SPHERICAL COMPLETENESS WITH INFINITESIMALS

by

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In the theory of nonarchimedean normed spaces over valued fields other than R or C, the property of spherical completeness is of utmost importance in several contexts, and it appears to play the role conventional completeness does in some topics of classical functional analysis. In this note we give various characterizations of spherical completeness for general ultrametric spaces, related to but different from the notions of pseudo-convergent sequence and pseudo-limit introduced by Ostrowski in [4], and apply them to obtain some new results. Although we use the language and methods of so-called infinitesimal (or non-standard) analysis, the way to rephrase some of our statements within non-infinitesimal analysis is pointed out conveniently.

The first part contains some notations and our main result. The second part deals with some applications. In the third part, one more characterization of spherical completeness is given.

1. In the sequel, (X, d) will denote an ultrametric space, i. e., a metric space where the strong triangular or ultrametric inequality

$$\forall x, y, z \in X, \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

holds; $\mathscr{X}$ will be a superstructure in the sense of [6], big enough to include all real numbers and all points of X as individuals; and by $^*\mathscr{X}$ we will mean a fixed non-standard $\aleph_1$-saturated model of $\mathscr{X}$, e. g., a ε-incomplete ultrapower. For hyperreal numbers $a, b \in {}^*R$, «a is infinitely close to b», $a \approx b$, means $a - b$ is infinitesimal; $a \lesssim b$ means $a < b$ *or* $a \approx b$; $a < \approx b$ means $a < b$ *and* $a \approx b$. If a is any finite hyperreal number, st a is the only real number infinitely close to a.

As usual, we identify a function with its non-standard extension. For $x, y \in {}^*X$, $x \approx y$ means $d(x, y) \approx 0$. See [6] for more infinitesimal concepts and notations.

Ingleton, [2], introduced the following notion in order to characterize those nonarchimedean valued fields such that for normed spaces over them an analog of the Hahn-Banach theorem is true (see [7] as a general reference for ultrametric spaces):

DEFINITION.—X will be said to be *spherically complete* whenever every shrinking sequence of nonempty closed balls has nonempty intersection.

(Notice its difference with Cantor property: no condition on the radii of the balls.)

THEOREM 1.—*The following conditions on* (X, d) *are equivalent:*

(1) (X, d) *is spherically complete.*

(2) *For every nonnegative real number* λ *and every sequence* $(\mathrm{x_n} \mid \mathrm{n} \in N)$ *in* X, *if* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}_{\alpha+1}) \lesssim \lambda$ *whenever* α *is an infinite subscript, then there exists an* $\mathrm{x} \in X$ *such that* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}) \lesssim \lambda$ *for all infinite* α.

(3) *For every nonnegative real number* λ *and every sequence* $(\mathrm{x_n} \mid \mathrm{n} \in N)$ *in* X, *if* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}_\beta) \approx \lambda$ *whenever* α, β *are different infinite subscripts, then there exists an* $\mathrm{x} \in X$ *such that* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}) \lesssim \lambda$ *for some infinite* α.

(4) *For every nonnegative real number* λ *and every sequence* $(\mathrm{x_n} \mid \mathrm{n} \in N)$ *in* X, *if* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}_\beta) \approx \lambda$ *whenever* α, β *are different infinite subscripts, then there exists an* $\mathrm{x} \in X$ *such that* $\mathrm{d}(\mathrm{x}_\alpha, \mathrm{x}) \approx \lambda$ *for all infinite* α.

PROOF.—*(1)* $\Longrightarrow$ *(2):* Assume X is spherically complete and let (x_n) be a sequence such that for all infinite subscripts α, $d(x_\alpha, x_{\alpha+1}) \lesssim \lambda$, where λ is a real number. Then the sequence

$$(d(x_n, x_{n+1}) \mid n \in \mathrm{N})$$

has all its accumulation points within $[0, \lambda]$, so that

$$\lambda' = \lim\sup d(x_n, x_{n+1}) \in [0, \lambda],$$

and for each finite natural n,

$$r_n = \sup \{d(x_m, x_{m+1}) \mid m \in N, \quad m \geq n\}$$

exists. Then $r_n \downarrow \lambda'$, and the sequence of closed balls

$$B_n = \{y \in X \mid d(y, x_n) \leq r_n\}, \quad n \in N$$

is decreasing: if $y \in B_{n+1}$, then

$$d(y, x_n) \leq \max \{d(y, x_{n+1}), d(x_{n+1}, x_n)\} \leq r_n.$$

By hypothesis, there exists $x \in \bigcap B_n$. Now, by a conventional application of Transfer (or Leibniz') Principle, for all natural $n \in {}^*N$, $d(x, x_n) \leqslant r_n$, therefore if $\alpha \in {}^*N$ is infinite, then

$$d(x, x_\alpha) \leq r_\alpha \approx \lambda' \leq \lambda,$$

so $d(x, x_\alpha) \lesssim \lambda$.

