ON SOME TOPOLOGICAL ALGEBRAS OF HOLOMORPHIC FUNCTIONS

por

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INTRODUCTION.—Let E be a complex Banach space, U a balanced open subset of E and $\mathcal{H}$ (U) the algebra of all holomorphic functions

$$f\colon U \subset E \longrightarrow C$$

endowed with the Nachbin topology τ_ω [6]. In [5] Mujica has proved that τ_ω is a multiplicative locally convex topology on $\mathcal{H}$ (U) and Aron in [1] has explicitely constructed a fundamental family of seminorms for τ_ω. A slight improvement is done here by selecting a subfamily which is fundamental for τ_ω and multiplicative, which might perhaps interest for studying the topological algebra ($\mathcal{H}$ (U), τ_ω).

When E has an inconditional Schauder basis, τ_ω is the same as τ_δ ([2] and [3]), hence the results also applie to ($\mathcal{H}$ (U), τ_δ).

We shall systematically use standard symbols in theory of Holomorphy [6].

For each non void subset S of U and each element $f \in \mathcal{H}$ (U) we shall write

$$\|f\|_S = \sup_{x \in S} |f(x)|$$

and for $S \subset U$ and $\rho \geq 0$, B_ρ (S) is defined by

$$B_\rho(S) = \bigcup_{x \in S} B_\rho(x)$$

where $B_\rho(x)$, stands for the open ball of E with center at x and radius ρ.

DEFINITION 1.—We shall say that a seminorm p on $\mathcal{H}$ (U) is ported by a compact subset K of U if for each open set V, $K \subset V \subset U$, there is a constant $C_V \geq 0$ such that

$$p(f) \leq C_V \|f\|_V$$

holds for all $f \in \mathcal{H}$ (U).

The topology τ_ω on $\mathcal{H}$ (U) is defined by the family of the seminorms which are ported by the compact subsets K of U.

Now, the following proposition is known [1]:

PROPOSITION 1.—For every non void conpact subset K of U and every sequence $\alpha = (\alpha_n)_{n \in N}$ of real numbers such that $\alpha_n \geq 0$ ($n \in$ N) and $\lim_{n \to \infty} \alpha_n = 0$, the application

$$p_{K,\alpha}: f \longrightarrow p_{K,\alpha}(f) = \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(0) \right\| B_{\alpha_n}(K) \qquad f \in \mathcal{H}(U)$$

is a seminorm ported by K and the family of the $P_{K,\alpha}$ when K and α are in the above conditions define the topology τ_ω on $\mathcal{H}$ (U)

LEMMA 1.—For each sequence $\alpha = (\alpha_n)_{n \in N}$ of real numbers such that $\alpha_n \geq 0$ ($n \in$ N) and $\lim_{n \to \infty} \alpha_n = 0$ there is another sequence $\beta = (\beta_n)_{n \in N}$ satisfying:

a).—β dominante α, that is, $\beta_n \geq \alpha_n$ for $n \in$ N.

b).—β is decreasing, that is, $\beta_{n+1} \geq \beta_n$ for $n \in$ N.

c).—β is a null sequence, that is, $\lim_{n \to \infty} \beta_n = 0$.

Indeed, we assume that α has infinitely many non null terms because otherwise the statement would obviously be true.

Now, let us define $\rho_1 = \sup_{n \geq 0} \alpha_n$, it is $\rho_1 > 0$ and, owing to the fact $\lim_{n \to \infty} \alpha_n = 0$, it is easy to see that the supremun is accessible and it is reached only for a finite number of indexes. Let n_1 be the last of those indexeses and define $\rho_2 = \sup_{n > n_1} \alpha_n$. It is $\rho_2 > 0$ and the supremum is reached only for a finite number of indexes, the last of which is denoted by n_2. By induction we get two sequences $(\rho_j)_{j \in N}$ and $(n_j)_{j \in N}$, and clearly

$$\beta = (\beta_n)_{n \in N} = \{\rho_1, \overset{(n_1}{\ldots}, \rho_1; \rho_2, \overset{(n_2}{\ldots}, \rho_2; \ldots\}$$

satisfies the conditions of the lemma.

DEFINITION 2.—We shall say that a seminorm p on the algebra $\mathcal{H}$ (U) is submultiplicative if

$$p(f \cdot g) \leq p(f)\, p(g)$$

holds for all $f, g \in \mathcal{H}$ (U).

We shall say that a family of seminorms on $\mathcal{H}$ (U) is submultiplicative if each one of its elements satisfies the above condition.

PROPOSITION 2.—The family of seminorms

$$p_{K,\beta}: f \longrightarrow p_{K,\beta}(f) = \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(\theta) \right\| B_{\beta_n}(K) \quad f \in \mathcal{H}(U) \qquad (2)$$

where K and $\beta = (\beta_n)_{n \in N}$ range respectively over the compact subsets of U and the decreasing null sequences, is submultiplicative and define τ_ω.

