PROPER PERIODS OF NORMAL N. E. C. SUBGROUPS WITH EVEN INDEX

by

E. BUJALANCE

1. Introduction

By a non-Euclidean crystallographic (N. E. C.) group we shall mean a discrete subgroup Γ of isometries of the non-Euclidean plane including those reverse orientation, reflections and glide-reflections.

In [1], we had compute the proper periods of normal N. E. C. sub groups of an N. E. C. group, when the index of the group with respect to the subgroup is odd. In this paper we shall compute the proper period of normal N. E. C. subgroups, when the index is even.

The corresponding problem for Fuchsian groups, which contain only orientable transformations, is essentially solved in the work of Maclachan [4].

2. Proper periods of normal N. E. C. subgroups

N. E. C. groups are classified according to their signatures, the signature of an N. E. C. group Γ is either of the form

$$(*) \quad (g; +; [m_1 \dots m_\tau]; \{(n_{11} \dots n_{1s_1}), \dots, (n_{k1} \dots n_{ks_k})\})$$

or

$$(**) \quad (g; -; [m_1 \dots m_\tau]; \{(n_{11} \dots n_{1s_1}), \dots, (n_{k1} \dots n_{ks_k})\})$$

The numbers m_i are the proper periods and the brackets $(n_{i1} \dots n_{is_i})$ the period-cycles.

The group Γ with signature $(*)$ has the presentation given by generators:

$$\begin{array}{lll} x_i & i=1\dots\tau & \\ e_i & i=1\dots k & \\ c_{ij} & i=1\dots k & j=0\dots s_i \\ a_j,\ b_j & j=1\dots g & \end{array}$$

and relations

$$\begin{array}{lll} x_i^{m_i}=1 & i=1\dots\tau & \\ c_{is_i}=e_i^{-1}c_{i0}\,e_i & i=1\dots k & j=1\dots s_i \end{array}$$

$$x_1\dots x_\tau\,e_1\dots e_k\,a_1\,b_1\,a_1^{-1}\,b_1^{-1}\dots a_g\,b_g\,a_g^{-1}\,b_g^{-1}=1$$

In a group Γ with signature $(**)$ we have the presentation given by generators

$$\begin{array}{lll} x_i & i=1\dots\tau & \\ e_i & i=1\dots k & \\ c_{ij} & i=1\dots k & j=0\dots s_i \\ d_j & j=1\dots g & \end{array}$$

and relations:

$$x_i^{m_i}=1 \qquad i=1\dots\tau$$

$$c_{is_i}=e_i^{-1}c_{i0}\,e_i \qquad i=1\dots k$$

$$c_{ij-1}^2=c_{ij}^2=(c_{ij-1}\cdot c_{ij})^{n_{ij}}=1 \qquad \begin{array}{l} i=1\dots k \\ j=1\dots s_i \end{array}$$

$$x_1\dots x_\tau\,e_1\dots e_k\,d_1^2\dots d_g^2=1$$

From now on, we will denote by x_i, e_i, c_{ij}, a_i, b_i, d_j the above generators associated to an N. E. C. group.

(2.1) Definition.—Let Γ be a N. E. C. group. A set $\{x_1, x_2, \dots, x_r\}$ of elliptic elements of Γ, neither of which is a product of two reflections in Γ, it is an e. c. s. (elliptic complete system) if the following conditions hold:

i) Each elliptic element of Γ which is not a product of two reflections of Γ is conjugate in Γ to a power of x_i $1 \leqslant i \leqslant r$.

ii) Every non-trivial power of x_i is never conjugate to a power of x_j $(i \neq j)$.

If Γ is an N. E. C. group and A, B are two e. c. s. of Γ, there exists a bijection f: A $\longrightarrow$ B such that if x belongs to A and has order m, then $f(x)$ has order m.

The elliptic generators of Γ, which are $x_1, x_2, \ldots, x_\tau$, form an e. c. s.; then if we know the orders of the elements of an e. c. s. we know the proper periods of the signature of Γ.

(2.2) THEOREM.—Let Γ be a N. e. C. group with signature

$$(g\,;\,+\,;\;[m_1 \ldots m_\tau]\,;\;\{(n_{i1} \ldots n_{is_i})_{i=1\ldots k}\})$$

and Γ_0 a normal subgroup of Γ such that $[\Gamma : \Gamma_0] = N$, N even. We supposse that:

i) $C = \{c_j, c'_j\}$ $j = 1 \ldots p$ is the set of pairs of reflections which are generators of Γ, not belonging to Γ_0, and such that $c_j \cdot c'_j$ is an elliptic elements of order n_j.

ii) p_i is the exponent of x_i modulo Γ_0 $(1 \leqslant i \leqslant \tau)$.

iii) q_j is the exponent of $c_j \cdot c'_j$ modulo Γ_0 $(1 \leqslant j \leqslant p)$

then, the proper periods of Γ_0 are.

$$\left[\left(\frac{m_i}{p_i}\right)^{N/p_i}_{\substack{i=1\ldots\tau\\ p_i \neq m_i}},\quad \left(\frac{n_j}{q_j}\right)^{N/2q_j}_{\substack{i=1\ldots p\\ q_j \neq n_j}}\right]$$

where by $(—)^r$ we mean that this proper period is repeated r times.

