ON THE UNIFICATION OF GENERALIZED HERMITE AND LAGUERRE POLYNOMIALS *

by

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1. INTRODUCTION.—From time to time various attempts have been made to generalize the Hermite polynomial (cf., e. g., [11])

$$H_n(x) = (-1)^n \exp(x^2)\, D^n \{\exp(-x^2)\}, \qquad (1.1)$$

where $D = \dfrac{d}{dx}$.

The first classical step in this direction was that of E. T. Bell [1] who replaced the exponent 2 in (1.1) by an arbitrary parameter r and later by an arbitrary function. But for a long time, until the investigations of Rajgopal [10] and Riordan [12], Bell's paper went unnoticed. Following the work of earlier researchers, Gould and Hopper [6] defined what they called the generalized Hermite polymonial,

$$H^r_n(x, \alpha, p) = (-1)^n x^{-\alpha} \exp(p\, x^r)\, D^n \{x^\alpha \exp(-p\, x^r)\}. \qquad (1.2)$$

They however mainly confined themselves to the study of operational formulas associated with this polynomial.

On the other hand in 1956, Chak [2] gave a generalization of the Laguerre polynomials [11]

$$L_n^{(\alpha)}(x) = \frac{\exp(x)\, x^{-\alpha}}{n!} D^n \{x^{\alpha+n} \exp(-x)\}, \qquad (1.3)$$

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in the fom

$$P_{n,r}^{(\alpha)}(x) = \frac{x^{-\alpha} \exp(x^r)}{n!} D^n \{x^{\alpha+n} \exp(-x^r)\}. \quad (1.4)$$

Later in 1959, Palas [9] considered the polynomial $T_{kn}(x)$ which satisfies the Rodrigues' formula

$$T_{kn}(x) = \frac{\exp(x^k)}{n!} D^n \{x^n \exp(-x^k)\}. \quad (1.5)$$

The aim of Chak and Palas was to generalize the results of Humbert [7] and Duffahel [5] who separately studied the polynomials

$$P_n(x) = \frac{\exp(x^2)}{n!} D^n \{x^n \exp(-x^2)\} =$$

$$= \exp(x^2)\, {}_2F_2 \left[\begin{matrix} \frac{1}{2}(n+2), \frac{1}{2}(n+1); \\ 1 \quad , \frac{1}{2} \quad ; \end{matrix} \; -x^2 \right]. \quad (1.6)$$

Recently, Chatterjea [3] and Singh and Srivastava [13] independently studied a class of polynomials which termedthet hey generalized Laguerre polynomials. In the notation of Singh and Srivastava the generalized Laguerre polymonial is given by the relation

$$L_n^{(\alpha)}(x, r, p) = \frac{x^{-\alpha} \exp(p\, x^r)}{n!} D^n \{x^{\alpha+n} \exp(-p\, x^r)\}. \quad (1.7)$$

It appears that the generalized Hermite polynomial referred to in (1.2) and the generalized Laguerre polynomial of Singh and Srivastava (also due to Chatterjea) are essentially the same. But on a closer examination one finds that it is not so. That is that they have remarkedly different properties. It was mainly this idea that led the authors to study a class of polynomials which could unify both the Hermite and the Laguerre polynomials. It is interesting to observe that the polynomial which we study here not only gives the generalization referred to above but also includes the generalized Bessel polynomial of Krall and Frink [8]

$$Y_n^{(\alpha)}(x) = {}_2F_0\left[-n, n+\alpha+1; -; -\frac{1}{2}x\right] =$$

$$= \frac{x^{-\alpha}\exp\left(\frac{2}{x}\right)}{2^n} D^n \left\{x^{\alpha+2n}\exp\left(-\frac{2}{x}\right)\right\}. \tag{1.8}$$

as a special case.

2. Definition.—We make the definition

$$J_n^{(\alpha)}(x, r, p, q) = C(q, n)\, x^{-\alpha}\exp(p\, x^r)\, D^n\{x^{\alpha+qn}\exp(-p\, x^r)\} \tag{2.1}$$

where for the sake of brevity

$$C(q, n) = \frac{(-1)^{\frac{1}{2}n(q-1)(q-2)}}{2^{\frac{1}{2}nq(q-1)}(1)_{nq(2-q)}},$$

q is a non-negative integer and

$$(a)_n = a(a+1)\ldots(a+n-1), \quad n \geq 1, \quad (a)_0 = 1.$$

It is easy to see that when $q = 0$, (2.1) reduces to (1.2) and when $q = 1$ it yields the relation (1.7). While if we let $q = 2$, we are led to the generalized Bessel polynomial mentioned above.