(2) $\Longrightarrow$ *(3)* is trivial.

(3) $\Longrightarrow$ *(4)*: Assume that the standard sequence (x_n) verifies

$$d(x_\alpha, x_\beta) \approx \lambda \in R$$

for all infinite α, β. Then by (3) there exists an $x \in X$ and an infinite α with $d(x, x_\alpha) \lesssim \lambda$. We claim that for every infinite

$$\beta, \ d(x, x_\beta) \approx \lambda.$$

Let us first suppose st $d(x, x_\alpha) < \lambda$ and show that if β is different from α, then $d(x, x_\beta) \approx \lambda$:

If st $d(x, x_\beta) < \lambda$, then

$$d(x_\alpha, x_\beta) \leq \max \{d(x_\alpha, x), \ d(x, x_\beta)\},$$

so $d(x_\alpha, x_\beta) \not\approx \lambda$, impossible.

If st $d(x, x_\beta) > \lambda$, then by a well-known consequence of the strong triangular inequality,

$$d(x_\alpha, x_\beta) = \max \{d(x_\alpha, x), \ d(x, x_\beta)\} = d(x, x_\beta),$$

so $d(x_\alpha, x_\beta) \not\approx \lambda$, impossible.

Next we prove that our assumption st $d(x, x_\alpha) < \lambda$ is absurd:

If there is a real number $r \in (d(x, x_\alpha), \lambda)$, apply the Transfer Principle to what we have just proved:

$$\exists\, \alpha \in {}^*\mathrm{N}, \quad \forall\, \beta \in {}^*\mathrm{N}, \quad \beta > \alpha \Longrightarrow d(x, x_\beta) > r$$

to obtain a finite n such that

$$\forall\, m \in \mathrm{N}, \quad m > n \Longrightarrow d(x, x_m) > r.$$

Applying Transfer to last sequence again, we conclude $d(x, x_\alpha) > r$, absurd.

Then $d(x, x_\alpha) \approx \lambda$. For any other infinite β, we know from the preceding discussion that st $d(x, x_\beta) \geqslant \lambda$. Now, if st $d(x, x_\beta) > \lambda$, then

$$d(x_\alpha, x_\beta) = \max\{d(x_\alpha, x), d(x, x_\beta)\} = d(x, x_\beta) \neq \lambda,$$

absurd.

(4) $\Longrightarrow$ *(1)*: Let $(\mathrm{B}_n \mid n \in \mathrm{N})$ be a shrinking sequence of closed balls,

$$\mathrm{B}_n = \{x \in \mathrm{X} \mid d(x, x_n) \leq r_n$$

and select a subsequence such that, relabeling, $x_n \notin \mathrm{B}_{n+1}$ (if this is impossible, the proof is over). Then for finite $m, n,\ m > n$,

$$d(x_m, x_{n+1}) \leq r_{n+1}, \qquad d(x_n, x_{n+1}) > r_{n+1},$$

so

$$d(x_m, x_n) = \max\{d(x_m, x_{n+1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n+1}).$$

Therefore, if α, β are infinite and n is finite,

$$d(x_\alpha, x_\beta) \leq \max\{d(x_\alpha, x_n), d(x_\beta, x_n)\} = d(x_n, x_{n+1})$$

hence $d(x_\alpha, x_\beta)$ is finite, and

$$d(x_\alpha, x_\beta) \lesssim \lambda = \inf\{d(x_n, x_{n+1}) \mid n \in \mathrm{N}\}.$$

On the other hand, if m, n are finite, $m > n$, then

$$d(x_m, x_n) = d(x_n, x_{n+1}) \geq \lambda,$$

so that if α, β are different infinite subscripts, then $d(x_\alpha, x_\beta) \gtrsim \lambda$. Hence $d(x_\alpha, x_\beta) \approx \lambda$.

