REAL COMMUTATIVE ALGEBRA. III. DEDEKIND-WEBER-RIEMANN MANIFOLDS

by

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The space S of all non-trivial real places on a real function field K | k of transcendence degree one, endowed with a natural topology analogous to that of Dedekind and Weber's Riemann Surface, is shown to be a one-dimensional k-analytic manifold, which is homeomorphic with every bounded non-singular real affine model of K | k. The ground field k is an arbitrary ordered, real-closed Cantor field (Definition below). The function field K | k is thereby represented as a field of real mappings of S which might be called «meromorphic» — each f in K | k has a convergent power series expansion at each of its finite points and a convergent Laurent series (finite negative order) in the vicinity of each of its finite set of infinities (or poles). The treatment is purely real up to the point where we want to show that K | k contains every «meromorphic» function on S. In order to do that we have had to take $k = \mathbb{R}$, the field of all real numbers, and appeal to complex function theory.

The best reference for the terminology and basic methods used here is Seidenberg's Elements of Algebraic Curves. Also helpful in several places, especially in connection with places, is Chevalley's Algebraic Functions of One Variable. For the purely real analogues in many cases, two earlier notes of the authors [3, 4] are frequently used, especially [3] for convergence properties in Cantor fields.

Throughout, k denotes a real-closed *Cantor field* — a Cantor field is an ordered field containing a *microbe*, and m is a microbe in k provided for every positive x in k, there exists n such that $0 < m^n < x$ for short, we say «powers of m are arbitrarily small in k». K | k will always denote a real function field of one variable, i.e. a finitely generated real extension of transcendence degree one. A *real place* on K is a place mapping onto $k \cup \{\infty\}$. The set of all real places on K is denoted by $S = S(K \mid k)$. It is topologized by the weak topology for $B = \cap \{R_h ; h \in S\}$, where h is a real place and R_h is its valuation ring. The maximal ideal of h is denoted by M_h, and the associated valuation by v_h. In the special case $k = \mathbb{R}$, the space $S(k \mid \mathbb{R})$ is homeomorphic with the space X (K) of [2].

A real function f of one real variable is *meromorphic* if it is the restriction to R of a complex meromorphic function. For a real topological one-manifold S′, a map f from an open subset U of S′ to R is *meromorphic at h* provided for every coordinate system x at h, $f \circ (x)^{-1}$ is a real meromorphic function; in case f is meromorphic at every point of U we say f is *meromorphic on* U.

1. Statement of Main Theorem. Let k be a real-closed Cantor field. Let X be a Hausdorff space. A d-*dimensional* k-*coordinate system* in X is a homeomorphism of an open set in X onto an open set in $k^{(d)}$. (X, C) is a d-*dimensional topological* k-*manifold* if X is covered by the domains of the coordinate systems in C, C being a set of d-dimensional k-coordinate systems in X. If, in addition, for each pair x and y of coordinate systems in C, $x \circ y^{-1}$ is an analytic function (described, locally, by convergent power series), then (X, C) is a d-*dimensional analytic* k-*manifold*.

THEOREM 1.—Suitable restrictions of the uniformizing variables form a family C of one-dimensional coordinate systems on S satisfying:

A. (S, C) is a topological k-manifold.

B. (S, C) is an analytic k-manifold. Every member f of $K \mid k$ is analytic on S except at its finite set of infinities, at each of which it has a Laurent expansion of finite negative order.

C. In case $k = \mathbb{R}$, S is compact and K is the field of all real meromorphic mappings of S.

D. In case $k = \mathbb{R}$, S is (topologically) a finite union of pairwise disjoint circles.

2. The Coordinate Systems. Observe that for all x in K,

$$(1 + x^2)^{-1} \quad \text{and} \quad x \cdot (1 + x^2)^{-1}$$

belong to B. Let h_0 be a place mapping into $k \cup \{\infty\}$. For any x, we have $x = u/v$, where

$$u = x/(1 + x^2), \quad v = 1/(1 + x^2),$$

if $x(h_0)$ is finite;

$$u = 1/(1 + x^{-2}), \quad v = x^{-1}/(1 + x^{-2})$$

if $x(h_0) = \infty$. In either case we see that x, as ratio of functions which are continuous and not both infinite nor both zero at h_0, is continuous at h_0. The same identities show that B separates points so S is Hausdorff:

LEMMA 1.—S is a Hausdorff space and every number of K is continuous on S.

