CALCULATIONS ABOUT MULTIPLICITIES (*)

by

MANUEL ABELLANAS

ABSTRACT

One gives a formula for the calculation of the local intersection multiplicity index of analytic varieties, analogous to the Bézout formula in the case of algebraic varieties in the projective space, in the case of normal crossings. One obtains also a recurrent process for the calculation of the local intersection multiplicity index of plane analytic curves

1. NORMAL CROSSINGS

Let $\mathcal{H}_p = \mathbb{C}\{x_1, \ldots, x_p\}$ be the $\mathbb{C}$-algebra of germs of analytic functions in the origin of the affin space $\mathbb{C}^p$, $p = 1, \ldots, n$, and let $\mathcal{M}_p$ be the maximal ideal of $\mathcal{H}_p$.

DEFINITION 1.—We will say that the ideal I of $\mathcal{H}_n$ is a *normal crossing's ideal* when there exists the germs g_i, $i = 1, \ldots, r$, $r \leqslant n$, $g_i \in \mathcal{H}_{n-i+1}$, such that

$$\mathrm{I}_{n-i} = \mathrm{I} \cap \mathcal{H}_{n-i} = (\mathrm{I} \cap \mathcal{H}_{n-i-1})\, \mathcal{H}_{n-i} + g_{i+1}\, \mathcal{H}_{n-i},\ i = 1, \ldots, r,$$

and

$$\mathrm{I} \cap \mathcal{H}_{n-r-1} = \{0\}.$$

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LEMMA 1.—*If I is a normal crossing's ideal, there exists a linear change of coordinates in* $\mathbb{C}^n$ *such that*

$$I = \mathcal{H}_n(P_1, \ldots, P_r), \qquad r \leq n,$$

where $P_i \in \mathcal{H}_{n-i}[x_{n-i+1}]$, $i = 1, \ldots, r$, *are Weierstrass polinomials.*

PROOF.—From the Def. 1 follows that

$$I = \mathcal{H}_n(g_1, \ldots, g_r), g_{i+1} \in \mathcal{H}_{n-i}.$$

If ord $g_i = r_i$, $i = 1, \ldots, r$, one can make a linear change of variables such that g_i be regular relatively to the variable x_{n-i+1}, $i = 1, \ldots, r$, and, by the Weierstrass preparation theorem, one has $g_i = u_i P_i$, u_i unities in $\mathcal{H}_{n-i+1}$, $i = 1, \ldots, r$, q. e. d.

LEMMA 2.—*In the hypothesis of L.1, if height* $I = r$, *one verifies that* $J_s = \mathcal{H}_n(P_1, \ldots, P_s)$, $s \leqslant r$ *is unmixed, of height* s *and* P_{s+1} *don't belong to any minimal prime divisor of* J_s.

PROOF.—Assume that $P_{s+1} \in \mathfrak{p}$, $\mathfrak{p}$ a minimal prime divisor (m.p.d.) of J_s, then $J_{s+1} \subset \mathfrak{p}$, and hence $\mathfrak{p}' \subset \mathfrak{p}$, where $\mathfrak{p}'$ is a m.p.d. of J_{s+1}. But since $J_s \subset J_{s+1}$, $\mathfrak{p}'$ divide to a m.p.d. $\mathfrak{p}_1$ of J_s, i.e. $\mathfrak{p}_1 \subset \mathfrak{p}' \subset \mathfrak{p}$ and $\mathfrak{p}_1 = \mathfrak{p}' = \mathfrak{p}$. Since $h(\mathfrak{p}') \leqslant s$, it follows that $h(J_{s+1}) \leqslant s$ and $h(I) = h(J_r) \leqslant r - 1$, contradiction. This last argument proves also that $h(J_s) = s$ and since $\mathcal{H}_n$ is a Cohen-Macaulay ring, that J_s is unmixed, $s = 1, \ldots, r$, q. e. d.