Apply hypothesis [4] to get an $x \in X$ such that $d(x, x_\alpha) \approx \lambda$ whenever α is infinite. We claim that $x \in \bigcap B_n$: for finite n, if α is infinite,

$$d(x, x_n) \leq \max\{d(x, x_\alpha), d(x_\alpha, x_n)\} \approx \max\{\lambda, d(x_n, x_{n+1})\} = \\ = d(x_n, x_{n+1}) \leq r_n,$$

and since both members are standard, we conclude $d(x, x_n) \leqslant r_n$.

Because of the strong triangular inequality, in [2] we can exchange «$d(x_\alpha, x_{\alpha+1}) \lesssim \lambda$ for all infinite α» by «$d(x_\alpha, x_\beta) \lesssim \lambda$ for all infinite α, β».

Underlying statements [2], [3] and [4] above there is, of course, a generalization fo Cauchy and convergent sequences: for $\lambda = 0$, any of the three properties would read: «every Cauchy sequence is convergent». Therefore, for $\lambda \geqslant 0$ it is fair to name λ-Cauchy or λ-convergent a sequence fulglling one of the conditions contained in [2], [3] or [4]. We have several possibilities for these definitions, and by reasons that will become apparent when trying to carry the same properties over to general metric spaces (which will be published elsewhre), we choose the following:

DEFINITION.—Let λ be any nonnegative real number. A sequence $(x_n \mid n \in N)$ in X will be said to be λ-*Cauchy* when for all infinite subscripts α, β,

$$d(x_\alpha, x_\beta) \lesssim \lambda.$$

and it will be said to be λ-*convergent* (*to* x $\in$ *X*) when for all infinite subscripts α,

$$d(x, x_\alpha) \lesssim \lambda.$$

A non-infinitesimal characterization of these definitions and a non-infinitesimal rephrasing of Theorem 1 can be obtained in a «standard» way, and we leave it to the reader: (x_n) is λ-Cauchy if and only if for every positive ε, there exists an n_ε such that whenever $m, n \geqslant n_\varepsilon$, $d(x_m, x_n) < \lambda + \varepsilon$; etc.

Properties [2], [3], [4] of Theorem 1 and last definitions are close to Ostrowski pseudo-convergence and pseudo-limits (see, e. g., [3] or [7]). Van Tiel, [8], has proved that a space is spherically complete if and only if every pseudo-convergent sequence has a pseudo-limit. Now, it is easy to see that in fact

Eevery psudo-convergent sequence has a pseudo-limit

amounts to say

For every sequence (x_n) *in X such that from some* n_0 *on,* $(d(x_n, x_{n+1}))$ *is decreasing, call* $\lambda = \lim d(x_n, x_{n+1})$ *(the «breadth» of the sequence, see* [3]); *then* (x_n), *which is* λ*-Cauchy, is also* λ*-convergent.*

2. The non-standard hull $(\hat{X}, \hat{d})$ of the metric space (X, d) is obtained in the following way: $\hat{X}$ is the quotient of the set fin $^*X$ of finite elements of $^*X$ (i. e., the set of points at a finite distance of some standard point) by the relation of infiniesimal nearness; and $\hat{d}(\hat{x}, \hat{y}) = \text{st}\, d(x, y)$ for $x, y \in \text{fin}\, ^*X$. It is well-known that $(\hat{X}, \hat{d})$ is a complete metric space. Since (X, d) is ultrametric, so is $(\hat{X}, \hat{d})$.

THEOREM 2.—*The non-standard hull of any ultrametric space is spherically complete.*

PROOF.—Let λ be any real number and $(\hat{x}_n \mid n \in N)$ a standard λ-Cauchy sequence in $\hat{X}$. Then $(x_n \mid n \in N) \subset \text{fin}\, ^*X$ can be extended to an internal sequence $(x_n \mid n \in {}^*N) \subset {}^*X$ by $\aleph_1$-saturation. Given any finite natural number p, there is a finite n_p and (by internality) an infinite α_p such that

$$\forall\, m, n \in {}^*N, \quad n_p \leq m, n \leq \alpha_p \Longrightarrow d(x_m, x_n) < \lambda + 1/p.$$

Again by $\aleph_1$-saturation, there is a (finite or infinite) natural number k such that for all finite p, $n_p \leqslant k \leqslant \alpha_p$. Then $x_k \subset \text{fin}\, ^*X$ and $\hat{x}_k$ is a λ-limit of the sequence $(\hat{x}_n)$.

