GENERAL NUMERATION I. GAUGED SCHEMES

by

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The paper deals with special partitions of whole numbers in the following form: given a sequence of pairs $\{[G_i ; D_i]\}$ of positive integers in which the G_i form a strictly increasing sequence, sums of the form $\Sigma\, n_i\, G_i$, with $0 \leqslant n_i \leqslant D_i$, are considered. The correspondence

$$[\, n_k \ldots n_0 \,] \longmapsto \sum_{i \leq k} n_i\, G_i$$

defines then a mapping $\mathfrak{a}$ from a set $\mathcal{M}$ of numerals, called «Neugebauer symbols», satisfying $0 \leqslant n_i \leqslant D_i$, into the set $\mathcal{W}$ of all non-negative integers. In $\mathcal{M}$, initial zeros are suppressed and $\mathcal{M}$ is ordered in the usual numerical order. Such a $\mathfrak{a}$ is called a *gauged scheme.*

Basic questions which are posed and answered in part, include: how does the structure of the sequence $\{[G_i ; D_i]\}$ affect the mapping $\mathfrak{a}$, especially as regards injectivity, surjectivity, preservation of order, additivity especially *carrying*, when the termwise sum of two numerals in $\mathcal{M}$ falls outside $\mathcal{M}$. The most important conditions involve comparison of G_k with

$$T_k \equiv 1 + \sum_{i < k} D_i\, G_i .$$

The condition that for all k, $G_k \geqslant T_k$, implies injectivity of $\mathfrak{a}$, it is implied by the condition that the addition of two summands involves nothing more complicated than carrying a one to the next place to the left, and it is equivalent to strict order preservation ($m > n$ implies $\mathfrak{a}\, m > \mathfrak{a}\, n$); the condition that for all k, $G_k \leqslant T_k$, is equivalent to surjectivity of $\mathfrak{a}$; the condition that for all k, $G_k = T_k$, is equivalent to bijectivity of $\mathfrak{a}$. Proofs are combinatorial.

The class of gauged schemes contains, along with almost all cultural schemes, a large number of unusual schemes which might conceivably be used for arithmetic and a large class of schemes which are quite impractical for either counting or calculating, exhibiting numerous weird features. In a sequel to this paper another class is presented, the class of *divided schemes*. It too, has most of the cultural schemes and many most unusual ones. Together, the papers constitute a mathematical theory of generalized numeration schemes. The advantages of such a theory are the usual ones of mathematical generalization. The familiar schemes appear against a background of related objects, rather than in isolation, standing out as the «nice objects» i. e. the regular, symmetric species. But appreciation of symmetry increases as asymmetry is better understood and the relations between the various nice, regular, fatures of the familiar types are elucidated. It is hoped that these commonplace mathematical advantages may also be helpful to teachers of arithmetic in base ten as well as other bases. For example, some gauged schemes are useful models of preconservation (in Piaget's sense) and other unusual or improper arithmetic behaviors [3].

There appears to be no precedent for such a theory as this. To be sure, many authors (see Menninger [1], and Neugebauer [2], and references given in these books) have given excellent classifications of numeration schemes, and the «numerals» used in the present paper are Neugebauer's invention. But none of these authors had any intention of making a mathematical theory of generalized numeration schemes.

1. Notations and definitions. The script capital $\mathcal{W}$ is the set of all non-negative integers (whole numbers). Finite sequences of whole numbers are listed in backward order, as are numerals (n_2, n_1, n_0), etc. For two finite sequences m and n, the relation «$m \sim n$» signifies that m and n are the same except possibly for initial zeros: i. e., there exists j such that for all i, if $i \leqslant j$ then m_i and n_i are defined and equal to each other, and if $i > j$ then each of m_i and n_i is either zero or undefined. A *numeral* (in the sense of Neugebauer) is an equivalence class of finite sequences. The set of all these Neugebauer numerals, ordered in the usual way, is denoted by the script capital letter $\mathcal{N}$.