2. INTERSECTION MULTIPLICITY OF TWO ANALYTIC VARIETIES

Let U, V be two Cohen-Macaulay [1] analytic varieties of the affin space $\mathbb{C}^n$, such that, locally, $U \cap V = \{0\}$, the origin of $\mathbb{C}^n$. Let $\mathcal{H}_n$ be the $\mathbb{C}$-algebra of germs of analytic functions with center 0 and $\mathfrak{p}_U$, $\mathfrak{p}_V$ the ideals defining U and V in $\mathcal{H}_n$, respectively. Then, the Serre's formula [3] gives:

$$i(\mathbb{C}^n, U \cap V, 0) = \dim_{\mathbb{C}} \mathcal{H}_n/\mathfrak{p}_U + \mathfrak{p}_V.$$

From the assumption follows that $\mathfrak{p}_U + \mathfrak{p}_V$ is a primary ideal of the maximal ideal $\mathcal{M}$ of $\mathcal{H}_n$ and, hence, $h(\mathfrak{p}_U + \mathfrak{p}_V) = n$. With the notations of 1., let

$$I_k = (\mathfrak{p}_U + \mathfrak{p}_V) \cap \mathcal{H}_k, \quad k = 1, \ldots, n.$$

LEMMA 3.—*Let A be a nötherian integral domain and J and I ideals of A such that J is unmixed, $I = J + \mathrm{f}\,\mathrm{g}\,A$, and let $I_1 = J + \mathrm{f}\,A$, $I_2 = J + \mathrm{g}\,A$, where g don't belong to any minimal prime divisor of J, then*

$$A/I \approx A/I_1 \oplus A/I_2$$

PROOF.—The sequence

$$0 \longrightarrow A/I_1 \xrightarrow{\varphi} A/I \xrightarrow{\psi} A/I_2 \longrightarrow 0$$

is exact, where $\varphi(h + I_1) = h\,g + I$ and ψ is the natural epimorphism. φ is obviously an A-homomorphism. φ is a monomorphism. Indeed, if $h\,g \in I$, then $h\,g = j + a\,f\,g$, $j \in J$, and $(h - a\,f)\,g \in J$, and by the assumptions $h - a\,f \in J$. Obviously $\psi \circ \varphi = 0$. If $\psi(h + I) = 0$, $h \in I_2$, i.e. $h = j + a\,g$, $j \in J$ and $h + I_1 = a\,g + I_1$ and $\varphi(a + I_1) = h + I$.

REMARK.—If $A = \mathcal{H}_n$ and I, I_1 and I_2 are primary ideals of $\mathcal{M}$ then

$$\mathcal{H}_n/I \approx \mathcal{H}_n/I_1 \oplus_{\mathbb{C}} \mathcal{H}_n/I_2.$$

PROPOSITION 1.—*In the above hypothesis, if $I_n = \mathfrak{p}_U + \mathfrak{p}_V$ is a normal crossing's ideal, one verifies:*

$$i(\mathbb{C}^n, U \cap V, 0) = \prod_{k=1}^{n} \operatorname{ord} I_k.$$

PROOF.—By L.1, there exists a linear change of variables and Weierstrass polynomials: $P_i \in \mathcal{H}_{n-i}[x_{n-i+1}]$ $i = 1, \ldots, n$ such that

$$I = \mathcal{H}_n(P_1, \ldots, P_n),$$

and by L.2, P_{s+1} don't belong to any minimal prime divisor of

$$J_s = \mathcal{H}_n(P_1, \ldots, P_s), \quad s \leq r.$$

Since $I_{n-i} \subset I_{n-i+1}$, it follows that

$$\operatorname{ord} I_{n-i} \geq \operatorname{ord} I_{n-i+1} \qquad (\operatorname{ord}(I) = \min \{\operatorname{ord} f, \quad f \in I\})$$

and taking into account that

$$I_{n-i} = \mathcal{H}_{n-i}(P_{i+1}, \ldots, P_n),$$

it follows that

$$\operatorname{ord} I_{n-i} = \operatorname{ord} P_{i+1} = (\operatorname{degr} P_{i+1})_{X_{n-i}} = r_{n-i}.$$