If E is a nonarchimedean normed space over a nontrivially valued nonarchimedean field K, then $\hat{E}$ is also a nonarchimedean normed space over $\hat{K}$ (straightforward proof), so over K, and by Theorem 2, $\hat{K}$, $\hat{E}$ are spherically complete spaces that contain K, E, respectively. This provides a new way of building a spherically complete field that contains K as a subfield and extends its valuation (cf. [7], pp. 149-150).

We will use the phrase «spherical completion» in the sense of [7]:

DEFINITION.—A *spherical completion* of a nonarchimedean normed vector space E is a pair (F, i) consisting of a spherically complete space F and a linear isometry $i : E \longrightarrow F$ such that F has no spherically complete proper linear subspace containing i (E).

LEMMA.—*Call* k (*respectively,* $\hat{k}$) *the residue class field of K respectively,* $\hat{K}$), *and* $|K|$ (*respectively,* $|\hat{K}|$) *the value group of K* (*respectively,* $\hat{K}$). *Then*

(1) $|\hat{K}| = {}^{*}|K|$.

(2) *If K is discretely valued, then* $\hat{k}$ *is cannonically (algebraicly) isomorphic to* *k.

PROOF.—(1) is easy. To prove (2) define a map from $\hat{k}$ onto $^{*}k$ by assigning the class of x to the class of $\hat{x}$ (notice that discreteness of $|K|$ ensures that $|x| \approx 1$ implies $|x| = 1$).

Part (2) of the Lemma is stronger than a result of Diarra ([1], Corollary to Theorem 1), obtained there in a different way.

The precise relationship between $\hat{k}$ and k when K isl densely valued, is an open problem.

Regard K as a subfield of $\hat{K}$ under the isometry $i(x) = \hat{x}$.

COROLLARY 1.—*Assume the valuation of K is discrete; then the following are equivalent:*

(a) *K is topologically dense in* $\hat{K}$.

(b) $\hat{K}$ *is a spherical completion of K.*

(c) k *is finite.*

(d) $\hat{K}$ *is locally compact.*

PROOF.—*(a)* $\Longrightarrow$ *(b)* follows easily from the definitions.

(b) $\Longrightarrow$ *(c)*: Assume $\hat{K}$ is a spherical completion of K; then $\hat{K}$ and K have the same residue class field, so by the Lemma, $k = \hat{k} = {}^{*}k$, hence k is finite.

(c) $\Longrightarrow$ *(d)*: Suppose $^*k = k$. Then by the Lemma, $\hat{k}$ is finite; and $\hat{K}$ is certainly complete. Now it suffices to observe that in case K is discretely valued, so is $\hat{K}$ (again, use the Lemma): it is well-known that under these conditions, $\hat{K}$ is locally compact.

(d) $\Longrightarrow$ *(a)*: If $\hat{K}$ is locally compact, its subfield K is locally precompact, so that fin $^*K$ = pns $^*K$. Then $\hat{K}$ is the completion of K.

Another consequence of the Lemma: since a field and its spherical completions have the same value group, in case $|K|$ is dense in R^+ but different from R^+, $\hat{K}$ is *not* a spherical completion of K.

THEOREM 3.—*If K is algebraicly closed, so is $\hat{K}$.*

PROOF.—From the Transfer Principle applied to the sentence that says K is algebraicly closed, the following is obtained: if $a_0, a_1, \ldots, a_n$ is a $*$-finite subset of $^*K$ ($n \in {}^*N$, $n \geqslant 1$, finite or infinite), $a_n \neq 0$, then there exists an $x \in {}^*K$ such that

$$a_n x^n + \ldots + a_1 x + a_0 = 0.$$

Now, if $n \geqslant 1$, is finite, $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n \in \hat{K}$, $\hat{a}_n \neq \hat{0}$, and $x \in {}^*K$ is a root of $a_n x^n + \ldots + a_0$, the x is necessarily of finite absolute value: if $|x|$ were infinite, for $i = 1, \ldots, n$,

$$|a_i x^i| = |a_i| \; |x| \; |x^{i-1}| > |a_{i-1} x^{i-1}| ,$$

then by the ultrametric inequality,

$$0 \approx (a_0 + \ldots + a_n x^n| = |a_n| \; |x|^n$$

hence $a_n \approx 0$, impossible. Then $\hat{x} \in \hat{X}$ is a root of $\hat{a}_n \hat{x}^n + \ldots + \hat{a}_0$.