Indeed, if τ is the topology define on $\mathcal{H}$ (U) by the seminorms (2) on account of proposition 1 one gets $\tau \leq \tau_\omega$. On the other hand, let $p_{K,\alpha}$ be one of the family (1) and assume that $\beta = (\beta_n)_{n \in N}$ is a decreasing null sequence which dominates $\alpha = (\alpha_n)_{n \in N}$. We have

$$\left\| \frac{1}{n!} \hat{d}^n f(\theta) \right\| B_{\alpha_n}(K) \leq \left\| \frac{1}{n!} \hat{d}^n f(\theta) \right\| B_{\beta_n}(K)$$

for all $f \in \mathcal{H}$ (U) and all $n \in$ N, so that

$$p_{K,\alpha}(f) = \sum_{0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(\theta) \right\| B_{\alpha_n}(K) \leq \sum_{0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n f(\theta) \right\| B_{\beta_n}(K) = p_{K,\beta}(f)$$

holds for all $f \in \mathcal{H}$ (U), that is $\tau_\omega \leq \tau$.

Now, let $p_{K,\beta}$ be an arbitrary seminorm of the form (2). For all $f, g \in \mathcal{H}$ (U) and all $n \in$ N we have,

$$\hat{d}^n (fg)(\theta) = \sum_{j=0}^{n} \binom{n}{j} \hat{d}^j f(\theta) \cdot \hat{d}^{n-j} g(\theta)$$

hence,

$$\left\| \frac{1}{n!} d^n(f \cdot g)(\theta) \right\| B_{\beta_n}(K) \leq$$

$$\leq \sum_{j=0}^{n} \left\| \frac{1}{j!} \hat{d}^j f(\theta) \right\| B_{\beta_n}(K) \cdot \left\| \frac{1}{(n-j!)} \hat{d}^{n-j} g(\theta) \right\| B_{\beta_n}(K) \qquad (3)$$

But β is a decreasing sequence, so that the relations $j \leq n$ and $n - j \leq n$ imply $\beta_n \leq \beta_j$ and $\beta_n \leq \beta_{n-j}$; therefore

$$\left\| \frac{1}{j!} \hat{d}^j f(\theta) \right\| B_{\beta_n}(K) \leq \left\| \frac{1}{j!} \hat{d}^j f(\theta) \right\| B_{\beta_j}(K)$$

$$\left\| \frac{1}{(n-j)!} \hat{d}^{n-j} g(\theta) \right\| B_{\beta_n}(K) \leq \left\| \frac{1}{(n-j)!} \hat{d}^{n-j} g(\theta) \right\| B_{\beta_{n-j}}(K) \tag{4}$$

From (3), (4) and the definition of $p_{K,\beta}$ one gets:

$$p_{K,\beta}(f \cdot g) = \sum_{n=0}^{\infty} \left\| \frac{1}{n!} \hat{d}^n (f \cdot g)(x) \right\| B_{\beta_n}(K) \leq$$

$$\leq \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left\| \frac{1}{j!} \hat{d}^j f(\theta) \right\| B_{\beta_j}(K) \left\| \frac{1}{(n-j)!} \hat{d}^{n-j} g(\theta) \right\| B_{\beta_{n-j}}(K) =$$

$$= \left(\sum_{0}^{\infty} \left\| \frac{1}{r!} \hat{d}^r f(\theta) \right\| B_{\beta_r}(K) \right) \left(\sum_{0}^{\infty} \left\| \frac{1}{s!} \hat{d}^s g(\theta) \right\| B_{\beta_s} \right) = p_{K,\beta}(f)\, p_{K,\beta}(g)$$

which completes de demostratión.

Definition 3.—We define R (U) to be the set consisting of the functions $f \in \mathcal{H}(U)$ such that $f(x) \neq 0$ for all $x \in U$.

It is known that for $f \in R(U)$, the function

$$\frac{1}{f} : x \longrightarrow \frac{1}{f}(x) = \frac{1}{f(x)} \qquad x \in U$$

verifies $\frac{1}{f} \in \mathcal{H}(U)$, so that R (U) is the set of the regular or invertible elements of the algebra $\mathcal{H}(U)$, hence a group relative to the natural multiplication.

Corollary 2.—R (U) endowed with the topology induced by $(\mathcal{H}(U), \tau_\omega)$ is a topological group, and the application

$$f \longrightarrow \frac{1}{f} \qquad f \in R(U)$$

is a topological automorphism of it.

Indeed, given a complex topological algebra with a unit element $e \neq \theta$, and assuming that its topology may be defined by a submultiplicative family of seminorms, it is known [4] that the set of the regular elements is a topological group relative to the induced topology and that the division operation is a topological automorphism.

BIBLIOGRAPHY

[1] ARON, R.: *Holomorphy types for open subsets of a Banach space.* «Studia Math.», 45 (1973), 273-289.

[2] DINEEN, S.: *Holomorphic functions on locally convex topological spaces. I. Locally convex topologies on* $\mathcal{H}(U)$. «Ann. Inst. Fourier Grenoble», XXIII, fasc. 1 (1973), 19-54.

[3] DINEEN, S.: *Holomorphic functions on* (c_0, X_b)*-modules.* «Math. Annalen»,196 (1972), 106-116.

[4] ISIDRO, J. M.: *Topologías en cuerpos,* to appear in «Rev. Mat. Hispano-Americana».

[5] MUJICA, J.: *Spaces of germs of holomorphic functions.* Doctoral Thesis, University of Rochester, 1974, to appear in «Advances in Math.».

[6] NACHBIN, L.: *Topologies on Spaces of holomorphic mappings.* «Ergebnisse der Mathematick», 47, Springer-Verlag, 1969.

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