PROOF.—Let us suppose

$$\frac{\Gamma}{\Gamma_0} = \{\Gamma_0 R_1,\ \Gamma_0 R_2, \ldots, \Gamma_0 R_N\}\,.$$

Given a x_i we know that $x_i^{p_i} \in \Gamma_0$, and that the order of $x_i^{p_i}$ in Γ_0 is m_i/p_i. Moreover in [1] we have proved that there exist a family of elements

$$S_{k_i}\, x_i^{p_i}\, S_{k_i}^{-1} \quad \left(1 \leq k_i \leq \frac{N}{p_i}\right)$$

conjugate to $x_i^{p_i}$, that verify the following conditions:

i) Each element conjugate to $x_i^{p_i}$ is conjugate to one of this family by an element of Γ_0.

ii) Every non-trivial power of an element of this family is never conjugate in Γ_0 to a power of other element of the family.

Given an element of the form $c_j \cdot c'_j$, we know that $(c_j \cdot c'_j)^{q_j} \in \Gamma_0$, and that the order of $(c_j \cdot c'_j)^{q_j}$ in Γ_0 is n_j/q_j. Let X be the set of all the conjugate elements of $(c_j \cdot c'_j)^{q_j}$, we shall study how many different classes of elements of X are not conjugate by elements of Γ_0. Given

$$k\,(c_j \cdot c_j')^{q_j}\,k^{-1} \in X,$$

there exists a $s \in \Gamma_0$ and a R_w $1 \leqslant w \leqslant n$, such that $k = s \cdot R_w$, therefore

$$k\,(c_j \cdot c_j')^{q_j}\,k^{-1} = s\,R_w\,(c_j \cdot c_j')^{q_j}\,R_w^{-1}\,s^{-1},$$

then $k\,(c_j \cdot c'_j)^{q_j}\,k^{-1}$ is conjugate to $R_w\,(c_j \cdot c'_j)^{q_j}\,R_w^{-1}$ for an element $s \in \Gamma_0$. Therefore now we shall study how many elements of the form $R_w\,(c_j \cdot c'_j)^{q_j}\,R_w^{-1}$ are not conjugate by elements of Γ_0. As $\Gamma_0\,c_j \cdot c'_j$ generate a cyclic group of order q in Γ/Γ_0. Then

$$\Gamma_0\,R_w = \Gamma_0\,T_x\,(c_j \cdot c_j')^{r_w}$$

where

$$\Gamma_0\,T_x\left(1 \leq x \leq \frac{N}{q_j}\right)$$

is a set of representatives of the classes of Γ/Γ_0 modulo $(\Gamma_0\,c_j \cdot c'_j)$; therefore

$$R_w\,(c_j \cdot c_j')\,R_w^{-1} = \gamma\,T_x\,(c_j \cdot c_j')^{q_j}\,T_x^{-1}\,\gamma^{-1},$$

where $\gamma \in \Gamma_0$. Hence we shall study how many elements of the form $Tx\,(c_j \cdot c_j)\,Tx^{-1}$ are not conjugate by elements of Γ_0. As $c_j \notin (\Gamma_0\,c_j \cdot c'_j)$, we have that

$$T_1 = 1 \quad T_2 = c_j \quad T_3 = \gamma_2 \quad T_4 = \gamma_2\,c_j \ldots T_{N/q_1} = \gamma_i\,c_j$$

are representatives of the classes of Γ/Γ_0 modulo $(\Gamma_0\, c_j \cdot c'_j)$, moreover

$$T_1\,(c_j \cdot c_j')^{q_j}\,T_1^{-1}$$

is the inverse element of

$$T_2\,(c_j \cdot c_j')^{q_j}\,T_2^{-1}, \quad T_3\,(c_j \cdot c_j')^{q_j}\,T_3^{-1}$$

is the inverse element of

$$T_4\,(c_j \cdot c_j')^{q_j}\,T_4^{-1},$$

and so on.