Now choose a uniformizing variable x at $h_0 : v(x) = 1$. As is true for any non-zero element, the set of all real zeros of x is finite [7]; say $h_0, h_1, \ldots, h_n$ is a complete list of (distinct) real zeros of x.—Since S is a Hausdorff space, there exists, for each $i > 0$, an element b_i in B such that

$$h_0 b_i = 0, \quad h_i b_i \neq 0.$$

Set $b = \Sigma_1^n b_i^2$. Then

$$h_0 b = h_0 \Sigma b_i^2 = \Sigma (h_0 b_i)^2 = 0,$$

and for

$$j > 0, \quad h_j b \geq (h_j b_j)^2 > 0.$$

The identity $x = u/v$ given above shows that K is the field of quotients of B. Since K is finite algebraic over $k(x)$ there exists w in

$$\bigcap \{R_{h_i}; \quad 0 \leq i \leq n\}$$

such that $K = k(x, w)$; put $v = w - h_0 w$, then

$$K = k(x, v), \quad h_0 v = 0, \quad v \in \bigcap \{R_{h_i}; \quad 0 \leq i \leq n\}.$$

Let

$$M = \max \{ | h_i(v) | ; \quad 0 \leq i \leq n\}, \quad m = \min \{ | h_i b | ; \quad 1 \leq i \leq n\}.$$

Then $M \geqslant 0$, $m > 0$. By the strong theorem of the primitive element we have for all large p in k,

$$K = k(x)(v, b) = k(x)(v + p\,b) = h(x, v + p\,b).$$

Choose p so large that $K = k(x, v + p\,b)$ and $p\,m > M$ (both m and M belong to k and $m > 0$). For $j > 0$,

$$h_j(v + p\,b) \geq p \mid h_i b \mid - \mid h_i v \mid \geq p\,m - M > 0, \quad h_0(v + p\,b) = 0.$$

Thus, for $y = v + p\,b$ we have:

i) $K = k(x, y)$. ii) x is a uniformizing variable at h_0. iii) $h\,x = h\,y = 0 \Longleftrightarrow h = h_0$ (for real h).

Let

$$y = \eta(x) \equiv \Sigma\, \eta_i x^i,$$

η_i in k, be the h_0-adic expansion for y (see [3]). Let Γ be the affine curve in $k^{(2)}$ with coordinate ring $k[x, y]$. Then $(t, \eta(t))$ is a branch representation centered at the origin. The bijective correspondence between branches and places and the simplicity of Γ at all points near the origin (the origin is an inner point of Γ, since there is a real branch there) now show (cf. [3]):

iv) For all points z near the origin on Γ, there is exactly one branch of Γ centered at z. (v) For all $z \neq (0, 0)$ on Γ near $(0, 0)$, z is simple.

Now we apply the generalized Puiseux theorem [3]; in view of iv) (recall $(t, \eta(t))$ is the unique branch at the origin) it asserts that for a small box W centered at the origin,

$$\Gamma \cap W = \{(a, \eta(a)); (a, 0) \in W\},$$

while simplicity of points near to but distinct from the origin shows that we may assume further that for $a \neq 0$, $(a, \eta(a))$ is simple and hence there is exactly one place h with $h\,x = a$, $h\,y = \eta(a)$. We have proved that x defines, for some $\varepsilon > 0$, a bijection of

$$U_\varepsilon = \{h; \ \mid h\,x \mid \leq \varepsilon, \ \mid h\,y \mid \leq \varepsilon\}$$

onto the closed interval

$$I_\varepsilon = \{a \in k; \ \mid a \mid \leq \varepsilon\}$$

in k; earlier we showed that x is continuous. We will show that $(x)^{-1}$ is continuous (restricted to I_ε for small enough ε); by $(x)^{-1}$ we mean the inverse of the function $x : h \longmapsto h(x)$, not the reciprocal $1/x$ of x. First we prove

LEMMA 2.—For a member $u(x, y)$ of the function field $k(x, y)$ if u is finite at h_0 then u is continuous on a neighborhood of $(0, 0)$ on Γ.

