ON RELATIVELY CONTRACTIVE RELATIONS IN PAIRS OF GENERALIZED UNIFORM SPACES

by

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ABSTRACT

J. C. Mathews and D. W. Curtis, [4], have introduced some structures which generalize structures of usual uniform types to the product of two sets, and they obtain a generalized version of Banach's contraction mapping theorem. In this note we prove that these structures are obtained from the usual analogues by means of a particular bijection; hence we have not a meaningful generalization. For example, this bijection provides, from a result of A. S. Davis, [1], an analogue of Banach's well-known contraction mapping theorem which trivially implies the main result of [4].

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0. INTRODUCTION

In all that follows, A and B will be nonvoid sets, $F \subset A \times B$ a fixed multifunction on A onto B, R_A on A and R_B on B the canonical equivalences associated with a relation $R \subset A \times B$, that is, $x R_A x'$ and $y R_B y'$ if and only if $R(x) = R(x')$ and $R^{-1}(y) = R^{-1}(y')$, respectively. F_d will be the difunctional closure of F, [7]. Of course F_d is a Riguet's multifunction, that is, a multifunction on A onto B such that

$$F_d \circ F_d^{-1} \circ F_d = F_d .$$

The Riguet's multifunctions G are characterized by

$$G^{-1} \circ G = G_A \quad \text{and} \quad G \circ G^{-1} = G_B .$$

We begin by recalling the F-dependent analogues of the inverse and composition of relations, which we shall term F-inverse and F-composition. If U, V $\subset$ A $\times$ B define

$$U_{-1} := F \circ U^{-1} \circ F \quad \text{and} \quad U * V := U \circ F^{-1} \circ V.$$

U is called an F-connector if $F \subset U$, and U is said to be a relation F-enlarged if

$$U_{-2} := (U_{-1})_{-1} = U, [4],$$

or equivalently

$$U = \bigcup \{ U_{-2n} : n \in N \} .$$

If F a Riguet's multifunction, it is clear that U F-enlarged is equivalent to $U = F_B \circ U \circ F_A$. Further, some notations will be used without explanation because their meanings are obvious; for instance, if $\mathcal{U}$ is a family of subsets of A $\times$ B, we write

$$F^{-1} \circ \mathcal{U} := \{ F^{-1} \circ U : U \in \mathcal{U} \} .$$

We can replace the terms connector, composition and inverse by its F-dependent analogues in the axioms of quasi-uniformities and uniformities. This is a natural way to define the F-quasi-uniform and F-uniform structures. An F-quasi-uniformity (F-uniformity) has an F-enlarged (F-symmetric) base, that is, a base of F-enlarged (F-symmetric) F-connectors; this fact enables to place, in this natural context, the «generalized quasi-F-uniformities» and «F-uniformities» considered by Curtis and Mathews, which are, respectively, the F-enlarged bases of all F-enlarged F-connectors of F-quasi-uniformities and F-uniformities, [2], here denominated canonical bases. On the other hand, for an F-(quasi-)uniformity $\mathcal{U}$, the families $F^{-1} \circ \mathcal{U}$ and $\mathcal{U} \circ F^{-1}$ are bases of (quasi-)uniformities $\mathcal{U}_A$ on A and $\mathcal{U}_B$ on B, respectively.

This paper consists of two parts. Firts we prove that F-structures are obtained from the usual analogues by means of a particular bijection; here, the word «usual» applies to the case A = B and F the diagonal D of A $\times$ A. Thus, F-structures cannot be considered as a meaningful generalization. Secondly we use the above bijection to obtain a result from a theorem of A. S. Davis, [1], which trivially implies theorem 2 of [4].

1. F-STRUCTURES: THE BASIC BIJECTION

Let $\mathcal{U}$ be an F-enlarged filter on $A \times B$, that is, $\mathcal{U}$ is a filter with an F-enlarged base. We consider the following F-axioms:

(1) $F \subset U$ for every $U \in \mathcal{U}$.

(2) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V * V \subset U$.

(3) $U \in \mathcal{U}$ implies $U_{-1} \in \mathcal{U}$.

LEMMA.—*Let $\mathcal{U}$ be a nonvoid family of subsets of $A \times B$. Then $\mathcal{U}$ is an F-enlarged filter if and only if $\mathcal{U}$ is an F_d-enlarged filter; in this case, the canonical base of $\mathcal{U}$ with respect to F and F_d is the same; this base $\mathcal{H}$ satisfies*

$$F^{-1} \circ \mathcal{H} = F_d^{-1} \circ \mathcal{H}$$

and

$$\mathcal{H} \circ F^{-1} = \mathcal{H} \circ F_d^{-1}.$$

Further, being $\mathcal{U}$ still an F-enlarged filter, the F-axiom (j) *is valid for $\mathcal{U}$ if and only if the F_d-axiom* (j) *is also valid, where* $j = 1, 2, 3$.

