THE N-DIMENSIONAL CAUCHY-RIEMANN EQUATIONS

por

M. O. GONZALEZ

1. INTRODUCTION. We propose to generalize the Cauchy-Riemann equations via the conformality property of analytic functions at points where the Jacobian $J_f(z) \neq 0$.

Consider in E^n a regular arc γ: $[a, b] \to E^n$ defined by

$$\mathbf{x} = \mathbf{x}(u) = x^i(u)\, \mathbf{e}_i,\ a \leqslant u \leqslant b,\ i = 1, \ldots, n$$

where the summation convention is used, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard orthonormal basis in E^n.

Let $\mathbf{f} = (f^1, \ldots, f^n)$ be a vector function defined in some region $\Omega \subset E^n$ containing γ^* (the graph of γ), with components f^i of class C^1 in Ω, and let

$$\mathbf{f}(\gamma) = \Gamma : \mathbf{y} = \mathbf{f}(\mathbf{x}) = y^i(u)\, \mathbf{e}_i,\ a \leqslant u \leqslant b$$

be the image of γ under $\mathbf{f}$, so that

$$y^i(u) = f^i(x^1, \ldots, x^n) \text{ with } x^i = x^i(u)$$

for each i. Since

$$\frac{dy^i}{du} = \frac{\partial f^i}{\partial x^j} \frac{dx^j}{du}$$

it follows that

$$J = J_f(x) = \left| \frac{\partial f^i}{\partial x^j} \right| = |a^{ij}| \neq 0$$

at $\mathbf{x} = \mathbf{p}$ is a sufficient condition for Γ to be also regular at the corresponding point $\mathbf{f}(\mathbf{p})$. In what follows we assume that the condition $J_f(\mathbf{p}) \neq 0$ is satisfied.

2. THE CAUCHY-RIEMANN EQUATIONS. The element of arc of γ at $\mathbf{p}$ is given by

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \sum_i (dx^i)^2$$

the dx^i being direction numbers of the tangent line to γ^* at $\mathbf{p}$. The corresponding element of arc of Γ at $\mathbf{f}(\mathbf{p})$ is given by

$$d\sigma^2 = d\mathbf{y} \cdot d\mathbf{y} = \sum_i (dy^i)^2 = \mathrm{M}_{jk}\, dx^j\, dx^k$$

$j, k = 1, \ldots, n$, the dy^i being direction numbers of the tangent line to Γ^* at $\mathbf{f}(\mathbf{p})$, and

$$\mathrm{M}_{jk} = \sum_i \frac{\partial f^i}{\partial x^j} \frac{\partial f^i}{\partial x^k}$$

Hence, the square of the magnification ratio $\rho = d\sigma/ds$ is given by

$$\rho^2 = \mathrm{M}_{jk}\, dx^j\, dx^k / \sum_i (dx^i)^2$$

This ratio is independent of the direction numbers dx^i (or, independent of the particular arc γ through $\mathbf{p}$) if, and only if,

$$\mathrm{M}_{jk} = 0 \text{ for } j \neq k, \text{ and } \mathrm{M}_{jj} = \mathrm{M} \tag{1}$$

M being a positive constant at p. Then we have

$$\rho^2 = \mathrm{M} \tag{2}$$

Equations (1) are also necessary and sufficient conditions for the mapping defined by $\mathbf{f}$ to be directly or inversely conformal at $\mathbf{p}$. This well known fact is easily derived from the formulas giving the cosine of the angle between two arcs.

With the notation $a^{ij} = \partial f^i / \partial x^j$, and using (2), equations (1) can be written as

$$\left\{ \begin{array}{l} (a^{11})^2 + \ldots + (a^{n1})^2 = \ldots = (a^{1n})^2 + \ldots + (a^{nn})^2 = \rho^2 \\ a^{11}\, a^{12} + \ldots + a^{n1}\, a^{n2} = 0 \\ a^{11}\, a^{13} + \ldots + a^{n1}\, a^{n3} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{1,n-1}\, a^{1n} + \ldots + a^{n,n-1}\, a^{nn} = 0 \end{array} \right. \tag{3}$$

Considering the n equations containing $a^{1j}, \ldots, a^{nj}$, namely,

$$\left\{ \begin{array}{l} a^{11}\, a^{1j} + a^{21}\, a^{2j} + \ldots + a^{n1}\, a^{nj} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{1j}\, a^{1j} + a^{2j}\, a^{2j} + \ldots + a^{nj}\, a^{nj} = \rho^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{1n}\, a^{1j} + a^{2n}\, a^{2j} + \ldots + a^{nn}\, a^{nj} = 0 \end{array} \right.$$

and solving for a^{ij} we get

$$a^{ij} = \frac{\rho^2}{J} A_{ij} \tag{4}$$

where A_{ij} is the cofactor of a^{ij} in J.

