TOPOLOGICAL CHARACTERIZATION OF $\xrightarrow{a}$ AND $\xrightarrow{b}$ CONVERGENCES

by

M.ª CARMEN DE LAS OBRAS-L.

S U M M A R Y

Given a Hilbert real separable space, H, it is used G (H) to denote the Geometry of the closed linear subspaces of H and $S = \{ E^{(n)} \mid n \in N \}$ a sequence in G (H). We construct the set M (S) and

$$\tilde{M}(S) = \bigcap_{S' \subset S} \tilde{A}_{S'},$$

in which

$$\tilde{A}_{S'} = \}x \in H \mid \exists \}x_m \mid m \in N\{ \subset \bigcup E^{(h_n)} \ni x_m \rightarrow x\{$$

The point x belongs to the set M (S) if every τ_w-neighborhood of x intersects almost any of the terms $E^{(n)}$.

The sets M (S) and $\tilde{M}$ (S) show a topological characterization of the $\xrightarrow{a}$ and $\xrightarrow{b}$ convergences (*).

1. Let H be the Hilbert real separable space. It is used [ ] to denote the closed linear hull, τ_w the weak topology [2], τ_s the strong topology and $\bar{A}^{\tau}$ the τ-closure of A.

DEFINITION 1.—Let $S = \{E^{(n)} \mid n \in N\}$ be a sequence of closed linear subspaces. The set

$$N(S) = \bigcap_{1}^{\infty} [E^{(n)}, E^{(n+1)}, \ldots]$$

(*) $\xrightarrow{a}$ and $\xrightarrow{b}$ convergences are defined in reference [3].

is called nucleus of S, and the set

$$\mathcal{N}_e(S) = \bigcap \, [E^{(h_1)}, \ldots, E^{(h_n)}, \ldots],$$

$(h_n) \subset (n)$ the strict nucleus of S. Both sets are closed linear subspaces and hold the following properties:

$$\mathcal{N}(S') \subset N(S)$$
$$\mathcal{N}_e(S') \supset N_e(S)$$
$$\mathcal{N}(S) = \bigcap_{S' \subset S} N(S')$$

for every subsequence S′ of S.

THEOREM 1.—The strict nucleus of S is characterized by the condition $\forall x \in \mathcal{N}_e(S)$, every subbasic τ_w-neighborhood of x, U [6], intersects almost any of the terms of the sequence S [5].

DEFINITION 2.—Given a sequence of closed linear subspaces $\{E^{(n)} \mid n \in N\}$ we say that $E^{(n)} \xrightarrow[\mathcal{N}e]{} E$ if and only if

$$E = \mathcal{N}_e(S') \; \forall \, S' \subset S.$$

The convergence above defined is a i^*-convergence verifying the three Frechet's axiomes [1].

If there exists $\lim\limits_{\xrightarrow{b}} E^{(n)}$ [3], then $\lim\limits_{\xrightarrow{b}} E^{(n)} \subset \mathcal{N}_e(S)$.

2. Given a sequence $S = \{E^{(n)} \mid n \in N\}$, the set

$$M(S) = \{x \in H \mid \text{every } \tau_w\text{-neighborhood cuts almost any } E^{(n)}\}$$

is a closed linear subspace.

We denote

$$A_{S'} = \bigcup_{1}^{\infty} E^{(h_n)}$$

for $S' \subset S$ and

$$S' = \{E^{(h_n)} \mid (h_n) \subset (n)\}.$$

It is easy to prove that M (S) is the set

$$\bigcap_{S' \subset S} \overline{A}_{S'}^{\tau_w}.$$

It is intuitively obvious that we may extend this property to other topologies, so,

$$\bigcap_{S' \subset S} \overline{A}_{S'}^{\tau}$$

is the set of elements of H such that every τ_w-neighborhood cuts almost any terms of S. Therefore, in particular

$$\underset{\xrightarrow{b}}{\text{li}}\ E^{(n)} = \bigcap_{S' \subset S} \overline{A}_{S'}^{\tau_s}$$

DEFINITION 3.—Given $S = \{E^{(n)} \mid n \in N\}$ and M (S), a point x of M (S) is a first class element when for every basic τ_w-neighborhood of x, U, there exist subsequences

$$S' = \{E^{(h_n)}\} \quad \text{and} \quad \{x_{h_n}\}$$

such that

$$x_{h_n} \in U \cap E^{(h_n)}, \quad \| x_{h_n} \| < K_{S'}.$$

Otherwise, x is a second class element.

LEMMA 1.—Let

$$S = \{E^{(n)} \mid n \in N\}, \quad E^{(n)} \xrightarrow{b} E, \quad \dim E = \infty.$$

If $F \subset E$ and $\text{codim}_E\, F = p$, it is verified

$$[\underset{\rightarrow}{\text{ls}}\, E^{(n)} \cap F] = F$$

for every $S' \subset S$.

PROOF.—Let us suppose that F is a hyperplane in E. Then there exists

$$x \in \mathrm{E} \cap \underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(n)}$$

but $x \notin \mathrm{F}$, hence it is possible the existence of the sequence

$$\{x_{h_n}\}, \quad x_{h_n} \in \mathrm{E}^{(h_n)}$$

such that $x_{h_n} \rightharpoonup x$.

We know that $\underset{\rightarrow}{\mathrm{ls}}\, \mathrm{S}'$ is the join of the planes specified by $\omega(x)$ (*) and the diferents rays of $\underset{\rightarrow}{\mathrm{ls}}\, \mathrm{S}'$. When these planes are cutted by F, a set of rays is obtained for which F is the closed linear hull.