COROLLARY 2.—*If the valuation of K is discrete, then $\hat{K}$ is never algebraicly closed. In case the valuation of K is dense, $\hat{K}$ is algebraicly closed if and only if so is its residue class fiel.*

PROOF.—Both parts follow from the following characterization of algebraic closedness for a spherically complete valued field ([7], Theorem 4.50): its value group is divisible and its residue class field is algebraicly closed. Now it suffices to apply the Lemma: if K is discrete then $|\hat{K}| = |K|$ is never divisible; and for K dense, $|\hat{K}| = R^+$ is divisible.

We end up this part with a different sorto of application of Theorem 1: a «λ-fixed point» theorem for spherically complete spaces:

THEOREM 4 (STANDARD).—*Let X be a spherically complete ultrametric space, and $T : X \longrightarrow X$. Assume for every* $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

where $\varphi : [0, +\infty) \longrightarrow R$ *is such that*

$$\lambda = \sup_t (\limsup \varphi^n(t))$$

exists, φ *is continuous from the right at* λ *and for all* $t \leqslant \lambda$, $\varphi(t) \leqslant \lambda$. *Then there is at least one* $x_0 \in X$ *such that* $d(x_0, Tx_0) \lesssim \lambda$.

PROOF. — Take any $x \in X$ and form the iterated sequence $(T^n x \mid n \in N)$. For all finite n,

$$d(T^{n+1} x, T^n x) \leq \varphi^n(d(Tx, x)),$$

so for infinite α, $d(T^{\alpha+1} x, T^\alpha x) \lesssim \lambda$. By Theorem 1, there exists $x_0 \in X$ such that $d(x_0, T^\alpha x) \lesssim \lambda$ whenever α is infinite. Then by the conditions imposed on φ,

$$\varphi(d(x_0, T^\alpha x)) \lesssim \lambda,$$

so

$$d(T^{\alpha+1} x, Tx_0) \lesssim \lambda$$

and

$$d(x_0, Tx_0) \leq \max\{d(x_0, T^\alpha x), d(T^\alpha x, T^{\alpha+1} x), d(T^{\alpha+1} x, Tx_0)\} \lesssim \lambda.$$

Now, $d(x_0, Tx_0)$ is standard, so $d(x_0, Tx_0) \leqslant \lambda$.

From the proof of Theorem 4 it follows that λ can be changed by

$$\lambda' = \limsup_n \varphi^n (d(\mathrm{T}\,x, x))$$

for some $x \in \mathrm{X}$.

The class of maps φ that verify the conditions of Theorem 4 include all increasing, right-continuous functions φ such that $\varphi(t) < t$ for all t in some infinite interval $[t_0, +\infty)$, since in that case $\lambda \leqslant \varphi(t_0)$. In particular, if $\varphi(t) = k\,t$ with $k \in (0, 1)$, then $\lambda = 0$; if $\varphi(t) = \sqrt{t}$, then $\lambda = 1$; if $\varphi(t) = k\sqrt{t}$ with $k \in (0, +\infty)$, then $\lambda = k^2$.

3. Now we give another characterization of spherical completeness through properties of some distinguished points of *X, more in the spirit of the well-know equivalence between «completeness» and «all approachable points are near-standard».

Definition.—For any nonnegative real number λ and any (possibly external) subset $\mathrm{Y} \subset {}^*\mathrm{X}$, call the set

$$\mathrm{B}[\mathrm{Y}, \lambda] + \{x \in {}^*\mathrm{X} \mid \text{there is a } y \in \mathrm{Y} \text{ such that } d(x, y) \lesssim \lambda\}$$

the *S-ball of radius* λ *around Y*.

