ON CERTAIN THEOREMS IN TRANSFORM CALCULUS

by

M. L. MAHESHWARI

1. Introduction.—The integral equation

$$\varnothing(p) = p \int_0^\infty e^{-pt} f(t)\, dt, \quad \operatorname{Re} p > 0, \tag{1.1}$$

is known as the Laplace transform of $f(x)$, provided the integral on the right converges.

Let

$$M^\gamma[f(x)] = \int_0^\infty (xy)^{\frac{1}{2}} K_\gamma(xy) f(x)\, dx = g(y) \tag{1.2}$$

be the K-transform of order γ of $f(x)$, where y is a real variable. Let

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\gamma(xy) f(y)\, dy, \tag{1.3}$$

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\gamma(xy) g(y)\, dy, \tag{1.4}$$

then the two functions so connected are said to be Hankel transforms of each other.

When $g(x) = f(x)$, the equations (1.3) and (1.4) give

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\gamma(xy) f(y)\, dy, \quad \text{Re}\,\gamma \geq -\frac{1}{2}$$

and in this case $f(x)$ is said to be self-reciprocal in the Hankel transform of order γ. Following Hardy and Tichmarsh we may say that $f(x)$ belongs to R_γ, if it is self-reciprocal in the Hankel-transform of order γ.

Let $\gamma = \pm \frac{1}{2}$, (1.2) reduces to (1.1) and (1.3) reduces to Fourier's Sine and Cosine tranforms respectively.

In this paper, we propose to use Hankel transform and K-transform for evaluating integrals involving products of Parabolic cylinder functions and Legendre's functions. The results given are believed to be new.

2. Theorem 1.—Let

(i)

$$M^\gamma [f(x)] = g(y) \tag{2.1}$$

(ii) $\varnothing(v)$ be the Hankel transform of order 2γ of $x^{-\frac{1}{2}} g(x^2)$. Then

$$\int_0^\infty x^{-\frac{1}{2}} f(x) \left[I_\gamma\left(\frac{a^2}{4x}\right) - \mathbb{L}_\gamma\left(\frac{a^2}{4x}\right) \right] dx = \frac{4}{\pi a^{1/2}} \varnothing(a), \tag{2.2}$$

provided $x^{\frac{1}{2}\pm\gamma} f(x) = 0\,(x^\beta)$, Re $\beta > -1$ for small x and $f(x)$, $g(x)$ are continuous and absolutely integrable in $(0, \infty)$ and Re $\gamma > -\frac{1}{2}$.

Let $y^{-\frac{1}{2}} g(y^2)$ be $R_{2\gamma}$. Then

$$\int_0^\infty f(x) \left[I_\gamma\left(\frac{a^2}{4x}\right) - \mathbb{L}_\gamma\left(\frac{a^2}{4x}\right) \right] \frac{dx}{x^{1/2}} = \frac{4}{\pi a} g(a^2), \tag{2.3}$$

provided the above mentioned conditions exist.

Proof.—Multiplying both sides of (2.1) by $y^{-\frac{1}{2}} J_{2\gamma}(a y^{\frac{1}{2}})$ and integrating with respect to y between limits $(0, \infty)$, we obtain

$$\int_0^\infty y^{-\frac{1}{2}} J_{2\gamma}(a y^{\frac{1}{2}})\, dy \int_0^\infty f(x) K_\gamma(xy)(xy)^{\frac{1}{2}}\, dx = \int_0^\infty J_{2\gamma}(a y^{\frac{1}{2}})\, g(y) \frac{dy}{y^{1/2}}. \tag{2.4}$$

On changing the order of integrations, which is justified by the conditions given in the theorem, and evaluating the y-integral on left hand side, we get (2.2).

Let $y^{-\frac{1}{2}} g(y^2)$ be $R_{2\gamma}$, then

$$\text{R. H. S. of (2.4)} = \frac{2}{a} g(a^2).$$

Hence we get (2.3).