The finiteness hypothesis guarantees that the Laurent series

$$\zeta(t) = u(t, \eta(t))$$

is actually a power series. Let φ be the map $s \longrightarrow (s, \eta(s))$, from a neighborhood of 0 in k into Γ. Bukowski's Theorem [3] show that η is continuous and that φ is (locally) a homeomorphism. Hence

$$u = \zeta \circ \varphi^{-1}$$

is continuous near $(0, 0)$, as asserted, and the Lemma is proved.

Next we note that near $(0, 0)$, any $u(x, y)$ in B satisfies

$$u = u(x, \eta \circ x),$$

and we just showed that u is continuous on Γ near $(0, 0)$. The formula thus shows that for some $\varepsilon > 0$ and any place h_1 in U_ε, the neighborhoods of the form

$$V_\delta(h_1) = (x)^{-1}\{h; x h_1 - \delta < x h < x h_1 + \delta\}$$

form a base for the neighborhoods of h_1. To prove continuity of $(x)^{-1}$ we need only show that the inverse image of $V_\delta(h_1)$ by $(x)^{-1}$ is open in k. But

$$((x)^{-1})^{-1} V_\delta(h_1) = x V_\delta(h_1)$$

is just the open interval of all a in k, between $x h_1 - \delta$ and $x h_1 + \delta$. This completes the proof or:

LEMMA 3.—$(x)^{-1}$ is continuous. In fact, there exists a positive ε in k such that the restriction of x to

$$U_\varepsilon = \{h; \mid h x \mid \leq \varepsilon, \ \mid h y \mid \leq \varepsilon\}$$

is a topological map onto the closed interval

$$I_\varepsilon = \{a; \mid a \mid \leq \varepsilon\} \quad \text{in} \quad k.$$

Thus the uniformizing variables, for h_0 define coordinate systems. Let C be the set of all these, as h_0 ranges over S. Clearly then (S, C) is a topological manifold, and part A of the Theorem is proved.

3. The analytic structure of S. To get the analytic structure recall that for any h_0 in S and a uniformizing variable x at h_0, we have i)-v), and also (see definition of U_ε, I_ε above) we know:

vi) For all a in I_ε there is exactly one point (a, b) on Γ near the origin.

Now ε can be chosen so small that:

vii) For all h in U_ε, $x - x\,h$ is a uniformizing variable at h. This holds for $h = h_0$ by ii). Let $h \neq h_0$. Then

$$(a, b) = (h\,x, h\,y) \neq (0, 0),$$

by iii). If h is near enough to h_0 then the tangent is not vertical at (a, b), and so the branch there has a representation

$$(a + t, b + b_1 t + \ldots)$$

Hence.

$$\nu_h (x - a) = \text{Ord}_t (a + t - a) = 1 : x - a$$

is a uniformizing variable at h, proving vii).

LEMMA 4.—Let f be a member of $K \mid k$, let h_0 be a real place and let x be a uniformizing variable at h_0. For all real places h close enough to h_0, except possibly h_0,

$$f\,h = \Sigma\, c_i\,(x\,h)^i,$$

where $\Sigma\, c_i\, x^i$ is the h_0-adic expansion of f. If f is finite at h_0, the equation goes for $h = h_0$.

