ON THE MAXIMUM MODULUS THEOREM FOR NONANALYTIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

by

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Let $w = f(z_1, \ldots, z_n) = u(x_1, \ldots, y_n) + i\, v(x_1, \ldots, y_n)$ be a complex function of the n complex variables $z_1, \ldots, z_n$, defined in some open set $A \subset \mathbb{C}^n$. The purpose of this note is to prove a maximum modulus theorem for a class of these functions, assuming neither the continuity of the first partial derivatives of u and v with respect to x_k and y_k, nor the conditions $f_{\bar{z}_k} = 0$ in A for $k = 1, 2, \ldots, n$ (the Cauchy-Riemann equations in complex form).

For the notation used in what follows, see reference [1].

THEOREM 1.—Suppose:

1) The combonents u and v are differentiable in A (in the real sense).

2) $|f_{z_j}| - |f_{\bar{z}_j}| \neq 0$ at each point of A for some $j \in \{1, 2, \ldots, n\}$, not necessarily the same j at different points.

Then $|f(z_1, \ldots, z_n)|$ does not attain a maximum value anywhere in A.

PROOF.—Let $z = (z_1, \ldots, z_n) \in$ A and $z + dz$ another point in the component of A, containing z.

Since u and v are differentiable in A, we have

$$\Delta u = \sum_{k=1}^{n} u_{x_k}\, d x_k + \sum_{k=1}^{n} u_{y_k}\, d y_k + \sum_{k=1}^{n} \varepsilon_k\, d x_k + \sum_{k=1}^{n} \varepsilon'_k\, d y_k$$

$$\Delta v = \sum_{k=1}^{n} v_{x_k}\, d x_k + \sum_{k=1}^{n} v_{y_k}\, d y_k + \sum_{k=1}^{n} \eta_k\, d x_k + \sum_{k=1}^{n} \eta'_k\, d y_k$$

where $\varepsilon_k, \varepsilon'_k, \eta_k, \eta'_k \to 0$ as $|dz| = \sqrt{|dz_1|^2 + \ldots + |dz_n|^2} \to 0$.
Hence,

$$\Delta w = \Delta u + i \Delta v = \sum_{k=1}^{n} (u_{x_k} + i v_{x_k}) dx_k + \sum_{k=1}^{n} (u_{y_k} + i v_{y_k}) dy_k + \\ + \sum_{k=1}^{n} \lambda_k dx + \sum_{k=1}^{n} \lambda'_k dy_k \tag{1}$$

where $\lambda_k = \varepsilon_k + i \eta_k \to 0$, $\lambda'_k = \varepsilon'_k + i \eta_k \to 0$ as $|dz| \to 0$.
From

$$f_{z_k} = \frac{1}{2}(f_{x_k} - i f_{y_k}), \qquad f_{\bar{z}_k} = \frac{1}{2}(f_{x_k} + i f_{y_k})$$

we obtain

$$f_{z_k} + f_{\bar{z}_k} = u_{x_k} + i v_{x_k}, \qquad i(f_{z_k} - f_{\bar{z}_k}) = u_{y_k} + i v_{y_k}$$

and using

$$dx_k = \frac{1}{2}(dz_k + d\bar{z}_k), \qquad dy_k = (-i/2)(dz_k - d\bar{z}_k)$$

equation (1) can be written as

$$\Delta w = \sum_{k=1}^{n} (f_{z_k} dz_k + f_{\bar{z}_k} d\bar{z}_k) + \sum_{k=1}^{n} \lambda_k dx_k + \sum_{k=1}^{n} \lambda'_k dy_k$$

or

$$f(z_1 + dz_1, \ldots, z_n + dz_n) = f(z_1, \ldots, z_n) + \sum_{k=1}^{n} (f_{z_k} dz + f_{\bar{z}_k} d\bar{z}_k) + \\ + \sum_{k=1}^{n} \lambda_k dx_k + \sum_{k=1}^{n} \lambda'_k dy_k \tag{2}$$