Assume that $p > 1$. If $x_1 \in \mathrm{E} \cap \underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(n)}$ and $x_1 \notin \mathrm{F}$, the set $\mathrm{F}_1 = [x_1, \mathrm{F}] \neq \mathrm{E}$ is constructed. Then there exist

$$\{\mathrm{E}^{(k_{1n})}\} \subset \mathrm{S} \wedge x_{k_{1n}} \in \mathrm{E}^{(k_{1n})}$$

such that $x_{k_{1n}} \rightharpoonup x_1$ hence $x_1 \in \underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(k_{1n})}$. Now we will have a x_2 in $\mathrm{E} \cap \underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(k_{2n})}$ but $x_2 \notin \mathrm{F}_1$. Thus there exists

$$\{\mathrm{E}^{(k_{2n})}\} \subset \{\mathrm{E}^{(k_{1n})}\}$$

and $\{x_{k_{2n}}\}$ where $x_{k_{2n}} \in \mathrm{E}^{(k_{2n})}$ such that $x_{k_{2n}} \rightharpoonup x_2$. Of course $\mathrm{F}_2 = [x_1, x_2]$ is contained in $\underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(k_{2n})}$.

Proceeding successively in this way, we obtain $\mathrm{F}_p = \mathrm{E} \ominus \mathrm{F}$.

Let us note that $\underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(k_{pn})}$ is defined by the join of $p+1$ dimensional subspaces though the subspace F_p. The intersection of these subspaces with F is a ray.

From this construction it is clear that F is the closed linear hull of these rays, otherwise

$$\underset{\rightarrow}{\mathrm{ls}}\, \mathrm{E}^{(k_{pn})} \neq \mathrm{E} \quad \text{and} \quad \mathrm{E}^{(n)} \overset{b}{\not\rightharpoonup} \mathrm{E}.$$

Finally this properties are hereditary [3].

(*) ω is the canonical application of H on the Projective space P (H).

THEOREM 2.—Let $S = \{E^{(n)} \mid n \in N\}$ be a sequence of closed linear subspaces, such that $E^{(n)} \xrightarrow{b} E$ and $\dim E = \infty$, then $E \subset M(S)$.

PROOF.—Let $x \in E$, $x \neq 0$, and U a basic τ_w-neighborhood of x such that it do not include the origin and with axe $x + F$ in which F is a finite codimension subspace. Let $F' = E \cap F$, then

$$x + F' = E \cap (x + F).$$

F' and $x + F'$ are affine hyperplanes in $[x, F']$, there exist rays of $\underrightarrow{\text{ls}}\, E^{(n)}$ in $[x, F']$ but outside F' intersecting $x + F'$. Consequently U intersects infinite terms of S. Since $E^{(n)} \xrightarrow{b} E$, U intersects almost any $E^{(n)}$.

THEOREM 3.—Let E be the closed linear subspace of the first class points of M (S′) with

$$S' \subset S, \quad \text{and} \quad S = \{E^{(n)} \mid n \in N\}.$$

Then $E^{(n)} \xrightarrow{b} E$.

PROOF.—We know that

$$\underrightarrow{\text{ls}}\, E^{(n)} = \bigcup_{S' \subset S} \underrightarrow{\text{li}}\, E^{(h_n)}, \quad S' = \{E^{(h_n)}\}.$$

Thus $[\underrightarrow{\text{ls}}\, E^{(n)}] \subset E$.

Let us suppose the existence of $x \neq 0$ in E, orthogonal to the linear hull of $\underrightarrow{\text{ls}}\, E^{(n)}$. Then there exists a basic closed τ_w-neighborhood of x, U, such that $U \cap \underrightarrow{\text{ls}}\, E^{(n)}$ is empty. Since x is a first class point, we have a weak convergent subsequence of which the limit belongs to $U \cap \underrightarrow{\text{ls}}\, E^{(n)}$, which contradicts the hypothesis.

COROLLARY.—$E^{(n)} \xrightarrow{b} E$ if and only if, E is the set of first class points of M (S′) for every $S' \subset S$.

3. For

$$S = \{E^{(n)} \mid n \in N\} \quad \text{and} \quad S' = \{E^{(h_n)}\} \subset S,$$

we denote

$$\tilde{A}_{S'} = \{x \in H \mid \exists \{x_m \mid m \in N\} \subset \bigcup E^{(h_n)} \ni x_m \rightharpoonup x\}.$$

DEFINITION 4.

$$E^{(n)} \xrightarrow{a} E \Longleftrightarrow \underset{\rightharpoonup}{\text{ls}}\, E^{(h_n)} = E \ \forall\ (h_n) \subset (n). \ [3]$$

DEFINITION 5.

$$\tilde{M}(S) = \bigcap_{S' \subset S} \tilde{A}_{S'}$$

PROPERTIES:

i) $\tilde{M}$ (S) is a closed linear subspace.

ii) $\tilde{M}(S) \subset M(S) \ \forall\ S$.

iii) $\tilde{M}(S) = M(S') \ \forall\ S' \subset S \Longleftrightarrow \underset{\rightharpoonup}{\text{ls}}\, S' = \underset{\rightharpoonup}{\text{ls}}\, S \ \forall\ S' \subset S$.

iv) $\tilde{M}(S) = \bigcap_{S' \subset S} \underset{\rightharpoonup}{\text{ls}}\, S'$.

THEOREM 1.—$E^{(n)} \xrightarrow{a} E$ if and only if $\tilde{M}(S) = \tilde{M}(S') \ \forall\ S' \subset S$, and $E = \tilde{M}(S)$.

PROOF.—It is inmediate from iii) and the definition of $\xrightarrow{a}$ convergence.

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Departamento de Matemáticas
Facultad de Ciencias
Oviedo (Spain)