Finite sequences may sometimes be treated as numerals, but sometimes not. If $m = (m_k \dots m_0)$ is a sequence then the numeral containing it is denoted $[m_k \dots m_0]$, or $[m]$. But if no confusion is probable, the letter m may itself be used for this numeral. The zero numeral is denoted $[0]$. For a non-zero numeral n, with, say, $n_k \neq 0$, $n_i = 0$ for all $i > k$, the *length of* n is k. The length of zero is -1.

Linguistic Axiom: $\mathcal{W} \cap \mathcal{N}$ is empty.

In other words, a numeral is never a number.

A *generalized numeration scheme* is any mapping

$$\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{W},$$

where $\mathcal{M}$ is a subset of $\mathcal{N}$ with the inherited order. When $\mathcal{C}\, n = \mathrm{S}$, the interpretations following may be used: «n is a representative of S», «n is a numeral for S».

The generalized scheme $\mathcal{C}$ is *univalent* provided the mapping is univalent in the usual sense, i. e. injective. $\mathcal{C}$ is said to be *complete* provided $\mathcal{C}(\mathcal{M})$ is an initial segment of $\mathcal{W}$, i. e. for some $k \leqslant +\infty$, $\mathcal{C}(\mathcal{M}) = \{\mathrm{S} ; 0 \leqslant \mathrm{S} < k\}$. The scheme is *finite* or *infinite* according as $\mathcal{M}$ is finite or infinite.

2. Gauges and gauged schemes. Let D_i be a sequence (finite or infinite) of positive integers, let G_i be a strictly increasing sequence (finite or infinite) of positive integers, $i = 0, 1, 2, \dots$ The pair of sequences is a *gauge*. It is denoted in two ways, by a script capital $\mathfrak{a}$, and script lower case g:

$$\mathfrak{a} : [\mathrm{G}_0 ; \mathrm{D}_0], \; [\mathrm{G}_1 ; \mathrm{D}_1], \dots$$

$$g : g_0, g_1, g_2, \dots$$

where the sequence g is constructed by listing the G_i in order with each G_i appearing D_i times in the list. Thus

$$g_i = \begin{cases} \mathrm{G}_0, & \text{if } 0 \leq i < \mathrm{D}_0 \\ \mathrm{G}_1, & \text{if } \mathrm{D}_0 \leq i < \mathrm{D}_0 + \mathrm{D}_1 \\ & \dots \\ \mathrm{G}_k, & \text{if } \sum\limits_{j<k} \mathrm{D}_j \leq i < \sum\limits_{j<k+1} \mathrm{D}_j \\ & \dots \end{cases}$$

EXAMPLE 1.—$\mathfrak{a}$: [1 ; 5], [4 ; 2], [17 ; 1]. Then g is the sequence 1, 1, 1, 1, 1, 4, 4, 17.

Given a gauge $\mathfrak{a}$ or g, and a number S, a *$\mathfrak{a}$-representation of* S is a numeral n, such that $0 \leqslant n_i \leqslant D_i$, and $\Sigma\, n_i\, G_i = S$. The gauge $\mathfrak{a}$ may be used to define a generalized numeration scheme, also denoted by $\mathfrak{a}$, as follows:

$$\mathfrak{a} : \mathcal{M} \longrightarrow \mathcal{W}, \quad \mathcal{M} = \{ n ;\ 0 \leq n_i \leq D_i \ \text{ for all } \ i \}$$

$\mathcal{M}$ being ordered, as usual, and $\mathfrak{a}[n] = \Sigma\, n_i\, G_i$. Thus, if $\mathfrak{a}[n] = S$ then n is a $\mathfrak{a}$-representative of S, and conversely.

DEFINITION.—A *gauged scheme* is a generalized numeration scheme defined as above via a gauge $\mathfrak{a}$ (or g).

EXAMPLE 2.—For the gauge $\mathfrak{a}$ defined in Example 1,

$$\mathfrak{a}[3] = 3, \quad \mathfrak{a}[1, 2] = 1 \cdot 4 + 2 \cdot 1 = 6, \quad \text{etc.}$$

This scheme is not univalent:

$$\mathfrak{a}[5] = 5 = \mathfrak{a}[1, 1],$$

and not complete: the equation $\mathfrak{a}[n] = 16$ has no solution.

