ON A FORM OF BI-VARIATE-t DISTRIBUTION

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1. SUMMARY.

Distribution of a statistic of the form bi-variate t has been obtained when the sample size is even and the probability integral has been tabulated for certain values of ρ and the sample size.

2. INTRODUCTION.

Among the many forms of bi-variate t available, some of the recent ones could be found in [2] and [3]. Suppose $\nu_1 = C_1 X/\sqrt{U_1}$ and $\nu_2 = C_2 Y/\sqrt{U_2}$ where X and Y are distributed at bi-variate Normal independent of U_1 and U_2 and C_1, C_2 are constants. Also U_1 and U_2 are distributed as chi-square random variables. In [3], distribution of ν_1 and ν_2 is obtained where U_1 and U_2 have the bi-variate chi-square distribution. In [2], in the place of U_2, a quantity $U_1 + U^*$ is used where U^* is $\chi^2(n-1)$ and U_1 is $\chi^2(m-1)$. In both these cases [2] and [3], expression for the probability integral $p[\nu_1 \leqslant a, \nu_2 \leqslant b]$ is given. So, also in [4] and [5], where this integral is being expressed in terms of the parabolic Cylinder functions in [4] and in terms of the normal probability integral in [5]. Evaluation in each of the above cases is quite heavy.

3. DISTRIBUTION OF THE STATISTIC ($\mu = 0$).

Here, the distribution of ν_1 and ν_2 is considered in relation to $U_1 - U_2$ when the sample size m is even. Then, new variables are

$$t_1 = \sqrt{m}\, \bar{x}/\sqrt{\nu} \text{ and } t_2 = \sqrt{m}\, \bar{y}/\sqrt{\nu} \tag{1}$$

where $\nu = U_1 - U_2$.

Taking the variances as unity, $(\bar{x}, \bar{y})$ has the density

$$f(\bar{x}, \bar{y}) = \frac{\sqrt{m}}{2\Pi\sqrt{1-\rho^2}} \exp\left[-m(\bar{x}^2 - 2\rho\,\overline{xy} + \bar{y}^2)/2(1-\rho^2)\right] \tag{2}$$

and the density of ν when m even is from [1]

$$f(\nu) = \sum_{i=0}^{\infty} \frac{(\rho^2)^i \, \Gamma^{-1}(m/2)}{i! \; 2^{m+2i-1}}$$

$$\cdot \sum_{j=0}^{\frac{m}{2}+i-1} \frac{e^{-\nu/2(1-\rho^2)} \; \nu^{\frac{m}{2}+i-j-1} \; \Gamma\left(\frac{m}{2}+i+j\right)}{j! \; (1-\rho^2)^{i-j} \; \Gamma\left(\frac{m}{2}+i-j\right)} \tag{3}$$

from (1), (2) and (3), we have

$$f(t_1, t_2, \nu) = \frac{1}{2\Pi \, (1-\rho^2)^{1/2}} \sum_{i=0}^{\infty} \frac{(\rho^2)^i \, \Gamma^{-1}(m/2)}{i! \; 2^{m+2i-1}} \cdot$$

$$\cdot \sum_{j=0}^{\frac{m}{2}+i-1} \frac{e^{-\nu(1+Q)/2(1-\rho^2)} \; \nu^{\frac{m}{2}+i-j} \; \Gamma\left(\frac{m}{2}+i+j\right)}{(1-\rho^2)^{i-j} \; j! \; \Gamma\left(\frac{m}{2}+i-j\right)} \tag{4}$$

where $Q = t_1^2 + t_2^2 - 2\rho \, t_1 t_2$.

From (4), we have,

$$f(t_1, t_2) = \frac{(1-\rho^2)^{\frac{m+1}{2}}}{2\Pi} \sum_{i=0}^{\infty} \frac{(\rho^2)^i \; \Gamma^{-1}(m/2)}{i! \; 2^{\frac{m}{2}+i-2}}$$

$$\cdot \sum_{j=0}^{\frac{m}{2}+i-1} \frac{\left(\frac{m}{2}+i-j\right) \Gamma\left(\frac{m}{2}+i+j\right)}{2^j \; j! \; (1+Q)^{\frac{m}{2}+i-j+1}} \tag{5}$$

$$P[t_1 \leqslant a, t_2 \leqslant b] = \int_{-\infty}^{a} \int_{-\infty}^{b} f(t_1, t_2)\, dt_1\, dt_2$$

is expressed in terms of polar coordinates as

$$= (1-\rho^2)^{\frac{m+1}{2}} \sum_{i=0}^{\infty} \frac{(\rho^2)^i\,(m/2)_i}{i!\,2^{\frac{m}{2}+i-1}} \sum_{j=0}^{\frac{m}{2}+i-1} \frac{\left(\frac{m}{2}+i\right)_j}{2^j\, j!}$$

$$\cdot \int_0^{\theta_0} \frac{d\theta}{1-\rho \operatorname{Sin} 2\theta} - \qquad (6)$$

$$- \int_0^{\theta_0} \frac{d\theta}{(1-\rho \operatorname{Sin} 2\theta)\,[1+r^2_0\,(1-\rho \operatorname{Sin} 2\theta]^{\frac{m}{2}+i-j}}$$

where $(a)_n = a(a+1) \ldots (a+n-1)$.

(6) is used to generate the probability integral for the values of ρ and m. The attached tables are for

$\rho = 0, .1, .5, .9$ and for

$m = 2, 6, 10, 20.$

Table of the integral for other values of $\rho = .2, .3, .4, .6, .7, .8$ and for $m = 4, 16, 30$ and 50 are available with the author.

4. Acknowledgement.

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5. References.

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