Let

$$f(z_1, \ldots, z_n) = M e^{i\theta}, \qquad f_{z_k} = A e^{i\alpha_k},$$
$$f_{\bar{z}_k} = B e^{i\beta_k}, \qquad dz_k = r_k e^{i\psi_k}$$

where any argument is chosen arbitrarily if the corresponding modulus is zero. Substituting in (2) we get

$$f(z_1 + d z_1, \dots, z_n + d z_n) = \mathrm{M}\, e^{i\theta} + \sum_{k=1}^{n} r_k \left[\mathrm{A}_k\, e^{i(\alpha_k + \psi_k)} + \mathrm{B}_k\, e^{i(\beta_k - \psi_k)}\right] +$$

$$+ \sum_{k=1}^{n} \lambda_k r_k \cos \psi_k + \sum_{k=1}^{k} \lambda'_k r_k \sin \psi_k \tag{3}$$

First suppose that for $k = j$ we have $| f_{z_j} | - | f_{\bar{z}_j} | = \mathrm{A}_j - \mathrm{B}_j > 0$ at the point $z \in \mathrm{A}$. Choose $r_k = 0$ for $k \neq j$ and $r_j \neq 0$. Also, choose ψ_j such that $\alpha_j + \psi_j = \theta$. Then (3) becomes

$$f(z_1, \dots, z_j + dz_j, \dots, z_n) = (\mathrm{M} + \mathrm{A}_j r_j)\, e^{i\theta} + \mathrm{B}_j r_j\, e^{i(\alpha_j + \beta_j - \theta)} +$$
$$+ r_j (\lambda_j \cos \psi_j + \lambda'_j \sin \psi_j)$$

and it follows that

$$| f(z_1, \dots, z_j + d z_j, \dots, z_n) | \geq \mathrm{M} + \mathrm{A}_j r_j - r_j (\mathrm{B}_j + | \lambda_j | + | \lambda'_j |) =$$
$$= \mathrm{M} + r_j (\mathrm{A}_j - \mathrm{B}_j - | \lambda_j | - | \lambda'_j |)$$

By taking r_j small enough, so that $\mathrm{A}_j - \mathrm{B}_j - | \lambda_j | - | \lambda'_j | > 0$, we obtain

$$| f(z_1, \dots, z_j + d z_j, \dots, z_n) | > \mathrm{M} = | f(z_1, \dots, z_j, \dots, z_n) |$$

Hence, the modulus of f does not attain a maximum at the point $z \in \mathrm{A}$.

Next suppose that $| f_{z_j} | - | f_{\bar{z}_j} | = \mathrm{A}_j - \mathrm{B}_j < 0$ at z.

Choose $r_k = 0$ for $k \neq j$, $r_j \neq 0$, as before, and ψ_j such that $\beta_j - \psi_j = \theta$. Then we obtain

$$f(z_1, \dots, z_j + d z_j, \dots, z_n) = (\mathrm{M} + \mathrm{B}_j r_j)\, e^{i\theta} + \mathrm{A}_j r_j\, e^{i(\alpha_j + \beta_j - \theta)} +$$
$$+ r_j (\lambda_j \cos \psi_j + \lambda'_j \sin \psi_j)$$

and it follows that

$$| f(z_1, \dots, z_j + d z_j, \dots, z_n) | \geq \mathrm{M} + r_j (\mathrm{B}_j - \mathrm{A}_j - | \lambda_j | - | \lambda'_j |)$$

Again, taking r_j small enough, so that

$$\mathrm{B}_j - \mathrm{A}_j - | \lambda_j | - | \lambda'_j | > 0,$$

there results

$$| f(z_1, \ldots, z_j + d z_j, \ldots, z_n) | > \mathrm{M}.$$

As a consequence of Theorem 1, the following form of the maximum modulus theorem holds.

THEOREM 2.—Suppose:

1) u and v are differentiable in A, where $A \subset \mathbb{C}^n$ is open and bounded.

2) $f = u + i v$ is continuous on the closure of A.

3) $| f_{z_j} | - | f_{\bar{z}_j} | \neq 0$ at each point of A for some $j \in \{1, 2, \ldots, n\}$, not necessarily the same j at different points.

Then the maximum of $| f(z_1, \ldots, z_n) |$ is attained on ∂A (the bodndary of A).

Similar theorems hold for the minimum modulus of f, provided f does not vanish anywhere in A.

REFERENCE

(1) NACHBIN, L.: *Holomorphic Functions, Domains of Holomorphy and Local Properties*. North-Holland Publishing Co., Amsterdam, 1970.