EXAMPLE 3.—Cultural numeration schemes.

A. Babylonian. It is base sixty. The gauge is

$$\mathcal{B} : [60^k ;\ 59], \quad k = 0, 1, \ldots .$$

B. Mayan (Priestly type). Mixed base, gauge is

$$[1, 19], \ [20; 17], \ \ldots, \ [360\,(20)^k ;\ 19], \ \ldots,$$

k ranging from 0 upwards.

C. Roman *without* subtractive features like XIV = 14. Let

$$i = 2j + k, \quad 0 \leq k \leq 1 . \quad \text{Take} \quad G_i = 2^j \cdot 5^{j+k}, \quad D_i = 2^{2-k} - k.$$

D. Modern symbolic numeration scheme:

$$[\,10^k;\ 9\,],\quad k = 0, 1, 2, \ldots\ .$$

E. Oral numeration in, e.g., U.S.A. and France (but not England). The common usage of «thirty-five hundred» but not «four million thirty-five hundred» makes the system fall, at best, into some mixed up imitation of a gauged scheme, which is necessarily *not* univalent. For the case where such usage is excluded there are two reasonable gauged schemes for modeling our oral scheme:

$$\mathrm{I}: [\,1; 9\,],\ [\,10; 9\,],\ [\,100; 9\,],\ [\,1000^k; 999\,]\quad (k = 1, 2, \ldots)$$

$$\mathrm{II}: [\,1000^i; 999\,]\quad i = 0, 1, 2, \ldots\ .$$

In the interpretation given by gauge number II, it is understood that the «digits», ranging between 0 and 999, are spoken in a base ten fashion, gauged, naturally, by [1 ; 9], [10 ; 9], [100 ; 9].

TECHNICAL LEMMAS.—Let $\mathfrak{a}$ be a gauged scheme, let g be the other notation for $\mathfrak{a}$. For each k, let $\mathfrak{a}_k$ be the gauged scheme obtained by restricting $\mathfrak{a}$ to the first k of the G_i: Let $g^{(k)}$ be the gauged scheme, generally different from $\mathfrak{a}_k$, defined by the sequence

$$g^{(k)}: g_0, g_1, \ldots, g_k.\quad \text{Let}\quad U_k = \sum_{i \leq k} D_i G_i,\quad u^{(k)} = \sum_{i \leq k} g_i\,.$$

LEMMA A.—Assume $\mathfrak{a}_k$ is univalent and that $G_{k+1} \geqslant 1 + U_k$. Then $\mathfrak{a}_{k+1}$ is also univalent.

LEMMA B.—Assume $g^{(k)}$ is complete and that $g_{k+1} \leqslant 1 + u^{(k)}$. Then $g^{(k+1)}$ is also complete.

LEMMA C.—Assume $G_{k+1} > 1 + U_k$. Then $\mathfrak{a}_{k+1}$ is not complete; in fact $1 + U_k$ has no representation.

LEMMA D.—Assume that $\mathfrak{a}_k$ is complete and that $G_{k+1} < 1 + U_k$. Then $\mathfrak{a}_{k+1}$ is not univalent.

PROOF A.—Suppose each of m and n represents the same number in $\mathfrak{a}_{k+1}$:

$$\sum_{i \leq k+1} n_i G_i = \sum_{i \leq k+1} m_i G_i\,.$$

It can be assumed that

$$m_{k+1} \geq n_{k+1}\,, \quad \text{say} \quad m_{k+1} - n_{k+1} = d \geq 0\,.$$

Subtraction of $n_{k+1}\, G_{k+1}$ from the displayed equality leaves

$$\sum_{i \leq k} n_i\, G_i = d\, G_{k+1} + \sum_{i \leq k} m_i\, G_i\,.$$

But by hypothesis, G_{k+1} is, all by itself, larger than the left side, so d must be zero, whence it follows that $m_{k+1} = n_{k+1}$. Moreover

$$m' = [\, m_k \ldots m_0 \,] \quad \text{and} \quad n' = [\, n_k \ldots n_0 \,]$$

both represent the same number in $\mathfrak{a}_k$, which is assumed to be univalent. Therefore $m_i = n_i$ for $i \leqslant k$, as well as for $i = k$; i. e. m and n are equal numerals. This proves univalence of $\mathfrak{a}_{k+1}$.