THEOREM 2.—Let

(i) $M^\gamma[f(x)] = g(y)$

(ii) $\varnothing(y)$ be the Hankel transform of order $2\gamma - 1$ of $x^{\frac{1}{2}} g(x^2)$. Then

$$\int_0^\infty x^{-\frac{3}{2}} f(x) \left[I_{\gamma-1}\left(\frac{a^2}{4x}\right) - \mathbb{L}_{\gamma-1}\left(\frac{a^2}{4x}\right) \right] dx = \frac{8}{\pi a^{3/2}} \varnothing(a), \tag{2.5}$$

provided $x^{\frac{1}{2} \pm \gamma} f(x) = 0(x^\alpha)$, Re $\alpha > -1$ for small x, and $f(x)$, $g(x)$ are continuous and absolutely integrable in $(0, \infty)$.

Let $y^{\frac{1}{2}} g(y^2)$ be $R_{2\gamma-1}$. Then

$$\int_0^\infty x^{-\frac{3}{2}} f(x) \left[I_{\gamma-1}\left(\frac{a^2}{4x}\right) - \mathbb{L}_{\gamma-1}\left(\frac{a^2}{4x}\right) \right] dx = \frac{8}{\pi a} g(a^2),$$

provided the above mentioned conditions are satisfied, and Re $\gamma > 0$.

THEOREM 3.—Let

(i) $g(y)$ be the Hankel transform of order γ of $f(x)$,

(ii)

$$M^\mu[x^{\gamma-\mu} f(x)] = \varnothing(y). \tag{2.7}$$

Then

$$\int_0^\infty \frac{y^{\left(\gamma+\frac{1}{2}\right)}}{(y^2+b^2)^{(1+\gamma-\mu)}}\, g(y)\, dy = \frac{b^{\left(\mu-\frac{1}{2}\right)}}{2^{(\gamma-\mu)}\,\Gamma(\gamma-\mu+1)}\, \varnothing(b), \quad (2.8)$$

provided $x^{(\gamma-\mu+\frac{1}{2}\pm\mu)} f(x) = 0\,(x^\beta)$, Re $\beta > -1$, and $f(x)$, $g(x)$ are continuous and absolutely integrable in $(0, \infty)$,

$$\text{Re}\,\gamma > -1, \quad \text{Re}\left(\gamma - 2\mu + \frac{3}{2}\right) > 0.$$

Let $f(x)$ be R_γ. Then

$$\int_0^\infty \frac{y^{\left(\gamma+\frac{1}{2}\right)}}{(y^2+b^2)^{(1+\gamma-\mu)}}\, f(y)\, dy = \frac{b^{\left(\mu-\frac{1}{2}\right)}}{2^{(\gamma-\mu)}\,\Gamma(1+\gamma-\mu)}\, \varnothing(b),$$

provided the conditions mentioned above exist.

Proofs of the theorems two and three are on similar lines as that of theorem 1.

Applications.—Example 1. Let

$$f(x) = 0, \quad 0 < x < b = x^{\mu-2}(x^2-b^2)^{-\mu/2}\, P^{\mu}_{\gamma-\frac{1}{2}}\left(\frac{x}{b}\right), \; b < x < \infty.$$

Then we get from (2.2)

$$\int_b^\infty x^{\left(\mu-\frac{5}{2}\right)}(x^2-b^2)^{-\mu/2}\, P^{\mu}_{\gamma-\frac{1}{2}}\left(\frac{x}{b}\right)\left[I_\gamma\left(\frac{a^2}{4x}\right) - \mathbb{L}_\gamma\left(\frac{a^2}{4x}\right)\right] dx =$$

$$= \frac{a^{2\gamma}\, 2^{\frac{1}{2}-2\gamma}}{\pi^{\frac{1}{2}}\,\Gamma\left(\gamma-\mu+\frac{5}{2}\right) b^{\left(\gamma+\frac{3}{2}\right)}}\; {}_1F_1\left[\begin{matrix} 1; & \\ & -\dfrac{a^2}{4b} \\ \gamma-\mu+\dfrac{5}{2}: & \end{matrix}\right],$$

Re $\gamma > -1$ and Re $\mu < 1$.