Consider X endowed with the S-topology introducted by Robinson, [5], namely the topology a subbasis of which consits of the S-balls $\mathrm{B}[\{x\}, \lambda]$, $x \in {}^*\mathrm{X}$, λ positive real. It is immediate that for any nonnegative real number λ,

$$\mathrm{B}[\mathrm{X}, \lambda] \subset \overline{\mathrm{B}[\mathrm{X}, \lambda]} \subset \bigcap \{\mathrm{B}[\mathrm{X}, \lambda + \varepsilon] \mid \varepsilon \text{ positive real number}\}.$$

When $\lambda = 0$, $\mathrm{B}[\mathrm{X}, 0]$ is the set ns *X of near-standard points, and

$$\mathrm{B}[\mathrm{X}, 0] = \bigcap_\varepsilon \mathrm{B}[\mathrm{X}, \varepsilon]$$

equals the set pos *X of approachable points. S-closedness of $\mathrm{B}[\mathrm{X}, 0]$ is equivalent to completeness of X (again, see [6]).

However, the picture is very different for $\lambda > 0$:

Definition.—For $\lambda > 0$, call a point in $\lambda - \mathrm{ns}\,{}^*\mathrm{X} = \mathrm{B}[\mathrm{X}, \lambda]$, λ-*near-standard*, and a point in

$$\lambda - \mathrm{pns}^*\,\mathrm{X} = \bigcap_\varepsilon \mathrm{B}[\mathrm{X}, \lambda + \varepsilon],$$

λ-*approachable*.

THEOREM 5.—*Let* (X, d) *be an ultrametric space. Then*

(a) *For* $\lambda > 0$, *the sets* λ-*ns* $^*X$ *and* λ-*pns* $^*X$ *are necessarily S-closed and S-open.*

(b) *Spherical completeness of X is equivalent to all* λ-*approachable points being* λ-*near-standard for each* $\lambda \geqslant 0$.

PROOF.—(a) Take $\lambda > 0$ and $x \in \overline{B[X, \lambda]}$. Then for each finite natural number n, there exist $x_n \in {}^*X$, $a_n \in X$ such that

$$d(x, x_n) \lesssim 1/n, \qquad d(x_n, a_n) \lesssim \lambda.$$

Then for

$$1/n \leq \lambda, \quad d(x, a_n) \leq \max\{d(x, x_n), \quad d(x_n, a_n)\} \lesssim \lambda.$$

hence $x \in B[X, \lambda]$. Thus λ-ns $^*X = B[X, \lambda]$ is S-closed and λ-pns $^*X$ is an intersection of S-closed sets, hence S-closed.

On the other hand, if $x \in \lambda$-ns $^*X$, then

$$B(x, \lambda/2) \subset \lambda - \text{ns}^* X,$$

and if $x \in \lambda$-pns $^*X$, then

$$B(x, \lambda/2) \subset \lambda - \text{pns}^* X,$$

as follows immediately from the strong triangular inequality.

(b) Assume X is spherically complete, and take $x \in \lambda$-pns $^*X$, $\lambda \geqslant 0$. Then for finite n there is an $a_n \in X$ such that

$$d(x, a_n) < \lambda + 1/n.$$

Since (a_n) is a standard sequence, the set

$$\{n \in {}^*\mathrm{N} \mid d(x, a_n) < \lambda + 1/n\}$$

is internal so it contains some infinite α: $d(x, a^\alpha) \lesssim \lambda$. On the other hand, the strong triangular inequality implies

$$d(a_n, a_{n+1}) < \lambda + 1/n$$

for all finite n, so for infinite n as well. Therefore (a_n) is λ-Cauchy, hence λ-convergent: there exists $a \in X$ such that $d(a, a_\alpha) \lesssim \lambda$, and then $d(x, a) \lesssim \lambda$. Thus, $x \in B[X, \lambda]$.

Conversely, suppose (a_n) is a λ-Cauchy sequence in X, $\lambda \geqslant 0$. Then for allinfinite $\alpha \in {}^*N$, a_α is λ-approachable, hence λ-near-standard. Fix α and take $a \in X$ such that $d(a, a_\alpha) \lesssim \lambda$. Then the ultrametric inequality and the λ-Cauchy condition guarantee that for any other infinite β, $d(a, a_\beta) \lesssim \lambda$. Hence (a_n) λ-converge to a.

It is well-known that the completion of a metric space X is the non-standard hull of pns *X, contained in $\hat{X}$ ([6]), Theorem (8.4.28)). A characterization of a spherical completion within $\hat{X}$, if it exists, is an open problem.

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