B. Let T be a number which doesn't exceed $u^{(k+1)}$. It is required merely to find a representative for T. The assumption of completeness of $g^{(k)}$ settles the case of smaller values so it is assumed that T exceeds $u^{(k)}$ which, in view of the assumed inequality $g_{k+1} \leqslant 1 + u^{(k)}$ implies that T is at least as large as g_{k+1}. Therefore it follows that

$$g_{k+1} \leq T \leq u^{(k+1)} = g_{k+1} + u^{(k)}$$

$$0 \leq T - g_{k+1} \leq u^{(k)}\,.$$

Completeness of $g^{(k+1)}$ guarantees a representation for $T - g_{k+1}$ in $g^{(k)}$, from which a representation of T in $g^{(k+1)}$ is immediately derived.

C. The given inequality $G_{k+1} > 1 + U_k$ shows clearly why $1 + U_k$ has no representation: $\mathfrak{a}_k$ can't get up to it (it reaches only to U_k) while the next gauge size up, namely G_{k+1}, in $\mathfrak{a}_{k+1}$, is too large to start.

D. Failure of univalence is proved by the existence of two distinct representations of G_{k+1}, namely $[1_{k+1}\, 0 \ldots 0]$ and a shorter representation in $\mathfrak{a}_k$, which exists by virtue of the assumed small size of G_{k+1} and the completeness of $\mathfrak{a}_k$.

DEFINITIONS. — Let C be a generalized numeration scheme, $\mathcal{M} \longrightarrow \mathcal{W}$. $\mathcal{C}$ is *order-preserving-in-the-strong-sense* provided for every, m, n in $\mathcal{M}$, if $m > n$ then $\mathcal{C}\, m > \mathcal{C}\, n$. $\mathcal{C}$ is *order-preserving* provided for all m, n in $\mathcal{M}$, if $m \geqslant n$ then $\mathcal{C}\, m \geqslant \mathcal{C}\, n$.

THEOREM 1.—Let $\mathfrak{a}$ be a gauged scheme, $\mathcal{M} \longrightarrow \mathcal{W}$.

A. $\mathfrak{a}$ is order-preserving in the strong sense if and only if condition ($\mathcal{L}$) holds:

$$(\mathcal{L}) \quad \text{For all} \quad k, \; G_k \geq 1 + \sum_{i<k} D_i G_i .$$

B. Condition ($\mathcal{L}$) implies that $\mathfrak{a}$ is univalent.

PROOF.—A. First, assume $\mathfrak{a}$ is order preserving in the strong sense. In the order of $\mathcal{N}$ and therefore of $\mathcal{M}$, $[1_k\, 0 \dots 0] = e$, exceeds every numeral of length less than k. Since $\mathfrak{a}$ preserves order, $\mathfrak{a}\,[e]$ must exceed

$$\mathfrak{a}\,[\,D_{k-1} \dots D_0\,]; \quad \text{i. e.} \quad G_k \geq 1 + \sum_{i<k} D_i G_i .$$

Since k is arbitrary, condition ($\mathcal{L}$) is valid. Now suppose, conversely, that condition ($\mathcal{L}$) is valid. Let m and n belong to $\mathcal{N}$, and assume $m > n$. There are two cases: Case (i). m and n have the same length. The given inequality implies that $m_i \geqslant n_i$ holds for all i, while for some i, $m_i > n_i$. These inequalities imply, since every G_i is positive:

$$\mathfrak{a}\, m = \Sigma\, m_i G_i > \Sigma\, n_i G_i = \mathfrak{a}\, n,$$

as required for strong order preserving. Case (ii). The length of n is k and m is longer than k. The smallest numeral of length $k + 1$ is $[1_{k+1}\, 0 \dots 0] = e$. By condition ($\mathcal{L}$), $G_{k+1} > \Sigma_{i \leq k} D_i G_i$, which implies $\mathfrak{a}\,[e] > [D_k \dots D_0]$, and so by Case (i), $\mathfrak{a}\,[e] > \mathfrak{a}\,[n]$. Now a routine induction yields the conclusion that $\mathfrak{a}\,[m] > \mathfrak{a}\,[n]$. The proof is complete.

