SYMMETRIES AND SOLVABILITY OF LINEAR DIFFERENTIAL EQUATIONS

by

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SUMMARY

The canonical form theorem, applied to a certain group of symmetry transformations of certain Fuchsian equations, leads automatically to the integration of them. The result can be extended to any n-order differential equation possessing a certain pointlike group of symmetries with a maximal abelian Lie-subgroup of order c.

The following points should be recalled as theoretical background in all what follows: A) A second order ordinary differential equation cannot have a Lie group of pointlike transformations of symmetry with more than eight parameters [1], B) An ordinary differential equation of order n does not possess, in general, groups (Lie groups) of pointlike transformations of symmetry [2], and C) The maximum number of parameters of a pointlike Lie group of symmetry transformations of an ordinary differential equation of order n ($n > 2$) is equal to $n + 4$ [3].

Consider now the following Fuchslike differential equation:

$$\sum_{n=0}^{N} a_n x^n y^{(n} (\alpha) = 0, \qquad a_n \in \mathbb{R} \tag{1}$$

It is clear that, in addition to the banal group of symmetries defined by

$$\left.\begin{array}{l} y' = \alpha y \\ x' = x \\ \alpha \in \mathbb{R}^* \end{array}\right\} \tag{2}$$

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equation (1) admits, as well, the following monoparametric Lie group of symmetries:

$$\left.\begin{array}{l} y' = y \\ x' = k\,x \\ k \in \mathbb{R},\, k \neq 0 \end{array}\right\} \tag{3}$$

Now, applying the canonical form theorem [4] to the vector field $\vec{X}$ associated with (3), we would be induced to the introduction of the new local coordinates (x', y') defined through the equations:

$$\left.\begin{array}{c} \vec{x} = 0\,\dfrac{\partial}{\partial y} + x\,\dfrac{\partial}{\partial x} \\ \dfrac{d\,y}{0} = \dfrac{d\,x}{x} = d\,t \end{array}\right\} \tag{4}$$

$$\left\{\begin{array}{l} x = e^{x'} \\ y = y' \end{array}\right\} \tag{5}$$

But since in this new local coordinates the group defined by (3) adopts the form:

$$\left\{\begin{array}{l} x' = x + \beta \\ y' = y \\ \beta \in \mathbb{R} \end{array}\right. \tag{6}$$

and, at the same time the change of coordinates defined by (5), does preserves the linear character of (1), it is obvious that the symmetry (6) of the transformed differential equation forces to this one to be (and we do not need to make any calculations in order to assert it) a linear differential equation *but with constant coefficients*). We have, therefore, before us, now, a purely algebraic problem of finding roots (and the multiplicities of them) of a certain polinomial (the characteristic polinomial associated with the final differential equation). In accordance with all this results, it is now quite clear the purely algebraic origin of the presence of the terms

$$x^{\alpha},\ x^{\alpha}(\log x),\ x^{\alpha}(\log x)^{2},\ldots,\qquad \alpha \in \mathbb{C}$$

in the general solution of (1). The root of this fact is seen to be purely algebraic, and is connected with the multiplicity of α in the final characteristic polinomial associated (by the substitution given by the equations (5)) to the linear differential equation (with constant coefficients) obtained by the procedure explained above. The same can be said on the behaviour, in the neighbourhood of $x = 0$, of the linear differential equation,

$$\sum_{n=0}^{N} a_n(x)\, x^n\, y^{(n}(x) = 0,$$

$a_n(x)$ being analytic functions such that $a_n(0) \neq 0$ for all the values of n.

It is interesting to recall here that the multiplicity of a certain root of the characteristic polinomial associated with a given linear differential equation of constant coefficients has, as well, important consequences in relation with the representations induced, by the general solution of the given equation, of the symmetry group, given by equations (6), that all these linear equations do possess. Indeed, when α is a simple root of the characteristic polinomial, then the solution exp (αx) induces the realization of (6) given by:

$$\beta \longmapsto e^{\alpha\beta}.$$

But if α is, for instance, a double root of the characteristic polinomial, then the symmetry group (6) is linearly represented in the two-dimensional vectorial space of the solutions of the form,

$$(C x + D) \exp(\alpha x).$$

In this case the representation of (6) obtained is given by:

$$\beta \longmapsto \begin{pmatrix} e^{\alpha\beta} & 0 \\ \beta e^{\alpha\beta} & e^{\alpha\beta} \end{pmatrix}.$$

This representation is reducible, but *not* completely reducible. This not completely reducible character is to be ascribed, as well, to the algebraic fact of the non-simplicity of the root α in the caracteristic polinomial. The non-completely reducible character can be understood given the *non*-compact nature of the group given by (6) (in the compact case the reducibility of the representation would automatically induce the complete reducibility of it).

Consider now a generic differential equation of order n,

$$E\,(x, y, y', \ldots, y^{(n)}) = 0 \tag{7}$$

Let G_p be a certain Lie group of pointlike transformation of symmetry of (7). Let G_I^n be the associated Lie group associated to G_p, by means of the standard prolongations procedures [5]. Calling now

$$\vec{X}_1{}^A, \ldots, \vec{X}_g{}^A$$

to the generators of it, then all of them (considered as vector fields acting on the $x, y, y', \ldots, y^{(n}$ space) must be tangent to the hypersurface of this space defined by (7). Now, let A_c be a certain maximal abelian Lie subgroup of G_I^n, and let c be the number of independent generators of it. In that case the canonical form theorem applied to A_c would introduce a set of n local coordinates over the manifold defined by (7) such that c of them would transform, under the action of G_I^n, in the way:

$$\left.\begin{array}{l} x_i' = x_i + c_i \\ i = 1, \ldots, c. \end{array}\right\} \tag{8}$$

Therefore in this new local variables we would obtain a new differential equation of order n (or possibly less than n) in which only $n - c + 1$ variables can appear. In that way we have now considerably reduced the problem of integrating eq. (7).

It is to be remarked that in contrast with the above example the new local coordinates used here (in order to apply the canonical form theorem) do involve *not only* the original variables (x, y) but also the derivatives $y', y'', \ldots, y^{(n}$. Note, as well, that our result can be repeated, without any modifications, for groups of contact transformations (of generators tangent to the hypersurface defined by (7)) not necessarily induced by a pointlike group of transformations G_p acting on the (x, y) plane, as considered above. It is for this class of groups that the abelian subgroups A_c have a certain entity and it is for this reason that the reader should be recalled, and induced, here to study the contact structures.

REFERENCES

[1] DICKSON, L. E.: «Ann. Math. Soc.» (2), 25, 287 (1924).
[2] CAMPBELL, J. E.: *Introductory Treatise on Lie's Theory of finite Continuous Transformations Groups*. Chelsea Pu. Co. Bronx NY 1966, Chapt. II.
[3] See Dickson Ref. 1, p. 353 (Footnote).
[4] HERMANN, R.: *Differential Geometry and the Calculus of Variations*, Ac. Press NY 1968, pages 63-72.
[5] See Campbell Ref. 2, Chapts. II and XIX.