Since

$$J^2 = |a^{ij}| \times |a^{ij}| = \begin{vmatrix} \rho^2 & 0 \; \ldots\ldots & 0 \\ 0 & \rho^2 \ldots\ldots & 0 \\ \ldots & \ldots\ldots\ldots & \ldots \\ 0 & 0 \; \ldots\ldots & \rho^2 \end{vmatrix} = \rho^{2n}$$

we have $\rho^2 = |J|^{2/n}$, and equations (4) become

$$\boxed{(sgn\ J)\ |J|^{(n-2)/n}\, a^{ij} = A_{ij}} \tag{5}$$

These are the n-dimensional Cauchy-Riemann equations. For $n = 2$ we have, with $\mathbf{f} = (f^1, f^2)$,

$$J = \begin{vmatrix} \dfrac{\partial f^1}{\partial x^1} & \dfrac{\partial f^1}{\partial x^2} \\ \dfrac{\partial f^2}{\partial x^1} & \dfrac{\partial f^2}{\partial x^2} \end{vmatrix}$$

and equations (5) reduce to

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \quad \frac{\partial f^1}{\partial x^2} = -\frac{\partial f^2}{\partial x^1}$$

which are the ordinary Cauchy-Riemann equations, or to

$$\frac{\partial f^1}{\partial x^1} = -\frac{\partial f^2}{\partial x^2}, \quad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$$

the so-called conjugate Cauchy-Riemann equations, depending on whether $J > 0$ or $J < 0$. Clearly, if $\mathbf{f} = (f^1, f^2)$ has positive Jacobian, then the conjugate function $\overline{\mathbf{f}} = (f^1, -f^2)$ has a negative Jacobian.

For $n = 3$ we obtain nine equations of the form

$$a^{ij} \sqrt[3]{J} = A_{ij}$$

These equations, in an essentially equivalent form, were given by Hedrick and Ingold in 1925 [2]. The case $n = 4$ was considered by J. Abercrombie in 1970 [1].

We note that in the case $n = 3$ there are four conjugates, namely $\bar{\mathbf{f}} = (f^1, f^2, -f^3)$, $\mathbf{f}^* = (f^1, -f^2, f^3)$, $\mathbf{f}^* = (-f^1, f^2, f^3)$, and $-\mathbf{f} = (-f^1, -f^2, -f^3)$, corresponding to symmetries with respect to each of the coordinates planes, or to all three simultaneously (i.e. with respect to the origin). In general, if n is even, the mappings defined by $\mathbf{f}$ and $-\mathbf{f}$ belong to the same class (both are directly or inversely conformal), not so if n is odd.

3. Another form of the Cauchy-Riemann equations. Using (3) and (4) we obtain

$$(6) \qquad (a^{i1})^2 + \ldots + (a^{in})^2 = \frac{\rho^2}{J} (a^{i1} A_{i1} + \ldots + a^{in} A_{in}) = \rho^2$$

and, for $i \neq j$,

$$(7) \qquad a^{i1} a^{j1} + \ldots + a^{in} a^{jn} = \frac{\rho^2}{J} (a^{i1} A_{j1} + \ldots + a^{in} A_{jn}) = 0$$

Since $\nabla f^i = a^{ij} \mathbf{e}_j$, equation (6) can be written as

$$|\nabla f^i| = \rho \qquad (i = 1, \ldots, n)$$

and equation (7) as

$$\nabla f^i \cdot \nabla f^j = 0 \qquad (i \neq j)$$

Hence, the mapping defined by a function $\mathbf{f} = (f^1, \ldots, f^n)$ of class C^1 in Ω at a point $\mathbf{p}$ where $J_{\mathbf{f}}(\mathbf{x}) \neq 0$, is conformal if and only if all the component functions have gradients with the same magnitude at that point, and the gradients of any two different components are orthogonal. The common magnitude of the gradients is precisely the magnification ratio at $\mathbf{p}$, and the second condition means that the hypersurfaces $f^i = c^i$ meet orthogonally at $\mathbf{p}$.

4. The generalized Laplace equation. From (5) we have

$$(sgn\ J)\ |J|^{(n-2)/n} \frac{\partial f^i}{\partial x^j} = A_{ij}$$

Assuming now that the components f^i are of class C^2 in Ω, this yields

$$(sgn\ J) \frac{\partial}{\partial x^j} \left(|J|^{(n-2)/n} \frac{\partial f^i}{\partial x^j} \right) = \frac{\partial}{\partial x^i} A_{ij}$$

and

$$(8) \qquad (sgn\ J) \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \left(|J|^{(n-2)/n} \frac{\partial f^i}{\partial x^j} \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x^j} A_{ij} = 0$$

since the last sum in (8) can be written as a symbolic determinant

$$\Delta_i = \begin{vmatrix} \dfrac{\partial f^1}{\partial x^1} & \cdots & \dfrac{\partial f^1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial}{\partial x^1} & \cdots & \dfrac{\partial}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial f^n}{\partial x^1} & \cdots & \dfrac{\partial f^n}{\partial x^n} \end{vmatrix}$$

where the partial operators

$$\frac{\partial}{\partial x^j}$$

occupy the i-th row, and it follows easily that $\Delta_i = 0$.

Thus, (8) becomes

$$\sum_{j=1}^{n} \frac{\partial}{\partial x^j} \left(|J|^{(n-2)/n} \frac{\partial f^i}{\partial x^j} \right) = 0$$

This is the Laplace-type equation satisfied by each component f^i of a conformal mapping of class C^2. Clearly, in the particular case $n = 2$ it reduces to the ordinary Laplace equation in two dimensions.

REFERENCES

1. J. D. Abercrombie, *Bicomplex Quatermionic Function Theory*, University of Alabama, Ph. D. Dissertation (1970).

2. E. R. Hedrick and L. Ingold, *Analytic functions in three dimensions*, Trans. Amer. Math. Soc., 27 (1925), 551-555.