We begin by considering the equation (2.1) and obtain

$$J_n^{(\alpha)}(x, r, p, q) = C(q, n)\exp(p\, x^r)(\alpha + \overline{q-1}\, n + 1)_n \cdot$$

$$\cdot \sum_{m=0}^{\infty} \frac{(-p)^m (\alpha + q\, n + 1)_{mr}}{m!\,(\alpha + \overline{q-1}\, n + 1)_{mr}} x^{\overline{q-1}\, n + mr}. \tag{2.2}$$

Since [11]

$$(a)_{nk} = k^{nk}\prod_{j=1}^{k}\left(\frac{\alpha+j-1}{k}\right)_n, \tag{2.3}$$

(2.2) can also be put in a more elegant form

$$J_n^{(\alpha)}(x, r, p, q) = C(q, n)(\alpha + \overline{q-1}\,n + 1)_n\, x^{\overline{q-1}n} \exp(p\,x^r) \cdot$$
$$\cdot\, {}_rF_r\left[\begin{matrix}\Delta(r, \alpha + q\,n + 1); \\ \Delta(r, \alpha + \overline{q-1}\,n + 1);\end{matrix} \quad -p\,x^r\right], \tag{2.4}$$

where $\Delta(r, \alpha)$ stands for the set of r parameters

$$\frac{\alpha}{r}, \frac{\alpha+1}{r}, \ldots, \frac{\alpha+r-1}{r}.$$

It is obvious that when $q = 1$, (2.3) reduces to the formula (1.7) o Chatterjea [4].

Further from (2.2) we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!\,C(q, n)} J_n^{(\alpha)}(x, r, p, q) =$$
$$= \exp(p\,x^r) \sum_{m=0}^{\infty} \frac{(-p)^m x^{mr}}{m!} \sum_{n=0}^{\infty} \frac{(\alpha + 1 + m\,r)_{qn}}{n!\,(\alpha + 1 + m\,r)_{\overline{q-1}n}} (x^{q-1}\,t)^n, \tag{2.5}$$

which in view of (2.3) simplifies to the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{C(q, n)\,n!} J_n^{(\alpha)}(x, r, p, q) =$$
$$= \exp(p\,x^r) \sum_{m=0}^{\infty} \frac{(-p)^m x^{mr}}{m!} {}_qF_{q-1}\left[\begin{matrix}\Delta(q, \alpha + m\,r + 1; \\ \Delta(q-1, \alpha + m\,r + 1);\end{matrix} \quad \frac{x^{q-1}\,q^q}{(q-1)^{q-1}}\,t\right]. \tag{2.6}$$

In the special case $q = 1$, our formula (2.6) yields the generating function for the generalized Laguerre polynomials [13],

$$(1-t)^{-1-\alpha} \exp\{p\,x^r - p\,x^r(1-t)^{-r}\} = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x, r, p), \tag{2.7}$$

whereas on putting $q = 0$ in (2.5), we obtain

$$x^{-\alpha}(x-t)^\alpha \exp\{p\,x^r - p(x-t)^r\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H^r_n(x, \alpha, p), \tag{2.8}$$

due to Gould and Hopper [6].

3. The differlntial equation.—In this section we employ the differential operator $\delta = x \dfrac{d}{dx}$, which possesses the following interesting properties

$$F(\delta)\, x^n = F(n)\, x^n \tag{3.1}$$

and

$$F(\delta)\,[\exp\{g(x)\} f(x)] = \exp\{g(x)\}\, F(\delta + x g')\, f(x). \tag{3.2}$$

In view of these two relations, from (2.2) we have

$$\begin{aligned}
&(\delta - p\, r\, x^r - \overline{q-1}\, n)\,(\delta - p\, r\, x^r + \alpha + 1 - r)_r\, J_n^{(\alpha)}(x, r, p, q) = \\
&\quad = r\,(\alpha + \overline{q-1}\, n + 1)_n\, C(q, n) \exp(p\, x^r) \cdot \\
&\quad \cdot \sum_{m=1}^{\infty} \frac{(-p)^m (\alpha + q\, n + 1)_{mr}}{(m-1)!\,(\alpha + \overline{q-1}\, n + 1)_{mr-r}}\, x^{\overline{q-1}n+mr} = - \\
&\quad = -p\, r\, x^r (\alpha + \overline{q-1}\, n\, 1 +)_n\, C(q, n) \exp(p\, x^r) \cdot \\
&\quad \cdot \sum_{m=0}^{\infty} \frac{(-p)^m (\alpha + q\, n + 1)_{mr} (\alpha + q\, n + 1 + m\, r)_r}{m!\,(\alpha + \overline{q+1}\, n + 1)_{mr}}\, x^{\overline{q-1}n+mr} = - \\
&\quad = -p\, r\, x^r (\delta - p\, r\, x^r + n + \alpha + 1)_r\, J_n^{(\alpha)}(x, r, p, q).
\end{aligned}$$

We therefore, finally have the differential equation

$$\begin{aligned}
&[(\delta - p\, r\, x^r - \overline{q-1}\, n)(\delta - p\, r\, x^r + \alpha + 1 - r)_r + \\
&\quad + p\, r\, x^r (\delta - p\, r\, x^r + n + \alpha + 1)_r]\, Y = 0,
\end{aligned} \tag{3.3}$$

satisfied by $J_n^{(\alpha)}(x, r, p, q)$.