Taking appropriate $f(x)$, we have the following integrals.

We get from (2.2)

1.

$$\int_b^\infty x^{\mu-\frac{3}{2}}(x^2-b^2)^{-\mu/2}\,P_{\gamma-\frac{3}{2}}\left(\frac{x}{b}\right)\left[I_\gamma\left(\frac{a^2}{4x}\right)-\mathbb{L}_\gamma\left(\frac{a^2}{4x}\right)\right]dx=$$

$$=\frac{\Gamma(2\gamma)\,a}{\pi^{\frac{1}{2}}\,\Gamma(2\gamma+1)\,\Gamma\left(\frac{3}{2}-\mu+\gamma\right)2^{\left(2\gamma-\frac{1}{2}\right)}\,b^{\left(\gamma+\frac{1}{2}\right)}}\;{}_2F_2\left[\begin{matrix}2\gamma,1; \\ 2\gamma+1,\frac{3}{2}-\mu+\gamma;\end{matrix}\;-\frac{a^2}{4b^2}\right],$$

$\operatorname{Re}\mu<1$, $\operatorname{Re}\gamma>0$.

2.

$$\int_0^\infty x^{-\frac{1}{2}}D_{\gamma-\frac{1}{2}}\left(b\,x^{-\frac{1}{2}}\right)D_{-\gamma-\frac{1}{2}}\left(b\,x^{-\frac{1}{2}}\right)\left[I_\gamma\left(\frac{a^2}{4x}\right)-\mathbb{L}_\gamma\left(\frac{a^2}{4x}\right)\right]dx=$$

$$=2\,\Gamma(2\gamma-1)\,(2b^2+a^2)^{\frac{1}{2}}\,P_{-\frac{1}{2}}^{-2\gamma}\left[\frac{2^{\frac{1}{2}}\,b}{(2b^2+a^2)^{\frac{1}{2}}}\right],\quad \operatorname{Re}\gamma>\frac{1}{2}.$$

We get from (2.5)

$$\int_0^\infty x^{\frac{3}{2}}D_{\gamma-\frac{1}{2}}\left(b\,x^{-\frac{1}{2}}\right)D_{-\gamma-\frac{1}{2}}\left(b\,x^{-\frac{1}{2}}\right)\left[I_{\gamma-1}\left(\frac{a^2}{4x}\right)-\mathbb{L}_{\gamma-1}\left(\frac{a^2}{4x}\right)\right]dx=$$

$$=\frac{4}{(2\gamma-1)\,a^{2\gamma}}\left[(2b^2+a^2)^{\frac{1}{2}}-2^{\frac{1}{2}}\,b\right]^{2\gamma-1},\quad \operatorname{Re}\gamma>1.$$

Let $f(x)=x^{\frac{1}{2}}\,J_\gamma\left[2(a\,x)^{\frac{1}{2}}\right]K_\gamma\left[2(a\,x)^{\frac{1}{2}}\right]$.

We obtain from (2.7)

$$M^\mu\left[x^{\gamma-\mu+\frac{1}{2}}\,J_\gamma\{2(a\,x)^{\frac{1}{2}}\}\,K_\gamma\{2(a\,x)^{\frac{1}{2}}\}\right]=$$

$$=\frac{2^{(\gamma-\mu+2)}\,y^{\left(\mu-\gamma-\frac{3}{2}\right)}}{\pi^{1/2}}\,G_{13}^{3'}\left(\frac{a^2}{y^2}\,\middle|\,\begin{matrix}\mu-\frac{1}{2}\gamma \\ \frac{\gamma}{2},0,\frac{1}{2}\end{matrix}\right),$$

$\operatorname{Re}\gamma>-1$, $\operatorname{Re}(\gamma-\mu)>-1$, $\operatorname{Re}(\gamma-2\mu)>-2$.

REFERENCES

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Department of Mathematics
Birla Institute of Technology and Science
Pilani (Rajasthan), India