PROOF OF B.—By induction. First, $\mathfrak{a}_0$ is surely univalent in any case. The induction is completed by application of Lemma A.

THEOREM 2.—A necessary and sufficient condition that the gauged slcheme $\mathfrak{a}$ be complete is

$$\text{(S)} \quad \text{For all} \quad k, \; G_k \leq \sum_{i<k} D_i G_i .$$

PROOF.—Sufficiency is readily proved by induction with the induction step supplied by Lemma B. To prove necessity, assume condition (S) fails; then it can be assumed that for some k,

$$G_j \leq 1 + \sum_{i<j} D_i G_i \; (0 \leq j \leq k)$$

$$G_k > 1 + \sum_{i<k} D_i G_i = 1 + U_{k-1}.$$

From Lemma C it follows that $1 + U_{k-1}$ has no representation in $\mathfrak{a}_k$; but in any $\mathfrak{a}_h$ for $h > k$, the new gauge values, $G_{k+1}, \dots, G_h$ are all larger than G_k, and so of even less use in the search for a representat've for $1 + U_{k-1}$. Thus $\mathfrak{a}$ is incomplete.

THEOREM 3.—Let $\mathfrak{a}$ be a gauge scheme. Each of the following conditions is necessary and sufficient for $\mathfrak{a}$ to be both complete and univalent:

$$(\mathscr{L}\text{S}) \quad \text{For all} \quad k, \quad G_k = 1 + \sum_{i<k} D_i G_i$$

$$(l\text{s}) \quad \text{For all} \quad k, \quad g_k = 1 + \sum_{g_i < g_h} g_i$$

$$(\mathscr{R}) \quad \text{For all} \quad k, \quad G_k = (1 + D_{k-1}) \, G_{k-1}$$

$$(\mathscr{P}) \quad \text{For all} \quad k, \quad G_k = \prod_{i<k} (1 + D_i).$$

PROOF.—The first two conditions are simply versions for $\mathfrak{a}$ and g, respectively, of the same condition. It is an exercise in elementary algebra to verify that ($\mathscr{L}$ S), ($\mathscr{R}$) and ($\mathscr{P}$) are equivalent. Finally it must be proved that these conditions are equivalent to the completeness and univalence. Assume ($\mathscr{L}$ S). Then both ($\mathscr{L}$) and (S) are valid so by Theorems 1 and 2, respectively, $\mathfrak{a}$ is univalent and complete. Conversely, suppose $\mathfrak{a}$ is both univalent and complete. Theorem 2 then assures the validity of (S). Then for all k, $G_k \leqslant 1 + U_{k-1}$, which, by Theorem 2 again, implies that $\mathfrak{a}_k$ is complete. Univalence of $\mathfrak{a}$ implies univalence of $\mathfrak{a}_{k+1}$ ($\mathfrak{a}_{k+1}$ is simply a restriction of the mapping $\mathfrak{a}$), which permits, by application of Lemma D, the conclusion that $G_{k+1} \geqslant 1 + U_k$. Thus the condition ($\mathcal{L}$) is valid, which, combined with the already verified condition (S), proves that condition ($\mathcal{L}$ S) is valid.

DEFINITION.—A *based scheme* is a gauged scheme which is complete, univalent, and has the same value for every D_i, say $D_i = D$ for all i. The *base* is $1 + D$.

THEOREM 4.—In order that $\mathfrak{a}$ be a based scheme, it is necessary and sufficient that for some $D \geqslant 1$, $D_i = D$ and $G_i = (1 + D)^i$ for all i.

PROOF.—This is a special case of Theorem 3; condition ($\mathcal{P}$) applies.