It will be remarked that if in (3.3) we let $q = 0$, we are led to the differential equation

$$\begin{aligned}
&[(\delta - p\, r\, x^r + n)(\delta - p\, r\, x^r + \alpha + 1 - r)_r + \\
&\quad + p\, r\, x^r (\delta - p\, r\, x^r + n + 1 + \alpha)_r]\, H^r_n(x, \alpha, p) = 0,
\end{aligned} \tag{3.4}$$

for the generalized Hermite polynomials, which does not seem to have been noticed earlier. On the other hand $q = 1$ reduces (3.3) to

$$\begin{aligned}
&[(\delta - p\, r\, x^r)(\delta - p\, r\, x^r + \alpha + 1 - r)_r + \\
&\quad + p\, r\, x^r (\delta - p\, r\, x^r + n + 1 + \alpha)_r]\, L_n^{(\alpha)}(x, r, p) = 0.
\end{aligned} \tag{3.5}$$

4. Some additional results.—We notice that (2.1) can be put in the form

$$J_{n+1}^{(\alpha)}(x, r, p, q) = C(q, n+1)\, x^{-\alpha} \exp(p\, x^r) \cdot \\ \cdot D^n \{(\alpha + n\, q + q)\, x^{\alpha+qn+q-1} \exp(-p\, x^r) - \\ - p\, r\, x^{\alpha+qn+q+r-1} \exp(-p\, x^r)\},$$

which simplifies to

$$J_{n+1}^{(\alpha)}(x, r, p, q) = \frac{C(q, n+1)}{C(q, n)} \cdot \\ \cdot \{(\alpha + n\, q + q)\, x^{q-1} J_n^{(\alpha+q-1)}(x, r, p, q) - \\ - p\, r\, x^{q+r-1} J_n^{(\alpha+q+r-1)}(x, r, p, q)\}, \quad (4.1)$$

where

$$\frac{C(q, n+1)}{C(q, n)} = \frac{(-1)^{\frac{1}{2}(q-1)(q-2)}}{2^{\frac{1}{2}q(q-1)}\,[1 + n\, q\,(2-q)]_{q(2-q)}}.$$

Secondly, differentiating both the sides of (2.1) with respect to x ve deduce

$$(x\, D - p\, r\, x^r + \alpha)\, J_n^{(\alpha)}(x, r, p, q) = \frac{x^{1-q}\, C(q, n)}{C(q, n+1)}\, J_{n+1}^{(\alpha-q)}(x, r, p, q). \quad (4.2)$$

Yet another recurrence relation in the form

$$J_n^{(\alpha+1)}(x, r, p, q) - J_n^{(\alpha)}(x, r, p, q) = \\ = \frac{n\, C(q, n)}{C(q, n-1)}\, x^q\, J_{n-1}^{(\alpha+q)}(x, r, p, q), \quad (4.3)$$

follows from (2.1) by making use of the difference operator

$$\Delta f(\alpha) = f(\alpha+1) - f(\alpha).$$

In case when $q = 0$, (4.2) corresponds to (3.4) of Gould and Hopper [6], whereas (4.1) yields

$$x\, H^r_{n+1}(x, \alpha, p) = p\, r\, x^r\, H^r_n(x, \alpha + r - 1, p) - \alpha\, H^r_n(x, \alpha - 1, p). \quad (4.4)$$

Next since

$$J_n^{(\alpha+\beta)}(x, r, p_1 + p_2, q) = C(q, n)\, x^{-\alpha-\beta} \cdot$$
$$\cdot \exp\{(p_1 + p_2) x^r\} D^n [x^{\alpha+qn+\beta} \exp\{-(p_1 + p_2) x^r\}] =$$
$$= C(q, n)\, x^{-\alpha-\beta} \exp\{(p_1 + p_2) x^r\} \cdot$$
$$\cdot \sum_{k=0}^{n} \binom{n}{k} D^{n-k}\{x^\alpha \exp(-p_1 x^r)\} D^k \{x^{\beta+qn} \exp(-p_2 x^r)\},$$

a little simplification will yield the doubly additive addition formula

$$J_n^{(\alpha+\beta)}(x, r, p_1 + p_2, q) =$$
$$= \sum_{k=0}^{n} \binom{n}{k} \frac{C(q, n)}{C(q, k)\, C(q, n-k)} J_k^{(\beta-qk)}(x, r, p_2, q) \,.$$
$$\cdot\, J_{n-k}^{(\alpha+qk)}(x, r, p_1, q), \tag{4.5}$$

which generalizes the formula (3.10) of Gould and Hopper [6] to which it reduces when $q = 0$. And in the special case when $q = 1$ we get the neat formula

$$L_n^{(\alpha+\beta)}(x, r, p_1 + p_2) = \sum_{k=0}^{n} L_{n-k}^{(\alpha+k)}(x, r, p_1)\, L_k^{(\beta-k)}(x, r, p_2), \tag{4.6}$$

for the generalized Laguerre polynomials.

Lastly from (2.3) we readily obtain the multiplication formula

$$J_n^{(\alpha)}(x, r, m\,p, q) = m^{\frac{(1-q)n}{r}} J_n^{(\alpha)}(m^{\frac{1}{r}} x, r, p, q). \tag{4.7}$$

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