Given the elements

$$T_1\,(c_j \cdot c_j')^{q_j}\,T_1^{-1},\; T_3\,(c_j \cdot c_j')^{q_j}\,T_3^{-1},\; \ldots,$$

$$,\ldots, T\left(\frac{N}{q_j} - 1\right)(c_j \cdot c_j')^{q_j}\,T^{-1}\left(\frac{N}{q_j} - 1\right)$$

we have that this elements verify the condition of that no element is conjugate in Γ_0 to other of this elements.

Let us suppose that we have two elements

$$T_i\,(c_j \cdot c_j')^{q_j}\,T_i^{-1} \qquad \text{and} \qquad T_h\,(c_j \cdot c_j')^{q_j}\,T_h^{-1}$$

for which there exists a $s \in \Gamma_0$ such that

$$s\,T_i\,(c_j \cdot c_j')^{q_j}\,T_i^{-1}\,s^{-1} = T_h\,(c_j \cdot c_j')^{q_j}\,T_h^{-1}$$

then

$$T_h^{-1}\,s\,T_i\,(c_j \cdot c_j')^{q_j}\,T_i^{-1}\,s^{-1}\,T_h = (c_j \cdot c_j')^{q_j},$$

therefore if we denote $T_h^{-1}\,s\,T_i$ by M, we have that M leave fixed the point $p \in D$ left fixed by $c_j \cdot c'_j$; from [2] the stabilizer of this point is the dihedral group generated by $c_j \cdot c'_j$, therefore M can only have one of the following forms:

1.°) $M = (c_j \cdot c'_j)^p$. 2.°) $M = (c_j \cdot c'_j)^p c_j$. 3.°) $M = c_j (c_j \cdot c'_j)^p$.
If $M = (c_j \cdot c'_j)^p$, we have that $T_h^{-1} s = (c_j \cdot c'_j)^p T_i^{-1}$. As

$$\Gamma_0 T_x \left(1 \leq x \leq \frac{N}{q_i}\right)$$

is the set of representatives of the classes of Γ/Γ_0 modulo $\Gamma_0 (c_j \cdot c'_j)$, the last equality can only hold if $T_h = T_i$. If we reason in the same form in the casses 2.°) and 3.°), we obtain that $T_h = T_i$

If we make the same with every of the $(c_j \cdot c'_j)^{q_j}$ with $q_j \neq n_j$, we have that the set

$$E = \{ S_{k_i} x_i^{p_i} S_{k_i}^{-1} \}_{\substack{i=1\dots\tau \\ p_i \neq m_i \\ 1 \leq k_i \leq N/p_i}} \cup \{ T_{x_j} (c_j \cdot c_j')^{q_j} T_{x_j}^{-1} \}_{\substack{j=1\dots p \\ q_j \neq n_j \\ 1 \leq x_j \leq N/q_j \\ x_j \neq 1}}$$

is an e. c. s. Therefore the proper periods of Γ_0 are

$$\left[\left(\frac{m_i}{p_i}\right)^{N/p_i}_{\substack{i=1\dots\tau \\ p_i \neq m_i}}, \quad \left(\frac{n_j}{q_j}\right)^{N/2q_j}_{\substack{j=1\dots p \\ q_j \neq n_j}} \right]$$

(2.3) Corollary.—Let Γ be an N. E. C. group with signature

$$(g ; \pm ; [m_1 \dots m_\tau] \{(n_{11} \dots n_{1s_1}) \dots (n_{k1} \dots n_{ks_k})\})$$

and Γ_0 a normal subgroup of Γ such that $[\Gamma : \Gamma_0] = 2$. We suppose that i) $c = \{c_j \cdot c'_j\}_{j=1\dots p}$ is the set of pairs of reflections of (2.2).

ii) $x_1 \dots x_d$ have order 2.

iii) $x_1 \dots x_\alpha$ $(\alpha \geqslant d)$ have exponent > 1 modulo Γ_0. Then, the proper periods of Γ_0 are

$$\left[\frac{m_{d+1}}{2}, \dots, \frac{m_\alpha}{2}, \quad (m_{\alpha+1})^2 \dots (m_\tau)^2 \quad n_1 \dots n_p \right]$$

An inmediat consequence of this corollary is the theorem 2 of [4].

References

[1] Bujalance, E.: *N. E. C. signatures which have associated a normal N. E. C. subgroup* (to appear) (1980).

[2] Macbeat, A. M.: *The Clasification of non-euclidean plane crystallographic groups.* «Cand. J. Math.» (1967).

[3] Maclachan, C.: *Maximal normal Fuchsian groups.* «Illinois J. Math.», 15 (1971).

[4] Singerman, D.: *On the structure of non-Euclidean crystallographic groups.* «Proc. Camb. Phil. Soc.» (1974).