum_{m=0}^{\infty} \left(\frac{\sigma t}{2\xi^2}\right)^m P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right) \sum_{n=0}^{\infty} \frac{(2\lambda+m)_n}{n!} \left(\frac{(1-\xi^2)t^2}{2\xi^2}\right)^n =$$

$$= \sum_{n=0}^{\infty} \left[\frac{\frac{t^2(1-\xi^2)}{2\xi^2}}{n!}\right]^n \cdot \sum_{m=0}^{\infty} \left(\frac{\sigma t}{2\xi^2}\right)^m (2\lambda+m)_n P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right) =$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(\frac{t}{\xi}\right)^{n+m} \frac{(2\lambda+m)_n \sigma^{n+m}}{(2\xi)^{n+m} n!} \cdot P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right)\left(\frac{1-\xi^2}{\sigma}t\right)^n =$$

$$= \sum_{n=0}^{\infty} \left(\frac{t}{\xi}\right)^n \sum_{m=0}^{n} \frac{(2\lambda)_n}{n!} \frac{\sigma^n}{(2\xi)^n} \binom{n}{m} \frac{m!}{(2\lambda)_m} \cdot P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right)\left\{\frac{(1-\xi^2)t}{\sigma}\right\}^{n-m} \tag{2.3}$$

It follows therefore from (2.1) and (2.3) that

$$P_n^\lambda\left(\xi x + \frac{1-\xi^2}{2\xi}t\right) =$$

$$= \sum_{m=0}^{n} \frac{(2\lambda)_n \sigma^n}{n!\,(2\xi)^n} \binom{n}{m} \frac{m!}{(2\lambda)_m} \cdot P_m^\lambda\left(\frac{2\xi^2 x}{\sigma}\right)\left\{\frac{(1-\xi^2)t}{\sigma}\right\}^{n-m}$$

which is the addition formula (1.5) for the Ultraspherical polynomials.

Lastly we note that

$$(1-2\,x\,t+t^2)^{-\lambda}=$$

$$=\left[\frac{\{2\,\xi^2-t^2\,(1-\xi^2)\}^2-4\,\xi^2\,x\,t\,\{2\,\xi^2-t^2\,(1-\xi^2)\}+\sigma^2\,t^2}{4\,\xi^4}\right]^{-\lambda}$$

where

$$\sigma=[4\,\xi^2-4\,\xi^2\,x\,t\,(1-\xi^2)-(1-\xi^2)^2\,t^2]^{1/2}=$$

$$=\left(1-\frac{t^2\,(4\,\xi^4-\sigma^2)}{R}\right)^{-\lambda}\left(\frac{4\,\xi^4}{R}\right)^{\lambda}$$

where

$$R=\{2\,\xi^2-t^2\,(1-\xi^2)\}^2-4\,\xi^2\,x\,t\,\{2\,\xi^2-t^2\,(1-\xi^2)\}+4\,\xi^4\,t^2=$$

$$=\left(\frac{4\,\xi^4}{R}\right)^{\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k\,!}\left(\frac{t^2\,(4\,\xi^4-\sigma^2)}{R}\right)^{k}=$$

$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k\,!}\left\{\frac{t^2\,(4\,\xi^4-\sigma^2)}{4\,\xi^4}\right\}^{k}\left\{\frac{2\,\xi^2-t^2\,(1-\xi^2)}{2\,\xi^2}\right\}^{-2\lambda-2k}\cdot$$

$$\cdot\left\{\frac{R}{[2\,\xi^2-t^2\,(1-\xi^2)]^2}\right\}^{-\lambda-k}=$$

$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k\,!}\left\{\frac{t^2\,(4\,\xi^4-\sigma^2)}{4\,\xi^4}\right\}^{k}\left\{\frac{2\,\xi^2-t^2\,(1-\xi^2)}{2\,\xi^2}\right\}^{-2\lambda-2k}\cdot$$

$$\cdot\sum_{m=0}^{\infty}P_m^{\lambda+k}(x)\left\{\frac{2\,\xi^2\,t}{2\,\xi^2-t^2\,(1-\xi^2)}\right\}^{m}=$$

$$=\sum_{m=0}^{\infty}P_m^{\lambda+k}(x)\,t^m\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k\,!}\left\{\frac{t^2\,(4\,\xi^4-\sigma^2)}{4\,\xi^4}\right\}^{k}\left\{\frac{2\,\xi^2-t^2\,(1-\xi^2)}{2\,\xi^2}\right\}^{-2\lambda-m-2k}=$$

$$=\sum_{m=0}^{\infty}P_m^{\lambda+k}(x)\,t^m\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k\,!}\left\{\frac{t^2\,(4\,\xi^4-\sigma^2)}{4\,\xi^4}\right\}^{k}\cdot$$

$$\cdot\sum_{n=0}^{\infty}\frac{\left[\dfrac{t^2\,(1-\xi^2)}{2\,\xi^2}\right]^n}{n\,!}\,(2\,\lambda+m+2\,k)_n=$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left[\dfrac{t^2\,(1-\xi^2)}{2\,\xi^2}\right]^n}{n\,!}\left(\frac{t\,\sigma}{2\,\xi^2}\right)^{m+2k}(2\,\lambda+m+2\,k)_n\frac{(\lambda)_k}{k\,!}$$

$$\cdot\,\alpha^{m+2k}\left(1-\frac{1}{\alpha^2}\right)^{k}P_m^{\lambda+k}(x)$$

where

$$\alpha = 2\,\xi^2/\sigma = \sum_{n=0}^{\infty} \frac{\left[\dfrac{t^2(1-\xi^2)}{2\,\xi^2}\right]^n}{n\,!} \sum_{m=0}^{\infty} \left(\frac{t\,\sigma}{2\,\xi^2}\right)^m (2\,\lambda + m)_n \cdot$$

$$\cdot \sum_{k=0}^{[m/2]} \frac{(\lambda)_k}{k\,!}\,\alpha^m \left(1 - \frac{1}{\alpha^2}\right)^k \cdot P_{m-2k}^{\lambda+k}(x) \tag{2.4}$$

Now in deriving (2.3) we have noticed

$$(1 - 2\,x\,t + t^2)^{-\lambda} = \sum_{n=0}^{\infty} \frac{\left[\dfrac{t^2(1-\xi^2)}{2\,\xi^2}\right]^n}{n\,!} \cdot \sum_{m=0}^{\infty} \left(\frac{\sigma\,t}{2\,\xi^2}\right)^m (2\,\lambda + m)_n\, P_m^{\lambda}(\alpha\,x). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$P_m^{\lambda}(\alpha\,x) = \sum_{k=0}^{[m/2]} \frac{(\lambda)_k}{k\,!}\,\alpha^m \left(1 - \frac{1}{\alpha^2}\right)^k P_{m-2k}^{\lambda+k}(x)\cdot$$

which is the multiplication formula (1.6) for the Ultraspherical polynomials.

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References

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