PROOF.—y can be chosen so i)-vii) and Lemma 3 hold. Let $\Sigma\, b_i\, x^i$ be the h_0-adic expansion of y. Now

$$f\,(t, \Sigma\, b_i\, t^i) = \Sigma\, c_i\, t^i$$

by formal power series considerations (regardless of groundfield). Bukowski's theorem (loc. cit.) shows that, f being continous,

$$f(x, \Sigma \eta_i s^i) = \Sigma c_i s^i$$

for $0 < |s| < \varepsilon$, if ε is small enough (see Lemma 3); and if h is the unique real place mapping x on s, we have

$$f h = h(f) = f(h x, h y) = f(s, \Sigma \eta_i s^i) = \Sigma c_i s^i = \Sigma c_i x(h)^i,$$

as claimed. In case f is finite at h_0, its expansion has no negative powers and the condition $s \neq 0$ is unnecessary. The Lemma is proved.

Suppose next that x and f are both uniformizing variables at h_0, and that the h_0-adic expansion of f is

$$\sum_{i=1}^{\infty} c_i x^i, \quad c_1 \neq 0.$$

By Lemma 4 we get for all small $s = x h$,

$$(f \circ (x)^{-1}) s = f(h) = \Sigma c_i x(h)^i = \Sigma c_i s^i.$$

Hence $f \circ (x)^{-1}$ is analytic at 0 ($x h_0 = 0$). Let h_1 be another real place, let x_1 be a uniformizing variable at h_1. Let h be in the intersection of the neighborhoods

$$U_{\varepsilon_0}(h_0), \quad U_{\varepsilon_1}(h_1)$$

(see Lemma 3), set $a_i = x_i h$. As before we write $(f)^{-1}$ for the inverse function. We have

$$x_1 \circ (x_0)^{-1} = x_1 \circ (x_1 - a_1)^{-1} \circ (x_1 - a_1) \circ (x_0 - a_0)^{-1} \circ (x_0 - a_0) \circ (x_0)^{-1} =$$
$$= T_{-a_1} \circ [(x_1 - a_1) \circ (x_0 - a_0)^{-1}] \, T_{a_0}.$$

The translations T_{-a_1} and T_{a_0} are analytic. The bracketed map is likewise: each $x_i - a_i$ is a uniformizing variable at h, by vii) applied to each x_i, and so by the preceding paragraph the bracketed composition is analytic. Hence $x_1 \circ (x_0)^{-1}$ is analytic. Analyticity of (S, C) is proved. Analyticity of the members f of K, as functions on (S, C), follows from Lemma 4. Note that f can't have more than a finite number of infinities.

4. Meromorphic mappings (Case $k = \mathbb{R}$). Let S^* be the set of all places of K^* into $\mathbb{C} \cup \{\infty\}$, $\mathbb{C} = \mathbb{R}(i) =$ complex field,

$$K^* = \{r + s\,i;\, r, s \in K\};$$

for h in S,

$$h^*(r + s\,i) = h(r) + i\,h(s)$$

is a place of K^*. Elements of K are now extended (as maps) to all of S^* as usual, $x(h) = h(x)$. For the inverse of the *map* x, we write $(x)^{-1}$, not to be confused with the multiplicative inverse.

Let h_0 be any member of S, let $x \in K$ be a uniformizing variable for h_0. Then certainly x is a uniformizing variable for h_0^*. For any f in K we know from the classical theory that $f \circ (x)^{-1}$ is a (complex) meromorphic function on a disc centered at 0 in the complex plane. Its restriction to the real axis is $f \circ (x)^{-1}$, which is therefore a *real* meromorphic function. Thus f is meromorphic at h_0. Since h_0 is arbitrary in S, we have shown that f is a meromorphic mapping of S. Hence, every member of K is a meromorphic mapping of S. Let f be a real meromorphic mapping on S, say $f \circ (x)^{-1}$ is the restriction to the reals of the complex meromorphic function g (we are still assuming x is a uniformizing variable at $h_0 \in S$):

$$f \circ (x)^{-1} = g\,|_{\mathbb{R}}.$$

Define f^* on S^* by $f^* = g \circ x$. Now $f^* \circ (x)^{-1} = g$, which is meromorphic; hence *f^* is a meromorphic mapping of S^**. For any h in S, $x(h)$ is real and hence