PROOF.—First we can see that

$$F_d \circ F^{-1} = F_d \circ F_d^{-1} = F_{dB} \quad \text{and} \quad F^{-1} \circ F_d = F_d^{-1} \circ F_d = F_{dA};$$

moreover, if $R \subset A \times B$, we have

$$\begin{aligned} F_{dB} \circ R \circ F_{dA} &= \bigcup \{(F \circ F^{-1})^m : m \in N\} \circ R \circ \bigcup \{(F^{-1} \circ F)^n : n \in N\} = \\ &= \bigcup \{(F \circ F^{-1})^n \circ R \circ (F^{-1} \circ F)^n : n \in N\} = \bigcup \{R_{-2n} : n \in N\}. \end{aligned}$$

Hence R is F-enlarged if and only if R is F_d-enlarged.

Now let $U \subset A \times B$ be F-enlarged. Then

$$F^{-1} \circ U = F^{-1} \circ F_d \circ F_d^{-1} \circ U = F_{dA} \circ F_d^{-1} \circ U = F_d^{-1} \circ U;$$

likewise $U \circ F^{-1} = U \circ F_d^{-1}$. Therefore U is F-connector if and only if it is F_d-connector, and $U_{-1} = U_{-1d}$ where U_{-1d} is the F_d-inverse of U. Hence the lemma follows easily.

From now on, by virtue of the result just proved, without restricting the generality we may suppose that F is a Riguet's multifunction, so we have $F_A = F^{-1} \circ F$ and $F_B = F \circ F^{-1}$.

The basic bijection

We use P (S) for the power set of the set S, and p for the natural map of A onto A/F_A. Let the mapping

$$b_0 : P(A \times B) \longrightarrow P(A/F_A \times A/F_A)$$

be defined by $b_0(R) := p \circ F^{-1} \circ R \circ p^{-1}$. Given R_c in $P(A/F_A \times A/F_A)$ we have

$$b_0^{-1}(R_c) = \{ R \in P(A \times B) : F_B \circ R \circ F_A = F \circ p^{-1} \circ R_c \circ p \} .$$

Hence b_0 is surjective and $F \circ p^{-1} \circ R_c \circ p$ is the unique F-enlarged relation in $b_0^{-1}(R_c)$. Thus, the restriction b of b_0 to the set $E(A \times B; F)$ of all F-enlarged relations is a bijection of $E(A \times B; F)$ onto $P(A/F_A \times A/F_A)$, such that $b(F) = D_c$ where D_c is the diagonal of $A/F_A \times A/F_A$. We say that b is the basic bijection.

THEOREM.—(i) *The collection of the canonical bases of the F-enlarged filters on $A \times B$ is one-to-one mapped by* b *onto the collection of the filters on the set $A/F_A \times A/F_A$.*

(ii) *Let $\mathcal{U}$ be an F-enlarged filter with $\mathcal{H}$ as its canonical base. Then $\mathcal{U}$ satisfies the F-axiom* (j) *if and only if* $b(\mathcal{H})$ *satisfies the usual axiom* (j), *where* $j = 1, 2, 3$. *Therefore $\mathcal{U}$ is F-(quasi-)uniformity if and only if* $b(\mathcal{H})$ *is (quasi-)uniformity.*

(iii) *Let $\mathcal{U}$ be an F-quasi-uniformity, $\mathcal{H}$ its canonical base, $\mathcal{T}_A$ the topology on A induced by $\mathcal{U}_A$, $\mathcal{T}_{Ac}$ the quotient topology by F_A, and $\mathcal{T}_c$ the topology on A/F_A induced by the quasi-uniformity* $b(\mathcal{H})$. *Then $\mathcal{T}_c = \mathcal{T}_{Ac}$ and $\mathcal{T}_A$ is the coarsest topology on A such that $\mathcal{T}_c$ is its quotient topology by F_A. Further, being $\mathcal{T}_B$ the topology on the set B induced by $\mathcal{U}_B$, we have $\mathcal{T}_B = \{F(G) : G \in \mathcal{T}_A\}$,* [3].

PROOF.—(i) It suffices to note that if $\mathcal{U}_c$ is a filter on $A/F_A \times \times A/F_A$, then $b^{-1}(\mathcal{U}_c)$ satisfies:

(a) If $U, V \in b^{-1}(\mathcal{U}_c)$ then $U \cap V \in b^{-1}(\mathcal{U}_c)$.

(b) If $U \in b^{-1}(\mathcal{U}_c)$, V F-enlarged and $U \subset V$, then $V \in b^{-1}(\mathcal{U}_c)$; but these conditions characterize the canonical base of an F-enlarged filter.