3. Calculations. The TB algorithm. Let $\mathfrak{a}$ be a gauged scheme, let S be a nonzero whole number. The TB algorithm produces a sequence of subscripts, $m(0)$, $m(1)$, ..., and a sequence of whole numbers, S_0, S_1, ..., as follows:

$$S_0 = S \geq 1.$$

$$m(0) = \max \{ i, g_i \leqq S_0 \}; \; S_1 = S_0 - g_{m(0)}.$$

$$m(1) = \max \{ i; i < m(0) \text{ and } g_i \leq S_1 \}, \; S_2 = S_1 - g_{m(1)}.$$

$$\cdots$$

$$m(k) = \max \{ i; i < m(k-1) \text{ and } g_i \leq S_k \}, \; S_{k+1} = S_k - g_{m(k)}.$$

$$\cdots$$

Since the $m(i)$ are forced to decrease, an empty set is bound to appear and then the algorithm is finished on the previous line. Suppose, in calculation of $m(p)$, no empty set appears but that the next set is empty:

$$m(p) = \max \{ i; i < m(p-1) \text{ and } g_i \leq S_p \}, \; S_{p+1} = S_p - g_{m(p)},$$

$$\{ i; i < m(p) \text{ and } g_i \leq S_{p+1} \} \textit{ is empty}.$$

The emptiness of the last set arises from just one of two causes:

i) $S_{p+1} = 0$. ii) $S_{p+1} \neq 0$ but there are no $g_i \leqslant S_{p+1}$ having $i < m(p)$. In any case set

$$S' = \sum_{i \leq p} g_{m(i)} .$$

If case i) holds then S′ is obviously equal to S, and the algorithm shows how to find a representation of S. In case ii), S′ is not equal to S and the algorithm has *not* produced a representation for S.

DEFINITION.—If, in a gauged scheme $\mathfrak{a}$, case i) holds for every S which has a representation, then it is said that «the TB algorithm is effective for $\mathfrak{a}$».

EXAMPLE.—Let g: 1, 3, 4, 5 be a gauge scheme. Now 7 has the representation [110], since $7 = 4 + 3$, but the TB algorithm produces [1001], which represents 6, not 7. Therefore the algorithm is *not effective* for g. However, see the next theorem.

THEOREM 5.—Assume $\mathfrak{a}$ is a complete gauged scheme. Then the TB algorithm is effective for $\mathfrak{a}$.

PROOF.—The proof, using the $g^{(k)}$; is done by induction on k. The case $g^{(0)}$ is evidently no problem. Suppose for all $i < k$ $(k \geqslant 1)$, the TB algorithm is effective for $g^{(i)}$. Let

$$1 \leq S \leq u^{(k)} = \sum_{i \leq k} g_i .$$

It must be shown that case i) holds when the algorithm is applied, in $g^{(k)}$, to S, i. e., that $S' = S$. The algorithm gives:

$$S_0 = S , \quad m(0) = \max \{ i ; \; i \leq k , \; g_i \leq S_0 \} , \quad S_1 = S_0 - g_{m(0)} ,$$

for the first round of calculations. The completeness criterion of Theorem 2 implies

$$g_{1+m(0)} \leq 1 + \sum_{i \leq m(0)} g_i ,$$

while the definition of $m(0)$ implies that $g_{m(0)} \leqslant S_0 < g_{1+m(0)}$. Combined, these inequalities imply

$$1)\quad g_{m(0)} \leq S_0 < g_{1+m(0)} \leq 1 + \sum_{i \leq m(0)} g_i$$

$$0 \leq S_0 - g_{m(0)} = S_1 \leq \sum_{i \leq m(0)} g_i - g_{m(0)}$$

$$2)\quad 0 \leq S_1 \leq \sum_{i < m(0)} g_i .$$

Thus, S_1 is no larger than $u^{(m(0)-1)}$, and $m(0) - 1$ is less than k. The induction hypothesis asserts that a representation of S_1 is obtained by applying the TB algorithm in $g^{(m(0)-1)}$. This algorithm is merely the *continuation* of the algorithm already started on S_0 in $g^{(k)}$. [It must be checked that the first computation for S_1 actually produces $m(1)$, since, taking place in $g^{(m(0)-1)}$, it restricts attention to $i \leqslant m(0) - 1$, i. e., to $i < m(0)$, as is required.] Since $S = g_{m(0)} + S_1$, and

$$S_1 = \sum_{i \geq 1} g_{m(i)} ,$$

S itself is equal to

$$\sum_{i \geq 0} g_{m(i)} ,$$

that is, to S′, and the algorithm does indeed produce a representation for S. Since S was an arbitrary member of $g^{(k)}$, the algorith is effective on $g^{(k)}$. The induction is complete.