$$g(x(h)) = (f \circ (x)^{-1})(x\,h) = f(h);$$

so

$$f^*(h) = g(x(h)) = f(h).$$

Thus $f^*|_S$ *is just* f. From the classical theory we haye u and v in K with

$$f^* \circ (x)^{-1} = (u + i\,v) \circ (x)^{-1},$$

since every meromorphic mapping of S^* is got from a member of K^* in this way. We thus have $f^* = u + iv$ (as functions on S^*). For h in S, $u(h)$ and $v(h)$ are real, and

$$(u + i v)\, h = u(h) + i\, v(h) = f^*(h) = f(h),$$

since $f^* |_S = f$. But $f(h)$ belongs to R (since $h \in S$). Hence $v(h) = 0$. This holds for all the (infinitely many) h in S so v is itself zero. Hence $f^* = u$, and in fact, $f = f^* |_S = u \in K$. Thus every meromorphic mapping of S belongs to K, proving the reverse inclusion, and par C is done.

5. Topological structure in Case $k = R$. Now S is compact and locally connected. In fact it is a finite union of pairwise disjoint circles (topologically). Compactness was proved in [2]. Being a manifold, S is locally connected. By compactness it is a finite disjoint union of compact and connected real analytic one-manifolds, each homeomorphic with a circle (see Bishop and Crittenden, p. 5). This proves part D and finishes the proof of the theorem.

6. Relation to nonsingular bounded models. Such models always exist [3]. We return to an arbitrary real-closed Cantor field.

THEOREM.—Let C be a bounded non-singular model in $k^{(n)}$ with function field $K \mid k$ and coordinate ring $k[x_1, \ldots, x_n]$. The correspondence $h \longrightarrow (h\, x_1, \ldots, h\, x_n)$ defines a topological equivalence $\varphi : S \longmapsto C$.

PROOF.—Continuity follows from Lemma 1 (continuity of elements of K as functions on S), surjectivity from the real place extension theorem, injectivity from non-singularity. Continuity of φ^{-1} comes from (cf. Lemma 2) continuity of members of K as functions on C, at least in regions where they are defined. We give details for the last claim — the earlier ones are transparent. Let h_0 be a real place centered at z^0 on C. We must show that φ^{-1} is continuous at z^0. Let G be a neighborhood of h_0, say for members $u_1, \ldots, u_m$ of K, $\varepsilon > 0$, $| u_j h_0 | < \varepsilon$, $1 \leqslant j \leqslant m$,

$$G = \{h;\ \ | u_j h - u_j h_0 | < \varepsilon,\ \ 1 \leq j \leq m\},$$

We need a neighborhood of z^0 which maps into G. We have

$$u_j h_0 = h_0 (u_j (x)) = u_j (h_0 x_1, \ldots, h_0 x_n) = u_j (z^0).$$

For z close to z^0 on C, z can be substituted in $u_j(x)$ so we may write

$$u_j h = h u_j = u_j(z), \quad z = (h x_1, \ldots, h x_n);$$

h being the (unique) place centered at z, $h = \varphi^{-1}(z)$. Thus u_j is finite at h so we can apply the continuity of u_j on C to assert that for $1 \leqslant j \leqslant m$ and for all z close to z^0 on C,

$$| u_j(z) - u_j(z^0) | < \varepsilon,$$

hence

$$| u_j h - u_j h_0 | < \varepsilon, \quad h \in G.$$

In other words some neighborhood of z^0 on C maps by φ^{-1} into G; continuity of φ^{-1} is proved.

Corollary.—Assume $k = \mathbb{R}$. Let ν be the number of (connected) components of S. Every nonsingular compact affine real model has ν components. Every compact affine real model has at most ν components after isolated paints are discarded. ν is at most one plus the genus.

Proof.—The last claim is Harnack's Theorem, since ν is also, by the first claim, the number of components of a nonsingular model (Harnack [11]). The first two assertions are consequences of the properties of the map φ of the Theorem: In case C has singular points, φ might not be injective. But it maps onto the set of all inner points.

Note.—All the results of this paper were obtained by the author in 1971, but none have been previously published.

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