From all these remarks and taking into account the remark of L.3, it follows that:

$$\begin{aligned} i(\mathbb{C}^n, U \cap V, 0) &= \dim_{\mathbb{C}} \mathcal{H}_n/I_n = \dim_{\mathbb{C}} \mathcal{H}_n/\mathcal{H}_n(P_1, \ldots, P_{n-1}, X_1^{r_1}) = \\ &= r_1 \cdot [\dim_{\mathbb{C}} \mathcal{H}_n/\mathcal{H}_n(P_1, \ldots, P_{n-1}, X_1)] = \\ &= r_1 [\dim_{\mathbb{C}} \mathcal{H}_n/\mathcal{H}_n(P_1, \ldots, P_{n-2}, X_2^{r_2}, X_1)] = \\ &= r_1 \cdot r_2 \cdot [\dim_{\mathbb{C}} \mathcal{H}_n/\mathcal{H}_n(P_1, \ldots, P_{n-2}, X_2, X_1)] = \ldots = \\ &= r_1 \cdot r_2 \ldots r_n \cdot \dim_{\mathbb{C}} \mathcal{H}_n/\mathcal{H}_n(X_n, \ldots, X_1) = r_1 \ldots r_n = \\ &= \prod_{k=1}^{n} \operatorname{ord} I_k. \end{aligned}$$

3. Plane curves

With the same notations as above, let U: $P_1 = 0$ and V: $P_2 = 0$ be two analytic plane curves without common components. We can assume that P_1 and P_2 are Weierstrass polynomials:

$$P_i = x_2^{r_i} + a_{i_1} x_2^{r_i - 1} + \ldots + a_{i r_i}, \quad i = 1, 2\ r_1 \leq r_2.$$

Let

(1) $\quad P_2 = Q P_1 + R, \ \deg R < \deg P_1, \ R = b_0 x^s{}_0 + \ldots + b_s, \ s < r_1.$

and

$$\operatorname{ord}(b_i) = m_i, \quad m_t = \min \{m_i\}_{i=0,\ldots,s} \, 0 \leq t \leq s, \quad m_t < m_j, j > t.$$

Since $b_i = u_i x_1^{m_i}$, u_i unity, $i = 0, \dots, s$, we have

$$R = x_1^{m_t} [(b'_0 x_2^t + \dots + b'_{t-1} x_2 + u_t) x_2^{-} + b_{t+1} x_2^{s-t-1} + \dots + b_s]$$

and by putting

$$G = u x_2^{s-t} + b_{t+1} x_2^{s-t-1} + \dots + b_s, \quad s - t < r_1 \tag{2}$$

where $u = b'_0 x_2^t + \dots + b'_{t-1} x_2 + u_t$ is an unity, it result

$$R = x_1^{m_t} G, \tag{3}$$

and since, by hipothesis P_1 and P_2 are relative prime elements of $\mathcal{H}_2$, it follows from (1) and (3) that P_1 and x_1 are also relative prime elements, and taking into account that

$$I_2 = \mathcal{H}_2 (P_1, P_2) = \mathcal{H}_2 (P_1, R) = \mathcal{H}_2 (P_1, x_1^{m_t} G)$$

and L.3, it follows that:

$$\mathcal{H}_2/I_2 \approx \mathcal{H}_2/\mathcal{H}_2 (P_1, X_1^{m_t}) \oplus \mathcal{H}_2/\mathcal{H}_2 (P_1, G), \tag{4}$$

and by the Prop. 1,

$$\dim_C \mathcal{H}_2/\mathcal{H}_2 (P_1, X_1^{m_t}) = r_1 \cdot m_t. \tag{5}$$

From (3) and (1) follows that G and P_1 are also relatively primes, and hence one can repeat the above process if $t < s$. If $t = s$, from (4) and (5) follows:

$$\dim_C \mathcal{H}_2/I_2 = r_1 \cdot m_s.$$

We have obtained the following:

Proposition 2.—*If $P_1 = 0$, $P_2 = 0$ are the local equations in a neighbourhood of the origin of the affin plane $\mathbb{C}^2$ of the analytic curves U and V, without common components, where P_1 and P_2 are Weierstrass polynomials, one can calculate*

$$(\mathbb{C}^2, U \cap V, 0)$$

by the process of succesive Weierstrass divisions described above.

References

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Universidad Nacional de Educación a Distancia
Departamento de Matemática Fundamental
Ciudad Universitaria. Madrid-3 (Spain)