NOTE.—The converse is false. The algorithm is effective for the incomplete scheme g : 1,3.

Finally, a few arithmetical results are collected. Let $\alpha : \mathcal{M} \longrightarrow \mathcal{W}$ be a gauged scheme; let $\mathcal{X} = \alpha(\mathcal{M})$. For each k, let α_k be defined as usual, let $\mathcal{M}_k$ = domain α_k, let $\mathcal{X}_k = \alpha_k(\mathcal{M}_k)$, let $U_k = \max \mathcal{X}_k$.

THEOREM 6.—A. For every k, if S belongs to $\mathcal{X}_k$ then $U_k - S$ also belongs to $\mathcal{X}_k$.

B. Let $x_i + y_i = z_i$, and suppose $[x]$, $[y]$ and $[z]$ all belong to $\mathfrak{M}$. Then $\mathfrak{a}[x] + \mathfrak{a}[y] = \mathfrak{a}[z]$.

C. For all k, and all d_k, if $[d_k\, 0 \ldots 0] \in \mathfrak{M}$, then

$$\mathfrak{a}[d_k\, 0 \ldots 0] = d_k \cdot \mathfrak{a}[1_k\, 0 \ldots 0].$$

D. Carrying. Assume that addition of two numerals in $\mathfrak{M}$ never involves carrying 2 or more to the next place, and never influences the 2nd place over. Then for all k

$$G_k \geq 1 + \sum_{i<k} D_i G_i .$$

E. Assume $\mathfrak{a}$ is univalent. $\mathfrak{a}$ is complete if and only if for all k, $\mathfrak{a}[1_{k+1}\, 0 \ldots 0]$ is a multiple of $\mathfrak{a}[1_k\, 0 \ldots 0]$. $\mathfrak{a}$ is a based numeration scheme if and only if for all k, $\mathfrak{a}[1_k\, 0 \ldots 0]$ is equal to the k^{th} power of $\mathfrak{a}[10]$.

Proof.—The first 3 parts are quite obviously true, and part E is an immediate consequence of Theorems 3 and 4. The carrying, part D, will be proved now, in contrapositive form. Assume the condition fails. Then for some k,

$$G_{k+1} \geq \sum_{i \leq k} D_i G_i \equiv U_k .$$

In adding U_k to itself, the numeral form

$$[D_k \ldots D_0] + [D_k \ldots D_0],$$

involves either a carry of «2» into the $(k+1)^{\text{st}}$ place, or, possibly, a carry into the $(k+2)^{\text{nd}}$ place, since $2\,U_k \geqslant 2\,G_{k+1}$; the latter number is represented by $[2_{k+1}\, 0 \ldots 0]$.

Examples.—The following three gauge schemes are all univalent.

1. g : 1, 4, 4, 6, 6. Addition of [20] to [10] given [200], a carry of 2.
2. g : 1, 9, 9, 12, 12, 12. Then [20] + [20] = [300], a carry of 3.
3. g : 1, 5, 7, 9. Here [10] + [10] = [1001], a carry of 1 into the *second* place over.

Famous examples. In Roman numerals, Neugebauer form, [10] × × [10] = [210]. In the Priestly Mayan, [10] × [10] = [120] (i. e. 20 × 20 = 1 (360) + 2 (20)).

NOTE.—There are some cultural schemes which fail to fit the gauged scheme models, apart from subtractive features. One such is the Australian Aboriginal numeration, which has only pebble tallies and spoken numerals. It is like a gauge scheme except that there is no upper bound on the number of 2's which may appear. The gauge then is g : 1, 2, 2, 2, ...

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