(ii) We first note that $\mathcal{U}$ satisfies (*j*) if and only if $\mathcal{H}$ satisfies (*j*). On the other hand, for $U, V \in E(A \times B; F)$ we have

$$b(U * V) = p \circ F^{-1} \circ U \circ F^{-1} \circ V \circ p^{-1} = b(U) \circ b(V),$$
$$b(U_{-1}) = p \circ F^{-1} \circ F \circ U^{-1} \circ F \circ p^{-1} = b(U)^{-1}.$$

Hence (ii) follows easily.

(iii) It suffices to observe that each open G in $\mathcal{T}_A$ is F_A-saturated because $F^{-1} \circ F = F_A$ and an F-quasi-uniformizable topology on A satisfies $F^{-1} \circ F(G) = G$ for each open set G, [3].

2. Contractions and fixed points

First we recall some definitions of [4] with our terminology.

Let $\mathcal{U}$ be an F-quasi-uniformity with $\mathcal{H}$ as its canonical base, $\mathcal{B}$ an F-enlarged base of $\mathcal{U}$, r and s positive integers such that $r < s$, and R a multifunction on A into B. For $U \in \mathcal{U}$, U^n denotes the F-composition of n terms equal to U. Then:

R F-admissible: R F-enlarged and $R \circ R^{-1} \circ F \subset F$.

R r/s-map relative to $\mathcal{B}$: $U^s \subset R_{-1} * U^r * R$ for each $U \in \mathcal{B}$.

R r/s-contractive relative to $\mathcal{B}$: $R \circ F^{-1} \circ U^s \circ R^{-1} \circ F \subset U^r$ for $U \in \mathcal{B}$.

If R is F-admissible, then R r/s-map relative to $\mathcal{B}$ is equivalent to R r/s-contractive relative to $\mathcal{B}$. Moreover, the conditions «$\mathcal{H}$ chains A» and «A $\mathcal{H}$-complete» of [4] mean «$(A, \mathcal{U}_A)$ well-chained» and «$(A, \mathcal{U}_A)$ sequentially complete» with the terminology of Davis in [1].

Theorem.—*Let $\mathcal{U}$ be an F-quasi-uniformity on $A \times B$, $\mathcal{B}$ an F-enlarged base of $\mathcal{U}$ and R a multifunction on A into B. Assume that $(A, \mathcal{U}_A)$ is a sequentially complete well-chained space,*

$$F_A = \bigcap \{ U_A \cap U_A^{-1} : U_A \in \mathcal{U}_A \}$$

and R is F-admissible and r/s*-contractive relative to $\mathcal{B}$. Then there is* a $\in$ *A such that*

$$F \cap R = F_A(a) \times F(a)$$

and (u, v) $\in F \cap R$ *implies*

$$\overline{(u, v)} = \overline{F_A(a)} \times \overline{F(a)}.$$

PROOF.—By observing that given $U \in \mathcal{H}$ and $x, a \in A$, we have $x \in F^{-1} \circ U(a)$ if and only if $p(x) \in b(U)(p(a)) = p \circ F^{-1} \circ U(a)$, it is easy to see that if one of $(A, \mathcal{U}_A)$ or $(A/F_A, b(\mathcal{H}))$ is sequentially complete well-chained then both are.

By means of elementary operations,

$$F_A = \bigcap \{ U_A \cap U_A^{-1} : U_A \in \mathcal{U}_A \}$$

is characterized by $\bigcap \{b(U) : U \in \mathcal{H}\}$ antisymmetric, and this is equivalent to $(A/F_A, b(\mathcal{H}))$ T_0-space.

R multifunction on A into B F-admissible is equivalent to R F-enlarged such that $F \subset R_{-1} * R$ and $R * R_{-1} \subset F$, that is, to $b(R)$ is a map. Moreover, R F-admissible and r/s-contractive relative to $\mathcal{B}$ is obviously equivalent to $b(R)$ r/s-map relative to $b(\mathcal{B})$.

Therefore, from theorem 2 of Davis in [1], $b(R)$ has a unique fixpoint, that is, there is a unique $p(a) \in A/F_A$ such that $(p(a), p(a)) \in$ $\in D_c \cap b(R)$, or equivalently such that

$$F \circ p^{-1} \circ (p(a), p(a)) \circ p = (F_A(a, F(a)) = F \cap R.$$

On the other hand, for each $(u, v) \in (F_A(a), F(a))$ we have [3],

$$\overline{(u, v)} = \overline{(u)} \times \overline{(v)} = \overline{F_A(a)} \times \overline{F(a)}.$$

We note that F closed in A × B is equivalent to

$$F = \bigcap \{U_{-1} * U : U \in \mathcal{H}\}$$

which characterizes to $(A/F_A, b(\mathcal{H}))$ as T_2-space. Thus, theorem 2 of [4] is a particular case of the above result.

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